Fractional isometric path number

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Abstract: An isometric path is merely any shortest path between two vertices. The isometric path number is defined to be the minimum number of isometric paths required to cover the vertices of a graph. In this paper, we consider its fractional analogue. For classes of graphs such as trees, cycles and hypercubes, we determine the fractional isometric path number exactly. For square grid graphs, we provide upper and lower bounds. For grid graphs, finding the fractional isometric path number is equivalent to solving a network flow problem involving two simultaneous flows.

Keywords: Isometric path, grid, network flow

1 Introduction

An isometric subgraph of a graph G is defined to be a subgraph H of G such that for all $x, y \in V(H)$, $d_H(x,y) = d_G(x,y)$. Hence, an **isometric path** is any shortest path between two vertices of a graph. A set of isometric paths is said to **cover** $V(G)$ if every vertex of G lies on at least one path in the set. The **isometric path number** of G, denoted $p(G)$, is defined to be the minimum number of isometric paths required to cover the vertices of G.

The isometric path numbers of all grids (Cartesian products of two paths), were found in [1]. For example, the $n \times n$ grid was found to have an isometric path number of $\lceil 2n/3 \rceil$. In [3], the isometric path numbers of hypercubes were examined. It was shown that when $n+1$ is a power of 2, $p(Q_n) = 2^{n - \log_2(n+1)}$.

The problem of finding the isometric path number of a graph can be formulated as the following integer program:

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$$
\begin{aligned}
&\text{minimize} && \sum_{P \in \mathcal{P}} w(P) && \\
&\text{such that} && \sum_{P \ni v \wedge P \in \mathcal{P}} w(P) \geq 1 && \text{for all } v \in V(G) \\
& && w(P) \in \{0,1\} && \text{for all } P \in \mathcal{P}
\end{aligned}
$$

where $\mathcal{P}$ denotes the set of all isometric paths of maximal length. Note that each feasible solution corresponds to a cover where P is in the cover if and only if $w(P) = 1$. We refer to $w(P)$ as the *weight of* P and $\sum_{P \in \mathcal{P}} w(P)$ as the *weight of the cover*.

We obtain the linear programming relaxation of this integer program by replacing the constraint $w(P) \in \{0,1\}$ with $0 \leq w(P) \leq 1$. Any feasible solution to the resulting linear program is called a **fractional** isometric path cover. The minimum weight of a fractional isometric path cover is the **fractional isometric path number** of G, denoted $p_f(G)$. This is a typical approach, for other instances see [4].

It is also useful to consider the dual of the linear program since any feasible solution to the dual problem serves as a lower bound on $p_f(G)$. The dual is formulated as follows:

$$
\begin{aligned}
&\text{maximize} && \sum_{v \in V(G)} c(v) && \\
&\text{such that} && \sum_{v \in P} c(v) \leq 1 && \text{for all } P \in \mathcal{P} \\
& && 0 \leq c(v) \leq 1 && \text{for all } v \in V(G)
\end{aligned}
$$

where $c(v)$ is referred to as the cost of vertex v. For brevity, let $c(G)$ denote $\sum_{v \in V(G)} c(v)$.

The next three theorems are extensions of previous results to the integer program. In [1], it was shown that $p(G) \geq \lceil \frac{|V|}{diam(G)+1} \rceil$, and that for each the graphs presented in Theorems 1.2 and 1.3, $p(G) = \lceil p_f(G) \rceil$. Although the results are similar, the upper bounds are obtained by different means. Hence, we include the proofs for completeness.

Theorem 1.1 *Let G be any connected graph with vertex set V. Then*

$$p_f(G) \geq \frac{|V|}{diam(G)+1}.$$

Proof: For each vertex $v \in V(G)$, let $c(v) = \frac{1}{diam(G)+1}$. Since no isometric path has more than $diam(G)+1$ vertices on it, this is a feasible solution to the dual problem. Hence, $p_f(G) \geq \frac{|V(G)|}{diam(G)+1}$. □

Theorem 1.2 *Let C_n and K_n be the cycle and complete graph, respectively, on n vertices. For any $n \geq 1$*

1. $p_f(C_n) = 2n/(n+2)$ *for n even,*
2. $p_f(C_n) = 2n/(n+1)$ *for n odd,*
3. $p_f(K_n) = n/2$.

Proof: For each of these examples, we give all non-diameter paths weight 0, and all diameter paths weight $\frac{1}{diam(G)+1}$. Since every vertex is on exactly $diam(G) + 1$ of the diameter paths, we have a fractional cover. Since there are n diameter paths, the weight of this fractional cover is $\frac{n}{diam(G)+1}$. Since this equals the lower bound given in Theorem 1.1, we have a fractional cover of minimum weight. The result follows. □

Although the lower bound provided in Theorem 1.1 proves to be exact for many classes of graphs, this is not always the case. When considering trees, for example, a different solution to the dual program, and thus a different lower bound, is required.

Theorem 1.3 *If T is any tree and ℓ is the number of leaves in T then $p_f(T) = \ell/2$.*

Proof: Suppose $\mathcal{P}$ is the set of all maximal isometric paths in a tree T. Let $w(P) = 1/(\ell - 1)$ for each $P \in \mathcal{P}$. Since every vertex of T lies on at least $\ell - 1$ paths in $\mathcal{P}$, this is a fractional cover. Furthermore, every maximal isometric path in T has leaves as both its endpoints, so there are $\ell(\ell-1)/2$ paths in $\mathcal{P}$. Hence the weight of this cover is $\ell/2$, and $p_f(T) \leq \ell/2$.

Now, for each vertex v in T, define $c(v)$ as follows: if v is a leaf $c(v) = 1/2$, otherwise $c(v) = 0$. This is a feasible solution to the dual problem. Hence, $p_f(T) \geq \ell/2$ and the result follows. □

In [3], it was shown that for the hypercube Q_n, where $n + 1$ is a power of two, the lower bound given in Theorem 1.1 is exact. Hence, it is also equal to $p_f(Q_n)$. We now show that this fractional result can be extended to all values of $n \geq 1$.

Theorem 1.4 *Let Q_n denote the hypercube on 2^n vertices. Then $p_f(Q_n) = \frac{2^n}{n+1}$.*

Proof: Every maximal isometric path in Q_n is a diameter of Q_n, and there are $2^{n-1}(n!)$ diameters in Q_n. Furthermore, every vertex lies on $(n+1)n!/2 = (n+1)!/2$ diameters. Hence, if we let $w(P) = 2/(n+1)!$ for every maximal isometric path P, the result is a fractional cover with weight $(2^{n-1}(n!)(2/(n+1)!) = 2^n/(n+1)$. Hence, $p_f(Q_n) \leq \frac{2^n}{n+1}$. By Lemma 1.1, we have $p_f(Q_n) \geq \frac{2^n}{n+1}$, and the result follows. □

2 Grids

Let G_n denote the Cartesian product $P_n \Box P_n$. The graph G_n is also referred to as an $n \times n$ grid. The vertices of G_n can be labeled as coordinates on the grid. Unless stated otherwise, we let $V(G_n) = \{(i,j) | 0 \leq i,j \leq n-1\}$ where the distance between vertices $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$ is given by $d(v_1, v_2) = |i_1 - i_2| + |j_1 - j_2|$. Using this labeling, a vertex (i,j) is called a **corner** if i and j are both in $\{0, n-1\}$. We say a vertex is on the **boundary** of the grid if i or j is in $\{0, n-1\}$.

Lower bounds on $p_f(G_n)$ will be determined by considering the dual of the original problem. We give two feasible sets of solutions: one that applies for all $n \geq 3$ and another that applies when n is odd. The first gives all vertices within a certain proximity of a corner a cost of zero, and all other vertices the same positive cost. The second second also gives a zero cost to those within a given distance to a corner, but then assigns zero and non-zero costs to the remaining vertices in a "checkerboard" pattern.

Upper bounds are obtained by finding finding a fractional isometric path cover. In G_n, all maximal isometric paths extend from one corner of the grid to the diagonally opposite corner of the grid. Hence, there are two basic types of isometric paths to consider: those from $(0,0)$ to $(n-1, n-1)$ and those from $(n-1, 0)$ to $(0, n-1)$. Furthermore, if vertex (i,j) lies on an isometric path from $(0,0)$ to $(n-1, n-1)$, it must be followed by either $(i+1, j)$ or $(i, j+1)$. Hence, a maximal isometric path of this type is also a maximal path in the directed graph with underlying graph G_n and edges directed either left to right, or upward. A similar directed graph is associated with isometric paths from $(n-1, 0)$ to $(0, n-1)$. As a result, the problem of finding a fractional isometric path cover can be restated as a pair of simultaneous network flow problems.

2.1 Lower Bounds

The first lower bound presented is obtained by assigning a cost of 0 to all vertices within a certain distance of one of the corners of the grid, and a nonzero cost to all others. In effect, this is equivalent to choosing a particular isometric subgraph of G_n and applying Theorem 1.1 to that subgraph.

Lemma 2.1 *For any integers n and t such that $n \geq 3$ and $0 \leq t \leq \frac{n}{2}$,*

$$p_f(G_n) \geq \frac{n^2 - 2t(t+1)}{2n - 2t - 1}.$$

Proof: Given integers $n \geq 3$ and $0 \leq t \leq \frac{n}{2}$, we let $c(v) = 0$ if v is distance less than t from some corner. Otherwise, let $c(v) = 1/(2n - 2t - 1)$. We can see that this is a feasible solution to the dual by considering a maximal isometric path P in G_n. The path P contains $2n - 1$ vertices of which at least $2t$ are distance less than t from some corner. Therefore, $\sum_{v \in P} c(v) \leq 1$ for all $P \in \mathcal{P}$.

For any integer t such that $1 \leq t \leq \frac{n}{2}$, there are exactly t vertices distance $t-1$ from a particular corner of G_n. Furthermore, no vertex in G_n is distance at most $t-1$ from more than one corner when $t \leq \frac{n}{2}$. Since there are $4(1+2+\cdots+t) = 2t(t+1)$ vertices with cost 0, $c(G_n) = \dfrac{n^2-2t(t+1)}{2n-2t-1}$. The result follows. $\square$

While Lemma 2.1 applies to all $n \times n$ grids, the following Lemma provides an improvement for the cases where n is odd. In this case, we again assign those vertices within a certain distance of any corner a cost of zero. However, with the remaining vertices, non-zero and zero costs are assigned alternately, creating a "checkerboard" pattern.

Lemma 2.2 *For any odd integer $n \geq 3$,*

1. $p_f(G_n) \geq \dfrac{n^2+1-8k^2}{2n-4k}$ *for any integer k such that $0 \leq k \leq m$ where $m = \frac{n-1}{4}$ when $n \equiv 1 \pmod 4$ and $m = \frac{n+1}{4}$ when $n \equiv 3 \pmod 4$,*

2. $p_f(G_n) \geq \dfrac{n^2-1-8(l^2+l)}{2n-4l-2}$ *for any integer l such that $0 \leq l \leq m$ where $n = \frac{n-1}{4}$ when $n \equiv 1 \pmod 4$ and $m = \frac{n-3}{4}$ when $n \equiv 3 \pmod 4$.*

Proof: The graph G_n is bipartite with bipartition (X, Y) where $X = \{(i,j) | (i+j) \bmod 2 = 0\}$ and $Y = \{(i,j) | (i+j) \bmod 2 = 1\}$. Note that $|X| = \frac{n^2+1}{2}$ and $|Y| = \frac{n^2-1}{2}$.

1. Let X_k denote the set of vertices in X that are distance less than $2k$ from any corner. Let $c(v) = 0$ for all $v \in Y \cup X_k$, and $c(v) = 1/(n-2k)$ otherwise. This is a feasible solution to the dual problem since any isometric path in G_n contains at most n vertices from X, of which at least $2k$ are in X_k. Hence, on any isometric path in G_n, at most $n-2k$ vertices have a cost of $1/(n-2k)$, and all others have cost 0.

Given any corner in G_n and any k such that $1 \leq k \leq m$, there are $1+3+\cdots+(2k-1) = k^2$ vertices in X that are distance less than $2k$ from that corner. Since no vertex is distance less than $2m$ from more than one corner, $|X_k| = 4k^2$ when $1 \leq k \leq m$. Hence, $|X - X_k| = \frac{n^2+1}{2} - 4k^2$, and $c(G_n) = \left(\frac{n^2+1}{2} - 4k^2\right)\left(\frac{1}{n-2k}\right)$. The result follows.

2. Let Y_ℓ denote the set of all vertices in Y that are distance less than $2\ell+1$ from any corner. Let $c(v) = 0$ for all $v \in X \cup Y_l$, and $c(v) = 1/(n-2\ell-1)$ otherwise. This is a feasible solution to the dual problem since any isometric path in G_n contains at most $n-1-2l$ vertices from $Y - Y_\ell$.

There are $2+4+\cdots+2\ell = \ell(\ell+1)$ vertices in Y that are distance less than $2\ell+1$ from $(0,0)$. Since no vertex is distance less than $2\ell+1$ from more than one corner when $\ell \leq m$, there are exactly $4(\ell^2+\ell)$ vertices in Y_ℓ when $\ell \leq m$. Hence, $|Y - Y_\ell| = \frac{n^2-1}{2} - 4(\ell^2+\ell)$, and $c(G_n) = \left(\frac{n^2-1}{2} - 4(\ell^2+\ell)\right)\left(\frac{1}{n-2\ell-1}\right)$. The result follows. $\square$

Example 2.3 *Consider the cases $n = 7$ and $n = 9$. For $n = 7$, the first lower bound of Lemma 2.2 with $k = 1$ provides the best result. However, for the case $n = 9$, the second lower bound with $\ell = 1$ is best. In Figure 1, the vertices coloured black are precisely those receiving a non-zero cost. For G_7 there are exactly 21 vertices, v, such that $c(v) = 1/(7-2)$. For G_9 there are exactly 32 vertices, v, such that $c(v) = 1/(9-2-1)$. Hence, Lemma 2.2 gives $p_f(G_7) \geq 21/5$ and $p_f(G_9) \geq 16/3$.*

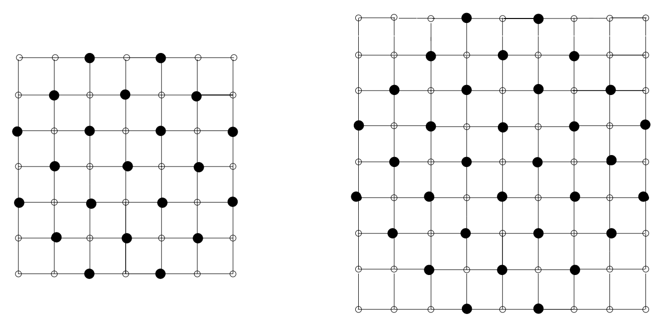

Figure 1: The "checkerboard" patterns on G_7 and G_9.

2.2 Upper Bounds

As previously mentioned, we can categorize each maximal isometric path in G_n as one of two types: those from $(0,0)$ to $(n-1,n-1)$ and those from $(n-1,0)$ to $(0,n-1)$. Hence, each maximal isometric path in G_n is a maximal directed path in one the two orientations illustrated in Figure 2. The directed graphs containing the two types of paths are denoted G'_n and G''_n, respectively.

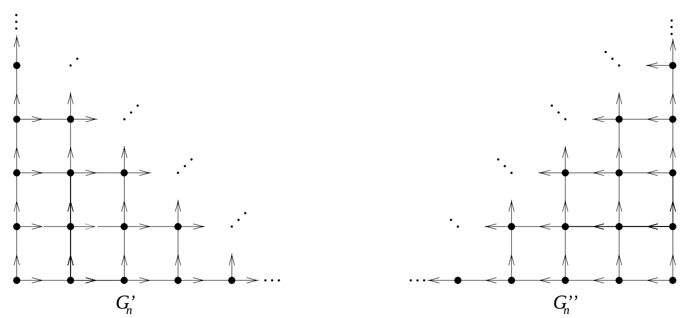

Figure 2: The directed graphs G'_n and G''_n.

The problem of finding the fractional isometric path number on G_n is equivalent to solving two simultaneous network flow problems, one on the graph G'_n with source $(0,0)$ and sink $(n-1,n-1)$ and the other on the graph G''_n with source $(n-1,0)$ and sink $(0,n-1)$.

Let f' and f'' denote feasible flows on G'_n and G''_n, respectively. We let $f'(e)$ denote the value assigned to every edge by f'. We will refer to the amount of flow *through* a vertex (i,j). If the vertex in question is the sink, the flow through it equals the flow into it, for all other vertices it equals the flow out of it. We will denote the amount of flow f' (respectively f'') through (i,j) as $f'(i,j)$ (respectively $f''(i,j)$). The values of f' and f'' are given by $\mathrm{val}(f') = f'(0,0)$ and $\mathrm{val}(f'') = f''(n-1,0)$, respectively.

Ultimately, we are interested in applying the flows f' and f'' simultaneously. Hence, we define the **combined flow**, or co-flow, f, and let $f(i,j) = f'(i,j) + f''(i,j)$. The value of the co-flow is given by $\mathrm{val}(f) = \mathrm{val}(f') + \mathrm{val}(f'')$.

Now, the problem of finding a fractional isometric path cover of G_n is equivalent to solving the following problem: *Find feasible flows f' and f'' on G'_n and G''_n, respectively, so that the co-flow f has the property that $f(i,j) \geq 1$ for all vertices (i,j) in G_n.* We will refer to such a co-flow as *feasible.* Our next task is to construct feasible flows f' and f'', with an eye toward minimizing $\mathrm{val} f$.

The flow f' will have symmetry about the diagonal running from $(0,0)$ to $(n-1,n-1)$, as well the diagonal running from $(n-1,0)$ to $(0,n-1)$. Specifically, for any $0 \leq i \leq n-2$ and $0 \leq j \leq n-1$, the three following arcs all have the same flow under f':

1. The arc from (i,j) to $(i+1,j)$,
2. The arc from (j,i) to $(j,i+1)$,
3. The arc from $(n-1-j, n-1-i)$ to $(n-1-j, n-1-(i+1))$.

The flow f'' is obtained by rotating f' by 90 degrees. Specifically, we have $f'(e_1) = f''(e_2)$ where e_1 is the arc from (i,j) to $(i+1,j)$ and e_2 is the arc from $(n-1-j,i)$ to $(n-1-j,i+1)$, with the remainder of f'' being determined by the same symmetries that applied to f'.

The total co-flow through the vertex (i,j) is given by $f(i,j) = f'(i,j) + f''(i,j) = f'(i,j) + f'(j,n-1-i) = f'(i,j) + f'(n-1-i,j)$. Hence, the co-flow f can be determined in its entirety by simply defining f' on the subgraph induced by $\{(i,j) | 0 \leq j \leq i \leq n-j-1\}$.

For example, flows on the directed graphs $G'_3, G'_4, \ldots G'_7$ are provided in Figure 3. When these flows are combined with the symmetric flows on G''_n for $n = 3, \ldots, 7$, the result is a feasible co-flow on each of the grids. Since the value of the co-flow provides an upper bound on the isometric path number, we have $p_f(G_3) \leq 2$, $p_f(G_4) \leq 12/5$, $p_f(G_5) \leq 3$, $p_f(G_6) \leq 32/9$ and $p_f(G_7) \leq 21/5$. When combined with the lower bounds previously presented, we find that these upper bounds are in fact the fractional isometric path numbers for these particular grids.

We now describe two methods for constructing feasible co-flows on the grid G_n for $n \geq 5$. Each method relies on using a known feasible co-flow on a smaller grid.

Method One

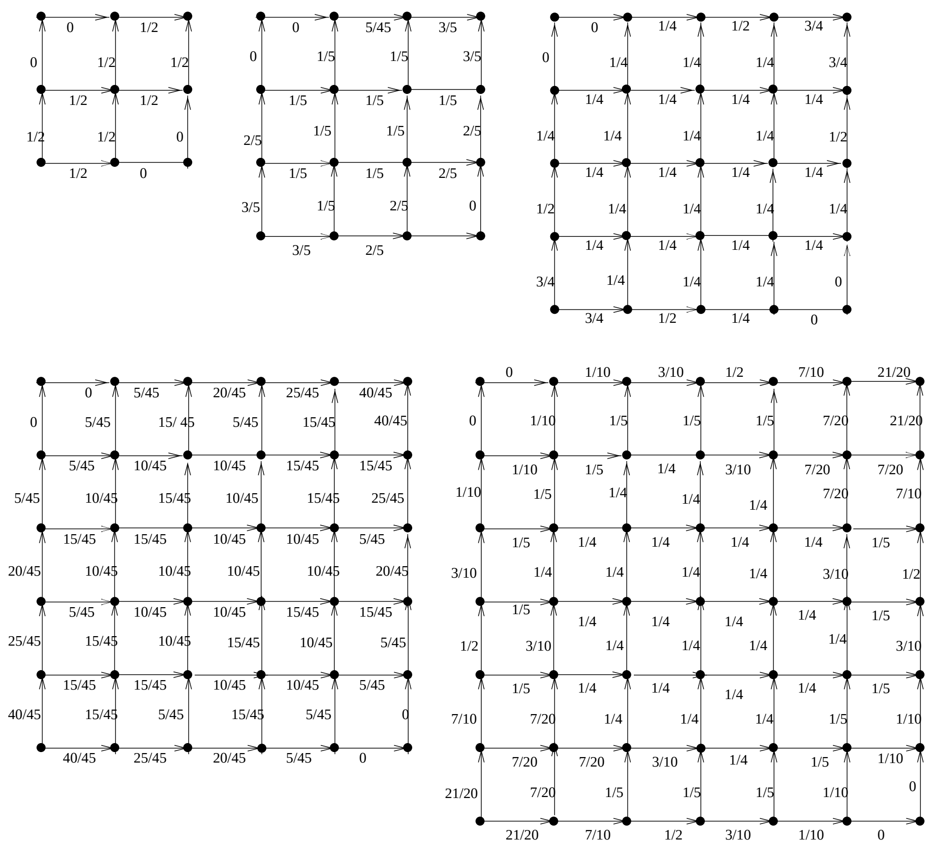

Figure 3: Flows on G'_n for $n = 3$ to 7

Let G_m be an $m \times m$ grid where $m \geq 5$. Let $V(G_m) = \{(i,j)|0 \leq i,j \leq m-1$. Define a flow, f'_0, on G'_m. In reference to f'_0, we let $r_0(i,j)$ denote the flow from (i,j) to $(i+1,j)$, and $u_0(i,j)$ denote the flow from (i,j) to $(i,j+1)$. We define f'_0 as follows:

1. $r_0(i,0) = \frac{m-i-2}{4}$ for all $i = 0, 1, \ldots, m-2$,
2. $r_0(i,j) = 1/4$ for all $i \geq j \geq 1$, $i + j < m - 1$,
3. $u_0(i,j) = 1/4$ for all $i > j \geq 0$, $i + j < m - 1$.

For example, the flow on G'_5 given in Figure 3 is the flow f'_0 for the case $m = 5$.

We now provide a recursive definition for the flow f'_{k+1} on the grid G'_{m+2k+2} $k \geq 0$. For integers $k \geq 0$, we define $r_k(i,j)$ and $u_k(i,j)$ in a similar manner as $r_0(i,j)$ and $u_0(i,j)$. Let f'_0 be defined as above, and let $a_0 = 1/4$. Note each subsequent flow formed has the same value as the previous flow. Hence, all flows formed using this recursive definition have a value of $(m-2)/2$.

Recursive Definition

If $(m-1)a_k > 1$ do the following:

1. The vertices of the grid G_{m+2k+2} are labeled from $(-k-1, -k-1)$ to $(m+k, m+k)$.
2. On the grid induced by the vertices from $(-k,-k)$ to $(m+k-1, m+k-1)$ place the flow f'_k,
3. For each $i = 1, 2, \ldots, m-2$, remove a flow of $a_{k+1} = \frac{(m-1)a_k - 1}{m-3}$ from the horizontal path that runs from $(-k,-k)$ to $(i,-k)$ and add that flow to the path that runs horizontally from $(-k-1,-k-1)$ to $(i,-k-1)$ then vertically to $(i,-k)$,
4. For each $i = -k+1, \ldots, 0$, remove a flow of $u_k(i,-k)$ from the horizontal path from $(-k,-k)$ to $(i,-k)$ and add that flow to the path that begins at $(-k-1,-k-1)$, runs horizontally to $(i,-k-1)$ and then vertically to $(i,-k)$ (when $k=0$, we omit this step),
5. Add necessary flow along the path $(-k-1,-k-1)(-k,-k-1)(-k,-k)$ in order to make f'_{k+1} feasible.

Now, using the recursive definition, we wish to find explicit expressions for for a_k, r_k and u_k, as well as determine the value of k for which the recursion terminates.

In order to form f'_{k+1}, we require that $(m-1)a_k > 1$. If $\ell(m)$ denotes the first value of k such that $(m-1)a_k \leq 1$, then the flow $f_{\ell(m)}$ on the grid $G_{m+2\ell(m)}$ is the last flow formed by this definition.

Given that $a_0 = 1/4$ and $a_{k+1} = \frac{(m-1)a_k - 1}{m-3}$, we obtain

$$\begin{aligned} a_k &= \frac{1}{4}\left(\frac{m-1}{m-3}\right)^k - \frac{1}{m-3}\left[\sum_{i=0}^{k-1}\left(\frac{m-1}{m-3}\right)^i\right] \\ &= \frac{1}{4}\left(\frac{m-1}{m-3}\right)^k - \frac{1}{2}\left[\left(\frac{m-1}{m-3}\right)^k - 1\right] \\ &= \frac{1}{2} - \frac{1}{4}\left(\frac{m-1}{m-3}\right)^k \end{aligned}$$

for all $k = 0, \ldots, \ell(m)$.

Now that we have this formula for a_k we can determine the exact value of $\ell(m)$. Solving the inequality $(m-1)a_k > 1$ results in

$$k < \frac{\ln 2}{\ln(m-1) - \ln(m-3)} - 1.$$

Hence, $\ell(m) < \frac{\ln 2}{\ln(m-1)-\ln(m-3)}$ and

$$\ell(m) = \left\lceil \frac{\ln 2}{\ln(m-1) - \ln(m-3)} \right\rceil - 1.$$

We now turn our attention to r_k and u_k. By induction, we assume the following:

- $r_0(i,j)$ and $u_0(i,j)$ are defined for all $0 \leq i,\ j \leq m-1$ according to the previous definition of f_0',
- For $k \geq 0$,

$$r_k(i,-k) = \begin{cases} (m-2)(a_k - a_{|i|} + 1/4), & i = -k, \ldots, -1 \\ (m-i-2)a_k, & i = 0, \ldots, m-3 \\ 0, & i = m-2, \ldots, m+k-2 \end{cases}$$

$$u_k(i,-k) = \begin{cases} (m-2)(a_{|i|} - a_{|i|+1}), & i = -k+1, \ldots, 0 \\ a_k, & i = 1, \ldots, m-2 \\ 0, & i = m-1, \ldots, m+k-2 \end{cases}$$

- If $k \geq 1$, then for all remaining vertices (i,j) where $r_k(i,j)$ and $u_k(i,j)$ are not determined by symmetry, $r_k(i,j) = r_{k-1}(i,j)$ and $u_k(i,j) = u_{k-1}(i,j)$.

We now show that $r_{k+1}(i,j)$ and $u_{k+1}(i,j)$ have the same form. The argument that follows shows that the flow f_k' is feasible (Lemma 2.4), and that the co-flow f_k is at least one through all vertices, except possibly those on the boundary (Lemmas 2.5 and 2.6).

Steps 3 and 4 of the recursive definition result in a transfer of flow from arcs of the type $(i,-k)(i+1,-k)$ to those of the type $(i,-k-1)(i+1,-k-1)$. For each $i = -k, \ldots, m+k-2$, the reduction in flow on the arc from $(i,-k)$ to $(i+1,-k)$ of the recursive definition is exactly equal to the increase in flow on arc from $(i,-k-1)$ to $(i+1,-k-1)$.

The value of the reduction on the arc $(i,-k)(i+1,-k)$ resulting from Step 3 is as follows:

- For each $i = -k, \ldots, -1$, $(m-2)a_{k+1}$ units,
- For each $i = 0, \ldots, m-3$, $(m-i-2)a_{k+1}$ units,
- For each $i = m-2, \ldots, m+k-2$, 0 units.

The value of the reduction on the arc $(i,-k)(i+1,-k)$ resulting from Step 4 is as follows:

- For $i = -k, \ldots, -1$, exactly $\sum_{j=i+1}^{0} u_k(j,-k) = (m-2)(1/4 - a_{|i|})$ units,
- For $i = 0, \ldots, m+k-2$, 0 units.

Hence,

$$r_{k+1}(i,-k) = \begin{cases} r_k(i,-k) - (m-2)(a_{k+1} + a_{|i|} - 1/4), & i = -k, \ldots, -1 \\ r_k(i,-k) - (m-i-2)a_{k+1}, & i = 0, \ldots, m-3 \\ r_k(i,-k), & i = m-2, \ldots, m+k-2 \end{cases}$$

Step 3 also results in an increase of $(m-2)a_{k+1}$ units of flow on the arc $(-k-1,-k-1)(-k,-k-1)$, and an increase of a_{k+1} units of flow on the arc $(i,-k-1)(i,-k)$ for each $i=1,\ldots,m-2$.

Step 4 also results in an increase of $(m-2)(1/4-a_k)$ units of flow on the arc $(-k-1,-k-1)(-k,-k-1)$, and an increase of $u_k(i,-k)$ units of flow on the arc $(i,-k-1)(i,-k)$ for each $i=-k+1,\ldots,0$.

Finally, Step 5 of the recursive definition requires additional flow in order to make the final flow feasible. After the first four steps, there is a flow of $(m-2)(a_k-a_{k+1})$ out of $(-k,-k)$, but no flow into $(-k,-k)$. Hence, we add this amount of flow along the path $(-k-1,-k-1)(-k,-k-1)(-k,-k)$.

Hence, we have the following values for r_{k+1} and u_{k+1}:

$$r_{k+1}(i,-k-1) = \begin{cases} (m-2)(a_{k+1}-a_{|i|}+1/4), & i=-k-1,\ldots,-1 \\ (m-i-2)a_{k+1}, & i=0,\ldots,m-3 \\ 0, & i=m-2,\ldots,m+k-1 \end{cases}$$

$$u_{k+1}(i,-k-1) = \begin{cases} (m-2)(a_{|i|}-a_{|i|+1}), & i=-k,\ldots,0 \\ a_{k+1}, & i=1,\ldots,m-2 \\ 0, & i=m-1,\ldots,m+k-1. \end{cases}$$

Note that for all vertices (i,j) not specifically mentioned above and not determined by the symmetry of f'_{k+1}, $r_{k+1}(i,j)=r_k(i,j)$ and $u_{k+1}(i,j)=u_k(i,j)$.

From the construction of f'_{k-1}, we now also have

$$r_{k+1}(i,-k) = \begin{cases} (m-2)(a_k-a_{k+1}), & i=-k,\ldots,-1 \\ (m-2-i)(a_k-a_{k+1}), & i=0,\ldots,m-3 \\ 0, & i=m-2,\ldots,m+k-2. \end{cases}$$

We now must verify that each of $f'_0, f'_1, \ldots, f'_{\ell(m)}$ is a feasible flow. It suffices to show that the flow on every arc is non-negative.

Lemma 2.4 *For each $k=0,1,\ldots,\ell(m)$, $f'_k(e)\geq 0$ for every arc e in G_{m+2k}.*

Proof: (By Induction) By definition, $f'_0(e)\geq 0$ for every arc in G_m. Assume f'_k has the desired property for some k where $0\leq k\leq \ell(m)-1$.

From the recursive definition, we have $f'_{k+1}(e)=f'_k(e)$ for every arc e with at least one end point in $\{(i,j)|-k+1\leq i,j\leq m+k-2\}$. Hence, by induction $f'_{k+1}(e)\geq 0$ for all such arcs.

Since $1/4=a_0>a_1>\cdots>a_{\ell(m)-1}>a_{\ell(m)}>0$, it is straightforward to show that all of the values of r_{k+1} and u_{k+1} are non-negative. Hence, $f'_{k+1}(e)\geq 0$ for every edge e of G_{m+2k+2}. By induction, the result follows. $\square$

We now turn our attention to the co-flow f_k formed by combining f'_k from the recursive definition with its counterpart f''_k, $k=0,\ldots,\ell(m)$. In order for f_k to be feasible we must have $f_k(i,j)\geq 1$ for all vertices (i,j). It turns out that

f_k is feasible for all $k = 0, \ldots, \ell(m) - 1$. The co-flow $f_{\ell(m)}$ is not necessarily feasible. However, $f_{\ell(m)}(i,j) \geq 1$ for all vertices (i,j) not on the boundary. We now verify these results.

Lemma 2.5 *For each $k = 0, 1, \ldots, \ell(m)$, $f_k(i,j) \geq 1$ for every vertex of G_{m+2k} in $\{(i,j) | -k \leq i, j \leq m + k - 1\}$.*

Proof: (By Induction) It is straightforward to show that $f_0(i,j) \geq 1$ for all (i,j) such that $0 \leq i, j \leq m - 1$. Now assume that for some k such that $0 \leq k \leq \ell(m) - 1$, $f_k(i,j) \geq 1$ for all (i,j) such that $-k \leq i, j \leq m + k - 1$.

Now consider the flow f'_{k+1}. By the recursive definition, $f'_{k+1}(i,j) = f'_k(i,j)$ for all (i,j) such that $-k + 1 \leq i, j \leq m + k - 2$.

Due to the symmetry of f'_{k+1} and f''_{k+1}, we have

$$f_{k+1}(i,-k) = \begin{cases} 2u_{k+1}(-k,-k-1), & i = -k \\ r_{k+1}(i-1,-k) + r_{k+1}(m-i-2,-k) + & \\ u_{k+1}(i,-k-1) + u_{k+1}(m-i-1,-k-1), & i = -k+1, \ldots, \frac{m-1}{2}. \end{cases}$$

Hence,

$$f_{k+1}(i,-k) = \begin{cases} 2(m-2)(a_k - a_{k+1}), & i = -k \\ (m-2)(a_k - a_{k+1} + a_{|i|} - a_{|i|+1}), & i = -k+1, \ldots, 0 \\ (m-1)a_k - (m-3)a_{k+1}, & i = 1, \ldots, \frac{m-1}{2}. \end{cases}$$

Since $a_k = \frac{1}{2} - \frac{1}{4}\left(\frac{m-1}{m-3}\right)^k$ for all $k = 0, \ldots, \ell(m)$, it follows that

$$1/4 = a_0 > \cdots > a_{\ell(m)} > 0,$$

$$a_k - a_{k+1} = \frac{1}{4}\left(\frac{m-1}{m-3}\right)^k \left(\frac{m-1}{m-3} - 1\right) = \frac{1}{2(m-3)}\left(\frac{m-1}{m-3}\right)^k,$$

$$\frac{1}{2(m-3)} = a_0 - a_1 < \cdots < a_{\ell-1} - a_{\ell(m)}.$$

From this we can easily verify that $f(i,-k) \geq 1$ for all $i = -k, \ldots, (m-1)/2$. Hence, by the symmetry of f_{k+1} we have the desired result. $\square$

Lemma 2.6 *For any $m \geq 5$ and each $k = 0, \ldots, \ell(m)$, $f_k(i,-k) \geq (m-1)a_k$ for all $i = -k, \ldots, m + k - 1$.*

Proof: (By Induction) The result is obviously true for f_0. Assume it is true for f_k where $0 \leq k \leq \ell(m) - 1$.

From the recursive definition and the symmetry of f_{k+1}, we have

$$f_{k+1}(i,-k-1) = \begin{cases} 2r_{k+1}(-k-1,-k-1), & i = -k-1 \\ r_{k+1}(i-1,-k-1) + & \\ r_{k+1}(m-i-2,-k-1), & i = -k, \ldots, \frac{m-1}{2}. \end{cases}$$

Therefore,

$$f_{k+1}(i,-k-1) = \begin{cases} (m-2)/2, & i=-k-1 \\ (m-2)(1/4+a_{k+1}-a_{|i|+1}), & i=-k,\dots,0 \\ (m-1)a_{k+1}, & i=1,2,\dots,(m-1)/2. \end{cases}$$

Using the fact that $(1/4+a_{k+1}-a_{|i|+1}) > (a_0-a_1+a_{k+1}) = \left(\frac{1}{2(m-3)}+a_{k+1}\right) > \frac{1}{2}+(m-2)a_{k+1} > (m-1)a_{k+1}$ when $-k \le i \le 0$, it is straightforward to verify the required result. $\square$

Corollary 2.7 *For any $m \ge 5$ and each $k = 0,\dots,\ell(m)-1$, f_k is a feasible co-flow and $p_f(G_{m+2k}) \le m-2$.*

Proof: Feasibility follows immediately from the definition of $\ell(m)$. Since $\mathrm{val} f_k = m-2$, we have $p_f(G_{m+2k}) \le m-2$. $\square$

Example 2.8 *As an example of this construction, we present the case $m = 9$. The flow f_0' is demonstrated in Figure 4 (0). Only the flow on one side of the boundary is provided, since the remaining flow is either 1/4, or determined by symmetry. (For a similar reason we have omitted parts of f_1' and f_2' as well.)*

Starting with f_0', we calculate $a_1 = 1/2 - 1/4(8/6) = 1/6$ and redirect some of the flow through vertices of the form $(i,-1)$. The resulting flow f_1' is seen in Figure 4 (1).

Once we have f_1', we calculate $a_2 = 1/2 - 1/4(8/6)^2 = 1/18$ and redirect some of the flow through vertices of the form $(i,-2)$. The result is f_2', seen in Figure 4 (2).

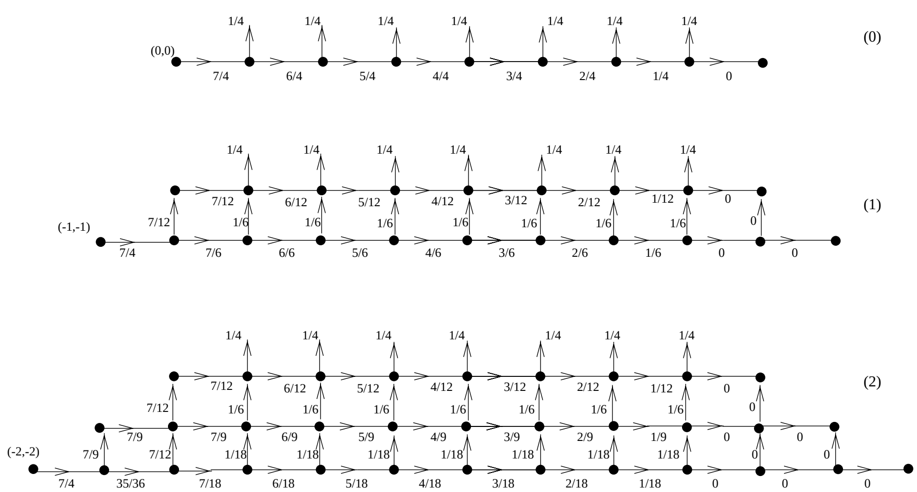

Figure 4: The flows f_0', f_1' and f_2' for the case $m = 9$.

As seen previously, there are vertices (i, j) on the boundary of $G_{m+2\ell(m)}$ such that $f_{\ell(m)}(i,j) = (m-1)a_{\ell(m)} \leq 1$. If $(m-1)a_{\ell(m)} < 1$, then $f_{\ell(m)}$ is not feasible. In order to create a feasible co-flow from $f_{\ell(m)}$, we could simply add flow through each of the four maximum isometric paths that run along the sides of the boundary. The second (and better) option is to take a linear combination of $f_{\ell(m)}$ together with a feasible co-flow that has excess flow through the boundary. This second co-flow will also result from the recursive definition using a different value of m. To avoid confusion, we will use the notation $f'_{m,0}, \ldots, f'_{m,\ell(m)}$ to denote the flows on $G'_m, \ldots, G'_{m+2\ell(m)}$ given by the recursive definition.

Now, suppose we wish to construct a feasible co-flow on the grid $G_{n+2\ell(n)}$ for some $n > 5$. We use the recursive definition to obtain the flow $f'_{n,\ell(n)}$ on the grid $G_{n+2\ell(n)}$. The resulting co-flow $f_{n,\ell(n)}$ has the property that some vertices on the boundary do not have a total flow of 1 through them, but all other vertices have a flow of at least 1. However, we do know the minimum flow through a vertex on the boundary is $(n-1)a_{\ell(n)} = (n-1)(1/2 - 1/4(\frac{n-1}{n-3})^{\ell(n)}) < 1$. Note that $\text{val} f_{n,\ell(n)} = (n-2)$.

Now, we use the recursive definition to obtain the flow $f'_{n+2,\ell(n)-1}$ on the grid $G'_{n+2\ell(n)}$. In this instance the minimum flow through a vertex on the boundary is $(n+1)(1/2 - 1/4(\frac{n+1}{n-1})^{\ell(n)-1})$. Note that $\text{val} f_{n+2,\ell(n)-1} = n$.

Let $f' = f'_{n,\ell(n)}$ and $g' = f'_{n+2,\ell(n)-1}$. We can take a linear combination of these two flows to form the new feasible flow h' as follows: for every arc e in $G_{n+2\ell(n)}$, let $h'(e) = xf'(e) + yg'(e)$ for x and y such that $0 \leq x, y \leq 1$. In order for the resulting co-flow h to be feasible, we require the following constraints to be satisfied:

$$x(n-1)\left(\frac{1}{2} - \frac{1}{4}\left(\frac{n-1}{n-3}\right)^{\ell(n)}\right) + y(n+1)\left(\frac{1}{2} - \frac{1}{4}\left(\frac{n+1}{n-1}\right)^{\ell(n)-1}\right) \geq 1$$

$$x + y \geq 1.$$

The first constraint guarantees a co-flow of at least one through vertices on the boundary, while the second does the same for the remaining vertices. The value of h is given by $\text{val} h = x \text{val} f + y \text{val} g = x(n-2) + yn$, and is minimized when both of the above inequalities achieve equality. The resulting value of h gives an upper bound on the fractional isometric path number. Hence, we have the following result:

Theorem 2.9 *For any integer $n \geq 5$,*

$$p_f(G_{n+2\ell(n)}) \leq \frac{(-n^2+n+2)\left(\frac{n+1}{n-1}\right)^{\ell(n)-1} + (n^2-n)\left(\frac{n-1}{n-3}\right)^{\ell(n)} + 4}{(-n-1)\left(\frac{n+1}{n-1}\right)^{\ell(n)-1} + (n-1)\left(\frac{n-1}{n-3}\right)^{\ell(n)} + 4}.$$

Example 2.10 *Consider the case where $m = 9$. By using the recursive definition, we can obtain the flow $f'_{9,2}$ on the grid G'_{13}. The formation f' was demonstrated in Figure 4. The resulting co-flow, f has a value of 7.*

Now repeat with $m = 11$. In this instance, we obtain the flow $f'_{11,1}$ on the grid G'_{13}. Call the resulting co-flow g. In this case every vertex not on the boundary has a co-flow of at least one through it, while the minimum flow through a vertex on the boundary is at least $(m-1)a_1 = 10(3/16) = 15/8$. The value of g is 9.

If we let $\frac{4}{9}x + \frac{15}{8}y = 1$ and $x + y = 1$, we obtain $x = 63/103$ and $y = 40/103$. We can now form the co-flow $xf + yg$ which is feasible and has value $7x + 9y = 801/103$.

Now, it is not the case that for any integer m there exists an integer n such that $m = n + 2\ell(n)$. Hence, not all grids can be given a feasible flow in this manner. In such cases, however, there is an n such that $m = n + 2k$ where $k \leq \ell(n) - 1$. By Corollary 2.7, the co-flow $f_{n,k}$ is feasible, and $p_f(G_m) \leq n - 2$

Method Two

The second method for constructing fractional isometric path covers is an extension of a construction appearing in [2]. The original result provides a means of constructing an isometric path cover on the grid G_{kn} whenever a cover for G_n is known. The technique is easily modified for the fractional problem.

The first step is to map each vertex (i, j) in G_n to a set of k^2 vertices in G_{kn}. Specifically $S(v) = \{(ki + \alpha, kj + \beta) | \alpha, \beta = 0, 1, \ldots, k-1\}$. For any isometric path P in G_n, there is a set of k parallel isometric paths P' in G_{kn} such that for every vertex v on P, the vertices in $S(v)$ each lie on exactly one of the corresponding paths in G_{kn}. Figure 5 demonstrates the map S, as well as the correspondence between isometric paths, for the case $k = n = 3$.

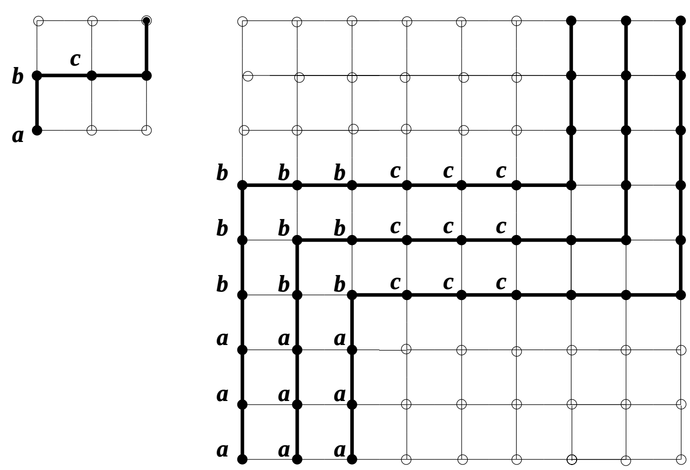

Figure 5: An isometric path in G_3 and the corresponding paths in G_9

Now consider any fractional isometric path cover, w, of G_n. For each isometric path P in G_n, a weight of $w(P)$ is assigned to each path in G_{kn} that appears in the corresponding set P'. All other isometric paths in G_{kn} are given weight zero. The result is a fractional isometric path cover of G_{kn}. Hence, we have the following:

Theorem 2.11 *If G_n has a fractional isometric path cover of weight p then G_{kn} has a fractional isometric path cover of weight kp.*

Corollary 2.12 *For any integers $k \geq 1$ and $n \geq 3$, $p_f(G_{kn}) \leq k \times p_f(G_n)$.*

Since the proof of Theorem 2.11 is almost identical to that of the analogous result in [2], we leave the details to the reader.

2.3 Experimental Results

For each value of m in the following table, upper and lower bounds on $p_f(G_m)$ have been determined. For all values of n presented, lower bounds result from equations presented in Section 2.1. For each odd value of n, the maximum of the three possible lower bounds is presented.

For $m = 3, 4, 5, 6$ and 7, the upper bounds were determined by constructing feasible flows, as seen in Figure 3. For the remaining values of m, upper bounds were determined using the two methods described in Section 2.2. In cases where both methods could be used (that is, when m is not prime), the minimum of the two upper bounds was chosen.

The table also shows the approximate ratio of the upper bound to the lower bound. Note that for cases where the ratio is one, we have determined the fractional isometric path number exactly.

m	3	4	5	6	7	8	9	10	11
lower	2	$\frac{12}{5}$	3	$\frac{32}{9}$	$\frac{21}{5}$	$\frac{52}{11}$	$\frac{16}{3}$	$\frac{88}{15}$	$\frac{13}{2}$
upper	2	$\frac{12}{5}$	3	$\frac{32}{9}$	$\frac{21}{5}$	$\frac{24}{5}$	$\frac{27}{5}$	6	7
ratio	1	1	1	1	1	1.0154	1.0125	1.0227	1.0769

m	12	13	14	15	16	17	18	19	20
lower	$\frac{120}{17}$	$\frac{23}{3}$	$\frac{156}{19}$	$\frac{97}{11}$	$\frac{216}{23}$	10	$\frac{264}{25}$	$\frac{145}{13}$	$\frac{340}{29}$
upper	$\frac{64}{9}$	$\frac{801}{103}$	$\frac{9971}{1198}$	9	$\frac{48}{5}$	$\frac{50659}{5001}$	$\frac{90249}{8432}$	$\frac{143903}{12781}$	12
ratio	1.0074	1.0144	1.0137	1.0206	1.0222	1.0130	1.0136	1.0094	1.0235

2.4 Asymptotic Results

We now compare upper and lower bounds on $p_f(G_m)$ for large values of m. First, we consider the case where $m = n + 2\ell(n)$ and use the upper bound obtained by our first method. In the next case, we consider $m = kn$ where $k \geq 3$ using the upper bound from the second method. In both cases, the lower bound from Lemma 2.1 is presented. (Asymptotically, for odd values of m, the two lower bounds from Lemma 2.2 provide the same result.) The infinite limits that follow were all evaluated using the software package MAPLE.

Let $m = n+2\ell(n)$. The first lower bound on $p_f(G_m)$ is given by the following function:

$$f(m,t) = \frac{m^2 - 2t\,(t+1)}{2m - 2t - 1}.$$

When $0 \leq t \leq m/2$, $f(t)$ is maximized at $t = -1/2+m-1/2\,\sqrt{-1+2\,m^2}$. Since t must be integer, we consider $f(m,t^*)$ where $t^* = \lfloor -1/2+m-1/2\,\sqrt{-1+2\,m^2} \rfloor$. We find

$$\lim_{n\to\infty} \frac{f(n+2\ell(n),t^*)}{n} = (\ln 2 + 1)(2-\sqrt{2}).$$

Now consider the upper bound on $p_f(G_{n+2\ell(n)})$ obtained by our first method. It is given by the function

$$g(n) = \frac{(-n^2+n+2)\left(\frac{n+1}{n-1}\right)^{\ell(n)-1} + (n^2-n)\left(\frac{n-1}{n-3}\right)^{\ell(n)} + 4}{(-n-1)\left(\frac{n+1}{n-1}\right)^{\ell(n)-1} + (n-1)\left(\frac{n-1}{n-3}\right)^{\ell(n)} + 4}$$

where $\ell(n) = \left\lceil \frac{\ln 2}{\ln(n-1)-\ln(n-3)} \right\rceil - 1$. We find

$$\lim_{n\to\infty} \frac{g(n)}{n} = 1.$$

Hence, the ratio of these upper and lower bounds on $p_f(G_{n+2\ell(n)})$ as n approaches infinity is

$$\lim_{n\to\infty} \frac{g(n)}{f(n+2\ell(n),t^*)} = \frac{1}{(\ln 2+1)(2-\sqrt{2})} \approx 1.0082.$$

Now we consider $m = kn$ where $p_f(G_n)$ is known. The lower bound is given by $f(m,t^*)$ where f and t^* are defined as in the previous case. We find

$$\lim_{n\to\infty} \frac{f(kn,t^*)}{kn} = 2-\sqrt{2}.$$

The upper bound is simply $k(p_f(G_n))$. Hence, as n approaches infinity the ratio of these upper and lower limits is

$$\lim_{n\to\infty} \frac{k(p_f(G_n))}{f(kn,t^*)} = \frac{p_f(G_n)}{n(2-\sqrt{2})}.$$

Using the flows in Figure 3, we can evaluate this limit for the specific cases $n = 3, 4, \ldots, 7$. The results are (approximately) 1.1381, 1.0243, 1.0243, 1.0116 and 1.0243, respectively.

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