Length of Finite Pierce Series: Theoretical Analysis and Numerical Computations

CS-96-21

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September 10, 1996[1]

[1]This research was done by April 4, 1996 as a course project in the course CS 760: Algorithmic Number Theory, taught by J. O. Shallit.

Abstract

Any real number $x \in (0,1]$ can be represented as a unique Pierce series

$$\frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \ldots$$

The series is finite if and only if the number x is rational. This paper discusses the length of the series and the final results are a new upper bound (Theorem 2) and a new lower bound (Theorem 3) on the length.

The numerical computations concerning the length are done by computer and the algorithms used and results are presented. The numerical results are an extension to the tables previously published.

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1 Introduction

Let the generalized binary operations div and mod be defined for all pairs of positive real numbers in the following way:

$$
\begin{aligned}
(\forall a, b \in \mathbb{R}^+) \qquad a \operatorname{div} b &\triangleq \max\{n \in \mathbb{Z} \ : \ bn \le a\}, \\
a \bmod b &\triangleq a - (a \operatorname{div} b) b.
\end{aligned}
$$

The result of the div operation is a nonnegative integer and the result of the mod operation is a nonnegative real number. Using the previous definitions it is easy to check that the following statements are true:

$$b > a \bmod b \ge 0, \tag{1}$$

$$a = b(a \operatorname{div} b) + a \bmod b, \text{ and} \tag{2}$$

$$\frac{a}{b} = a \operatorname{div} b + \frac{a \bmod b}{b}. \tag{3}$$

Remark: The priority of the operations div and mod is the same as the priority of multiplication or division.

Now, let x be any real number from the interval $(0, 1]$. If we denote $x_0 = x$ and calculate

$$q_1 = 1 \operatorname{div} x_0 \qquad \text{and} \qquad x_1 = 1 \bmod x_0$$

then using the relations (2) and (3) we have

$$
\begin{aligned}
x &= x_0 = \frac{1}{1/x_0} \overset{(3)}{=} \frac{1}{q_1 + x_1/x_0} = \frac{1}{q_1} - \left(\frac{1}{q_1} - \frac{1}{q_1 + x_1/x_0}\right) \\
&= \frac{1}{q_1} - \frac{q_1 + x_1/x_0 - q_1}{q_1(q_1 + x_1/x_0)} = \frac{1}{q_1} - \frac{1}{q_1} \cdot \frac{x_1}{x_0 q_1 + x_1} \overset{(2)}{=} \frac{1}{q_1} - \frac{1}{q_1} \cdot x_1.
\end{aligned}
$$

Also, using inequalities (1) and $1 \ge x$, we can state

$$0 < q_1, \qquad x = x_0 > x_1 \ge 0, \text{ and}$$

$$x = \frac{1}{q_1} - \frac{1}{q_1} \cdot x_1.$$

If x_1 is not zero then we can repeat the process by calculating

$$q_2 = 1 \text{ div } x_1 \qquad \text{and} \qquad x_2 = 1 \bmod x_1.$$

We have

$$(2) \quad \Rightarrow \quad x_0 q_1 + x_1 = 1 = x_1 q_2 + x_2 \overset{x_1 > x_2}{\Longrightarrow} x_0 q_1 < x_1 q_2 \overset{x_0 > x_1}{\Longrightarrow} q_1 < q_2.$$

In an analogous way to the previous step, we conclude

$$0 < q_1 < q_2, \qquad x = x_0 > x_1 > x_2 \geq 0, \text{ and}$$

$$x = \frac{1}{q_1} - \frac{1}{q_1} \cdot \left(\frac{1}{q_2} - \frac{1}{q_2} \cdot x_2\right) = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2} \cdot x_2.$$

If x_2 is nonzero then we can repeat the iteration and if x_3 is nonzero do it again, and so on. After k steps (if we succeed in making them) we will have k integers $0 < q_1 < q_2 < \ldots < q_k$ and k real numbers $x_1 > x_2 > \ldots > x_k \geq 0$ ($x_i < x_0 < 1$, $i = 1 \ldots k$) such that

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \ldots + \frac{(-1)^{k+1}}{q_1 q_2 \cdots q_k} + \frac{(-1)^k}{q_1 q_2 \cdots q_k} \cdot x_k. \tag{4}$$

If the process stops at some point, i.e. $x_k = 0$ for some k, then we know that x is a rational number and

$$x = \sum_{i=1}^{k} \frac{(-1)^{i+1}}{q_1 q_2 \cdots q_i}.$$

Therefore, for all irrational numbers x the process continues forever. The equality $q_{k+1} = 1 \text{ div } x_k$ implies $x_k < 1/q_{k+1}$, which gives

$$\left| x - \sum_{i=1}^{k} \frac{(-1)^{i+1}}{q_1 q_2 \cdots q_i} \right| = \frac{1}{q_1 q_2 \cdots q_k} \cdot x_k < \frac{1}{q_1 q_2 \cdots q_{k+1}} \leq$$

$$\leq \frac{1}{(k+1)!} \to 0 \qquad (k \to \infty),$$

so we can write

$$x = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{q_1 q_2 \cdots q_i}.$$

We still do not know whether there is a rational number x such that the process does not stop after a finite number of steps. To analyze this situation, let us assume that $x = b/a < 1$ for some positive integers a and b (the case $x = 1$ is trivial). In this case, the previous iteration looks like this:

$$\begin{aligned} q_1 &= 1 \text{ div } x_0 = 1 \text{ div } \frac{b}{a} = \max\{n \in \mathbb{Z} \ : \ n \cdot \frac{b}{a} \leq 1\} \\ &= \max\{n \in \mathbb{Z} \ : \ nb \leq a\} \\ &= a \text{ div } b, \end{aligned}$$

$$\begin{aligned} x_1 &= 1 \bmod x_0 = 1 - x_0(1 \text{ div } x_0) \\ &= 1 - \frac{b}{a}(a \text{ div } b) = \frac{a - b(a \text{ div } b)}{a} \\ &= \frac{a \bmod b}{a}. \end{aligned}$$

If we denote $b_0 = b$ and $b_1 = a \bmod b_0$ then $q_1 = a \text{ div } b_0$ and $x_1 = b_1/a$. If b_1 is not zero (i.e. x_1 is not zero) we can repeat the iteration, obtain b_2, q_2, and $x_2 = b_2/a$, and so on. Thus, in case that $x = b/a$ is a rational number, the ith iteration can be written as

$$b_i = a \bmod b_{i-1}, \quad q_i = a \text{ div } b_{i-1}, \quad \text{and} \quad x_i = b_i/a. \tag{5}$$

The sequence $\{x_i\}$ is strictly decreasing. Since $x_i = b_i/a$, the sequence $\{b_i\}$ is also decreasing. Actually, it is a decreasing sequence of positive integers, so it has to be finite, i.e. after a finite number of iterations we will get $b_k = 0$. Hence, if x is a rational number the sequence $\{x_i\}$ is finite.

We saw that any real number $x \in (0, 1]$ can be represented as a finite or infinite sum $\sum_i (-1)^{i+1}/(q_1 q_2 \ldots q_i)$. A natural question is: if a strictly increasing sequence of positive integers $\{q_i\}$ is given, how does the series $\sum_i (-1)^{i+1}/(q_1 q_2 \ldots q_i)$ behave? Since that series is alternating with the decreasing sequence of absolute values of its summands converging to 0 the series converges. The odd and even partial sums of that series are upper and lower bounds of its sum, respectively, so it is easy to see that the sum is in the interval $(0, 1]$.

The next question is: should we pose more constraints on the sequence of positive integers $\{q_i\}$, besides $q_i < q_{i+1}$, in order to guarantee uniqueness of that sequence when the number x is fixed?

Suppose that the number $x \in (0,1]$ can be expressed in two ways

$$x = \sum_i \frac{(-1)^{i+1}}{q_1 q_2 \dots q_i} = \sum_i \frac{(-1)^{i+1}}{m_1 m_2 \dots m_i}$$

where $\{q_i\}$ and $\{m_i\}$ are two distinct, finite or infinite, increasing sequences of positive integers. Then, there exists an integer j such that $q_i = m_i$ for $i < j$, and $q_j \neq m_j$. We have

$$\begin{aligned}
\sum_i \frac{(-1)^{i+1}}{q_1 q_2 \dots q_i} &= \sum_i \frac{(-1)^{i+1}}{m_1 m_2 \dots m_i} \Leftrightarrow \\
\sum_{i \geq j} \frac{(-1)^{i+1}}{q_1 q_2 \dots q_i} &= \sum_{i \geq j} \frac{(-1)^{i+1}}{m_1 m_2 \dots m_i} \Leftrightarrow \\
\sum_{i \geq j} \frac{(-1)^{i+1}}{q_j q_{j+1} \dots q_i} &= \sum_{i \geq j} \frac{(-1)^{i+1}}{m_j m_{j+1} \dots m_i} \Leftrightarrow \\
\frac{1}{q_j} - \frac{z}{q_j q_{j+1}} &= \frac{1}{m_j} - \frac{y}{m_j m_{j+1}},
\end{aligned}$$

where z and y are real numbers from the interval $[0,1]$. If q_{j+1} or m_{j+1} or both do not exist we can assume $q_{j+1} = q_j + 1$ or $m_{j+1} = m_j + 1$ (because $z = 0$ or $y = 0$). If we denote $w = 1/q_j - z/(q_1 q_{j+1})$, then w is a positive real number and

$$\begin{aligned}
q_j w &= 1 - \frac{z}{q_{j+1}} \leq 1, \\
(q_j + 1) w &= 1 - \frac{z}{q_{j+1}} + \frac{1}{q_j} - \frac{z}{q_j q_{j+1}} \geq 1 - \frac{1}{q_{j+1}} + \frac{1}{q_j} - \frac{1}{q_j q_{j+1}} \\
&\geq 1 - \frac{1}{q_j + 1} + \frac{1}{q_j} - \frac{1}{q_j (q_j + 1)} = 1.
\end{aligned}$$

We have two cases: if $(q_j + 1) w > 1$ then $q_j = 1 \bmod w$; otherwise, $(q_j + 1) w = 1$ and $q_j = 1 \bmod w - 1$. The second case will happen if and only if the sum is finite, $z = 1$, and $q_{j+1} = q_j + 1$. The analogous statement can be maid about m_j. We can assume $q_j < m_j$. This implies $q_j = 1 \bmod w - 1$,

$m_j = q_{j+1} = 1 \bmod w$, q_{j+1} is the last element of the sequence $\{q_i\}$, and m_j is the last element of the sequence $\{m_i\}$. In this and only in this situation, two distinct sequences $\{q_i\}$ and $\{m_i\}$ can represent the same number x. Since we want our algorithm (5) to always work, we will choose the shorter option, i.e. we will put the condition that if the sequence $\{q_i\}$ is finite (i.e. $1 \leq i \leq k$) then $q_k - q_{k-1} > 1$.

Finally, we can define that for any real number $x \in (0, 1]$ the expansion

$$x = \frac{1}{q_1} - \frac{1}{q_1 q_2} + \frac{1}{q_1 q_2 q_3} - \dots\ , \tag{6}$$

where $\{q_i\}$ is a strictly increasing sequence of positive integers, is called the *Pierce expansion.* The series on the right side of equation (6) is called the *Pierce series.* The expansion (6) can be finite or infinite. If the expansion is finite then one more condition on the sequence $\{q_i\}$ has to be satisfied: $q_k - q_{k-1} > 1$ where q_k is the last element of the sequence.

The facts proven in this section can be gathered in the following theorem:

Theorem 1 *Every real number in the interval* $(0, 1]$ *has a unique Pierce expansion. The rational numbers have finite Pierce expansions and the irrational numbers have infinite Pierce expansions. Any increasing sequence of positive numbers* $\{q_i\}$*, finite or infinite (if finite then the condition* $q_k - q_{k-1} > 1$ *is satisfied), represents the Pierce expansion of a real number from the interval* $(0, 1]$.

The sequence $\{q_i\}$ *which determines the Pierce expansion of a number* x *can be obtained using the recursive formulae*

$$x_0 = x, \quad x_i = 1 \bmod x_{i-1}, \quad \textit{and} \quad q_i = 1 \operatorname{div} x_{i-1} \qquad (i = 1, 2, \dots).$$

If the number x *is rational, i.e.* $x = b/a$ *for some positive integers* a *and* b $(a \geq b)$*, then the previous algorithm (in form of formulae) can be rephrased as*

$$b_0 = b, \quad b_i = a \bmod b_{i-1}, \quad q_i = a \operatorname{div} b_{i-1},$$

$$\textit{and} \quad x_i = \frac{b_i}{a} \qquad (i = 1, 2, \dots).$$

If the number x is rational, i.e. $x = b/a$ for some positive integers b and a ($b \leq a$), then the number of elements in the finite Pierce expansion of x, i.e. the number of elements of the finite sequence $\{q_i\}$ is called the *length* of that expansion, i.e. the length of that series, and is denoted as $P(a, b)$. The number $x_{P(a,b)-1}$ is the last element of the sequence $\{x_i\}$. We will denote $P(a) = \max\{P(a, b) \ : \ 1 \leq b \leq a\}$. It is easy to see that for all n ($1 \leq n \leq P(a)$) there exists such a number b ($1 \leq b \leq a$) such that $n = P(a, b)$. That is the reason why we are primarily interested in $P(a)$ when talking about the "length of the Pierce series."

This type of series was analyzed by Sierpiński in 1911 [8]. The expansion is due to Pierce in 1929 [4]. He made a short analysis of the expansion and showed how the expansion can be used to obtain approximations of the irrational roots of algebraic equations. Shallit gave the two above algorithms in 1983 (published 1986) [6]. In that paper, a thorough analysis of the Pierce expansions (more precisely, the metric theory of Pierce expansions) is given and we will refer to some of its results in the text that follows. Mays in 1985 (published 1987) [3] discussed indirectly the finite Pierce expansion and its length. Mays does not explicitly mention the Pierce expansion but his Algorithm 6 is basically the algorithm for producing the finite Pierce expansion and was given in [6]. We will refer to some of the results from that paper, too. The last paper discussing the matter is a 1991 paper [1] of Erdős and Shallit. In that paper, new lower and upper bounds are determined for the finite Pierce series and a table for the "Worst Cases for Pierce Expansions" up to $a \leq 830939$ (originally a is denoted as b) and $P(a) \leq 43$ is given.

2 Geometrical Representation

If we define a function $f\colon (0, 1] \to [0, 1]$ to be $f(x) = 1 \bmod x$, then the sequence $\{x_i\}$ can be simply expressed in terms of an iterative process:

$$\begin{aligned} x_0 &= x, \\ x_i &= f(x_{i-1}) \qquad \text{for } i = 1, 2 \ldots \end{aligned}$$

or we can write

$$x_i = f^i(x).$$

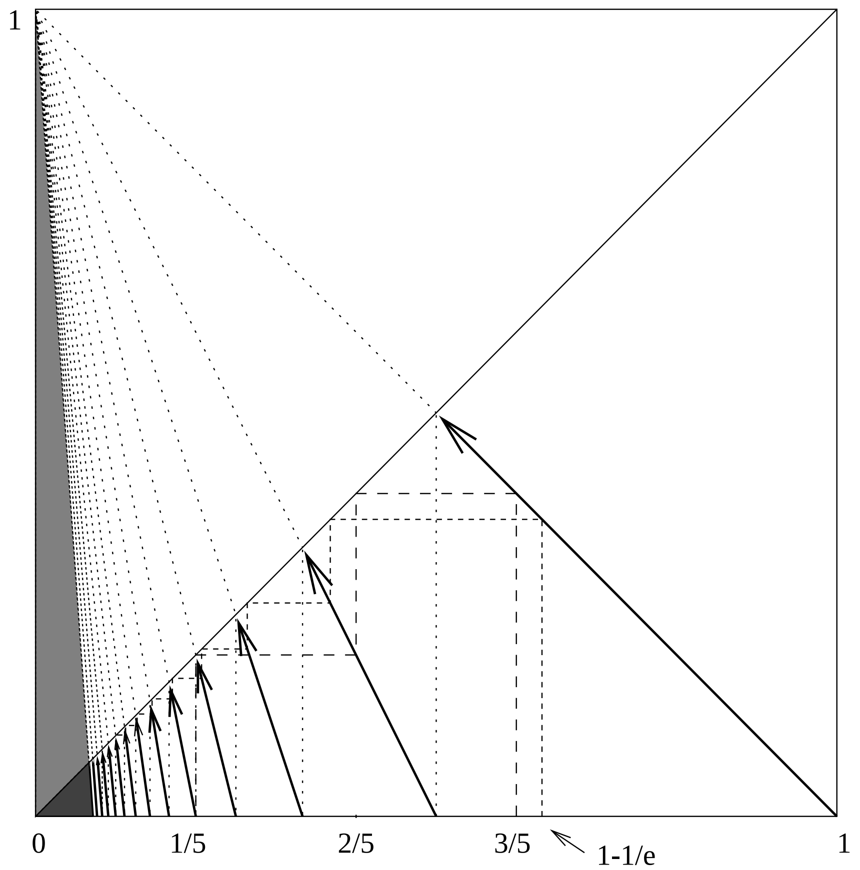

Figure 1: Geometrical representation

This approach gives us means to graphically illustrate the described algorithm (in general case). The illustration is given in Figure 1. The graph of the function f is a set of semi-open line segments

$$f(x) = 1 - nx, \qquad \text{for } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]. \tag{7}$$

The arrows on the figure represent the open ends of line segments. The limitations of the physical world do not allow us to print the complete graph, but we can get an idea. The figure also shows the sequences $\{x_i\}$ obtained in the Pierce expansions of numbers $3/5$ and $1 - e^{-1}$. The representation of the sequence $\{q_i\}$ is associated with relation $x_i \in \Big[1/(q_{i+1}+1), 1/q_{i+1}\Big)$.

The number $1 - e^{-1}$ is unique in the sense that $q_i = i$, i.e. that the numbers x_0, x_1, x_2, ... "hit" each segment of the form $[1/(i+1), 1/i)$. Intuitively, this means that the sequence $\{x_i\}$ obtained in the expansion of $1 - e^{-1}$ decreases with the slowest possible rate.

Since, the process stops if x_i reaches zero, without any harm to our analysis we can define

$$f(0) = 0.$$

This will save us from some distracting technical details.

If we note that $f([0,1]) = [0,1/2)$, $f([0,1/2)) = [0,1/3), \ldots$ or, in general, $f([0,1/i)) = [0,1/(i+1))$, then we get

$$f^n\left(\left[0,\frac{1}{i}\right)\right) = \left[0,\frac{1}{n+i}\right).$$

In the special case $i = 1$, we have

$$f^n([0,1)) = \left[0,\frac{1}{n+1}\right)$$

which implies

$$x_n = f^n(x) < \frac{1}{n+1} \tag{8}$$

for all $n = 1, 2, \ldots$.

If we take any real number $y \in (0,1)$, then we have

$$\left\lceil\frac{1}{y}\right\rceil - 1 < \frac{1}{y} \leq \left\lceil\frac{1}{y}\right\rceil \;\Rightarrow\; \frac{1}{\lceil 1/y\rceil - 1} > y \geq \frac{1}{\lceil 1/y\rceil}.$$

Using inequality (8), we get

$$y \geq \frac{1}{\lceil 1/y\rceil} > x_{\lceil 1/y\rceil - 1}. \tag{9}$$

3 Upper Bound

In this section, we will set an upper bound on the function $P(a)$ ($P\colon \mathbb{N} \to \mathbb{N}$). Since $P(a/a) = P(1) = 1$, we will not always make special remarks when a statement does not hold only for that case.

Let $x = b/a \in (0,1)$ (i.e. $a > b > 0$) be a rational number where a and b are positive integers (not necessarily relatively prime). We will use the same notation as in the first section, i.e. the following sequences will have the same definitions: $\{b_i\}_{i=0}^{k-1}$, $\{x_i\}_{i=0}^{k-1}$ and $\{q_i\}_{i=1}^{k}$, where $P(a,b) = k$ (b_{k-1}, x_{k-1} and q_k are the last nonzero elements obtained in the algorithm (5)). Knowing that all elements of the strictly decreasing sequence $\{b_i\}$ are from the finite set $\{1, 2, \ldots, a-1\}$, we easily conclude that there are no more than $a-1$ elements in that sequence. That means $P(a,b) = k \leq a-1$ which implies our first upper bound

$$P(a) \leq a - 1.$$

If we take a look at Figure 1 or at the inequality (8) we can note that the sequence $\{x_n\}$ decreases very fast in the beginning. We have not used that fact when we got the previous upper bound. In order to use it, we can choose any real number $y \in (0,1)$ and write

$$P(a,b) = \#\{x_i \ : \ i \geq 0\} = \#\{x_i \ : \ x_i \geq y\} + \#\{x_i \ : \ x_i < y\}, \tag{10}$$

where $\#X$ means: the number of elements of the finite set X. The inequality (9) implies

$$\#\{x_i \ : \ x_i \geq y\} \leq \left\lceil \frac{1}{y} \right\rceil - 1 < \frac{1}{y}. \tag{11}$$

On the other hand

$$\#\{x_i \ : \ x_i < y\} = \\ \#\{\frac{b_i}{a} \ : \ \frac{b_i}{a} < y\} = \#\{b_i \ : \ b_i < ay\} < ay,$$

since b_i's are distinct positive integers. Now, using the last two inequalities and (10), we have

$$P(a,b) < \frac{1}{y} + ay.$$

In order to choose the best value of y (so that the last bound reaches minimum), we calculate the derivation

$$\frac{d}{dy}(\frac{1}{y} + ay) = -\frac{1}{y^2} + a = 0 \Rightarrow y = \frac{1}{\sqrt{a}},$$

and we get the second upper bound

$$P(a,b) < 2\sqrt{a}\,,$$

i.e.

$$P(a) < 2\sqrt{a}\,.$$

This bound was proved by Shallit [6].

If $P(a,b)$ were close to the last upper bound then the set $\{b_i \ : \ b_i < ay\}$ would contain many close integers. The argument that follows does not allow that and we can get a better upper bound.

Let us denote

$$\begin{aligned} \Delta_i &= x_i - x_{i+1} && \text{for } i = 0, 1, \ldots, k-2, \\ \Delta_{k-1} &= x_{k-1}, \\ r_i &= b_i - b_{i+1} && \text{for } i = 0, 1, \ldots, k-2, \text{ and} \\ r_{k-1} &= b_{k-1}. \end{aligned}$$

Then $\Delta_i = r_i/a$. The definition of r_{k-1} and Δ_{k-1} is natural, since we can always assume $x_k = b_k = 0$ and it will not affect the following analysis.

If we recall the algorithm (5) for the finite Pierce series from the first section, we have

$$\begin{aligned} r_i = b_i - b_{i+1} &\Rightarrow b_i - r_i = b_{i+1} = a \bmod b_i \\ &\Rightarrow b_i \mid a - (b_i - r_i) = a + r_i - b_i \\ &\Rightarrow b_i \mid a + r_i. \end{aligned}$$

Since there are not too many divisors of $a + r_i$, we cannot have too many r_i's being equal. In order to use this limitation, we will choose two real numbers $1/a < z < y < 1$ and reformulate the equality (10):

$$\begin{aligned} P(a,b) &= \#\{x_i \ : \ x_i \geq y\} + \\ &\quad \#\{x_i \ : \ x_i < y \wedge \Delta_i < z\} + \#\{x_i \ : \ x_i < y \wedge \Delta_i \geq z\}. \quad (12) \end{aligned}$$

If $j = \#\{x_i \ : \ x_i < y \wedge \Delta_i \geq z\}$ and we list all elements of that set

$$y > x_{i_1} > x_{i_2} > x_{i_3} > \ldots > x_{i_j} > 0$$

then

$$\begin{aligned} y &> x_{i_1} \geq x_{i_2} + \Delta_{i_1} \geq x_{i_3} + \Delta_{i_2} + \Delta_{i_1} \geq \ldots \geq 0 + \sum_{l=1}^{j} \Delta_{i_l} \geq jz \\ &\Rightarrow j < \frac{y}{z} \Rightarrow \#\{x_i \ : \ x_i < y \wedge \Delta_i \geq z\} < \frac{y}{z}. \qquad (13) \end{aligned}$$

We showed above that $b_i \mid a + r_i$. If we note that

$$\Delta_i < z \Rightarrow \frac{r_i}{a} < z \Rightarrow r_i < az$$

then we have

$$\begin{aligned} \frac{b_i}{a} &\in \{x_i \ : \ x_i < y \wedge \Delta_i < z\} \\ &\Rightarrow b_i \text{ divides } a + r_i, \text{ where } r_i \text{ is an integer and } 0 < r_i < az \\ &\Rightarrow b_i \text{ divides a number from interval } \Big[a+1, a+\lceil az \rceil - 1\Big]. \end{aligned}$$

Since all numbers b_i are different we have

$$\#\{x_i \ : \ x_i < y \wedge \Delta_i < z\} \leq d\Big(\{a+1, a+2, \ldots, a + \lceil az \rceil - 1\}\Big) \qquad (14)$$

where $d(A) = \#\{n \in \mathbb{N} \ : \ (\exists a \in A) \ n \mid a\}$ $(d(\emptyset) = 0)$.

To make a bound on the function d, we can use a result from [2] (Theorem 315, page 260):

$$d(n) = O(n^{\delta}),$$

where $d(n)$ is the number of divisors of n and δ is any positive real number. This means that if we choose a positive real number δ, then there is a positive real constant c_1 such that

$$d(n) \leq c_1 n^{\delta}$$

for all n. Using (14) and this fact we get

$$\begin{aligned} &\#\{x_i \ : \ x_i < y \wedge \Delta_i < z\} \leq d\Big(\{a+1, a+2, \ldots, a+\lceil az \rceil - 1\}\Big) \leq \\ &\leq d(a+1) + d(a+2) + \ldots + d(a + \lceil az \rceil - 1) \leq \\ &\leq c_1(a+1)^{\delta} + c_1(a+2)^{\delta} + \ldots + c_1(a + \lceil az \rceil - 1)^{\delta} \leq \\ &\leq (\lceil az \rceil - 1) \cdot c_1(a + \lceil az \rceil - 1)^{\delta} < az \cdot c_1(a + az)^{\delta} = c_1 a^{1+\delta} z(z+1)^{\delta}. \end{aligned}$$

We can add a constraint $\delta < 1$ (the idea is to have small δ anyway), and since $z < 1$ we get from the last equation

$$\#\{x_i \;:\; x_i < y \wedge \Delta_i < z\} < 2c_1 a^{1+\delta} z. \tag{15}$$

If we combine (12), (11), (13), and (15) we get

$$\begin{aligned} P(a,b) &= \#\{x_i \;:\; x_i \geq y\} + \\ &\quad \#\{x_i \;:\; x_i < y \wedge \Delta_i < z\} + \#\{x_i \;:\; x_i < y \wedge \Delta_i \geq z\} \\ &< \frac{1}{y} + \frac{y}{z} + 2c_1 a^{1+\delta} z. \end{aligned} \tag{16}$$

In order to make the last expression minimal, we will try to choose y and z so that the partial derivations are zero:

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{-1}{y^2} + \frac{1}{z} = 0, \\ \frac{\partial}{\partial z} &= \frac{-y}{z^2} + 2c_1 a^{1+\delta} = 0 \end{aligned}$$

The first equation implies $y = \sqrt{z}$ and we easily get

$$y = \left(\frac{1}{2c_1 a^{1+\delta}}\right)^{1/3} \text{ and } z = \left(\frac{1}{2c_1 a^{1+\delta}}\right)^{2/3}.$$

If we substitute these values of y and z in (16) then we get

$$\begin{aligned} P(a,b) &< \frac{1}{(2c_1 a^{1+\delta})^{-1/3}} + \frac{\left(2c_1 a^{1+\delta}\right)^{-1/3}}{(2c_1 a^{1+\delta})^{-2/3}} + 2c_1 a^{1+\delta} \left(2c_1 a^{1+\delta}\right)^{-2/3} \\ &= 3\left(2c_1 a^{1+\delta}\right)^{1/3} = \left(3 \cdot 2^{1/3} c_1^{1/3}\right) \cdot a^{1/3+\delta/3} \end{aligned}$$

Since c_1 is a constant and δ is an arbitrary positive real number, the last inequality implies

$$P(a) = O(a^{1/3+\delta})$$

for any positive number δ. This is the third upper bound and it was proven by Erdős and Shallit [1].

However, we can improve the last result by a more strict use of inequality (14).

Firstly, let us note that the inequality (14) can be made more strict. Namely, since $b_i < a$ for all i, there can never be $b_i = a + j$ for any $j = 1, 2, \ldots, \lceil az \rceil - 1$ although $a + j \mid a + j$. So, instead of (14) we can write

$$\#\{x_i \ : \ x_i < y \wedge \Delta_i < z\} \leq d\Big(\{a+1, a+2, \ldots, a+\lceil az \rceil - 1\}\Big) - \lceil az \rceil + 1 \tag{17}$$

Let n and r be two positive integers such that $r < n$. We want to get an upper bound on the number $d\Big(\{n+1, n+2, \ldots, n+r\}\Big)$. If we denote $A = \{n+1, n+2, \ldots, n+r\}$ then

$$\begin{array}{rl} 1 & \text{divides exactly } \lfloor r/1 \rfloor \text{ elements of } A, \\ 2 & \text{divides at least } \lfloor r/2 \rfloor \text{ elements of } A, \\ 3 & \text{divides at least } \lfloor r/3 \rfloor \text{ elements of } A, \\ & \ldots \\ r-1 & \text{divides at least } \lfloor r/(r-1) \rfloor \text{ elements of } A, \text{ and} \\ r & \text{divides exactly } \lfloor r/r \rfloor \text{ elements of } A. \end{array}$$

Having this in mind, we get

$$\begin{aligned} d(A) &\leq d(n+1) + d(n+2) + \ldots + d(n+r) \\ &\quad -\Big(\Big\lfloor \frac{r}{1} \Big\rfloor - 1\Big) - \Big(\Big\lfloor \frac{r}{2} \Big\rfloor - 1\Big) \ldots - \Big(\Big\lfloor \frac{r}{r} \Big\rfloor - 1\Big) \\ &= \sum_{i=1}^{n+r} d(i) - \sum_{i=1}^{n} d(i) - \sum_{i=1}^{r} \Big\lfloor \frac{r}{i} \Big\rfloor + r. \end{aligned}$$

According to [2] (page 264), $\sum_{i=1}^{r} \lfloor r/i \rfloor = d(1) + d(2) + \ldots + d(r)$, so we get

$$d(A) \leq \sum_{i=1}^{n+r} d(i) - \sum_{i=1}^{n} d(i) - \sum_{i=1}^{r} d(i) + r.$$

There is a remark in [2] (page 272, § 18.2) which claims that Van der Corput in 1922 proved

$$d(1) + d(2) + \ldots + d(n) = n \log n + (2\gamma - 1)n + o(n^{33/100}),$$

where γ is Euler's constant and where log represents the natural logarithm ($\log \equiv \log_e$). Then we have

$$\begin{aligned} d(A) &\leq (n+r)\log(n+r) + (2\gamma-1)(n+r) + o((n+r)^{33/100}) - n\log n \\ &\quad -(2\gamma-1)n - o(n^{33/100}) - r\log r - (2\gamma-1)r - o(r^{33/100}) + r \\ &= n\log\left(1+\frac{r}{n}\right) + r\log\left(1+\frac{n}{r}\right) + o(n^{33/100}) + r. \end{aligned}$$

(We used the inequality $r < n$.) That means that for any real positive constant c_2

$$d(A) < n\log\left(1+\frac{r}{n}\right) + r\log\left(1+\frac{n}{r}\right) + c_2 n^{33/100} + r,$$

for n large enough. Now, starting from (17) we have

$$\begin{aligned} &\#\{x_i \;:\; x_i < y \wedge \Delta_i < z\} \leq \\ &\leq d\Big(\{a+1, a+2, \ldots, a+\lceil az\rceil - 1\}\Big) - \lceil az\rceil + 1 \\ &< a\log\left(1+\frac{\lceil az\rceil - 1}{a}\right) + (\lceil az\rceil - 1)\log\left(1+\frac{a}{\lceil az\rceil - 1}\right) + \\ &\quad + c_2 a^{33/100} + \lceil az\rceil - 1 - \lceil az\rceil + 1 \\ &< a\log(1+z) + az\log\left(1+\frac{a}{az-1}\right) + c_2 a^{33/100}. \end{aligned}$$

We should always keep in mind that z is chosen such that $z > 1/a$. Using, as before, (12), (11), (13), and the last inequality, we get

$$\begin{aligned} P(a,b) &= \#\{x_i \;:\; x_i \geq y\} + \\ &\quad \#\{x_i \;:\; x_i < y \wedge \Delta_i < z\} + \#\{x_i \;:\; x_i < y \wedge \Delta_i \geq z\} \quad (18) \\ &< \frac{1}{y} + \frac{y}{z} + a\log(1+z) + az\log\left(1+\frac{a}{az-1}\right) + c_2 a^{33/100}. \end{aligned}$$

We want to determine values of y and z so that the following partial derivations are zero:

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{-1}{y^2} + \frac{1}{z} = 0, \\ \frac{\partial}{\partial z} &= \frac{-y}{z^2} + \frac{a}{z+1} + a\log\left(1+\frac{a}{az-1}\right) + az \cdot \frac{az-1}{az-1+a} \cdot \frac{-a^2}{(az-1)^2} \\ &= 0. \end{aligned}$$

From the first equation we have $y = z^{1/2}$. We will not solve the second equation. Instead, we will approximate it by another one. (We don't have to solve the equation, we could even guess values of y and z.)

Since we know that we want to obtain $1/y = z^{-1/2} = o(\sqrt{a})$ and $az = o(\sqrt{a})$, we have $az \to \infty$ and $z \to 0$ when $a \to \infty$. Using this, we can modify the second equation by replacing all terms with their orders of magnitude:

$$
\begin{gathered}
-z^{-3/2} + a - a \log z - a = 0 \Rightarrow \\
z^{-3/2} = -a \log z.
\end{gathered}
$$

An approximate solution to the last equation is $z = 1.5^{2/3} a^{-2/3} (\log a)^{-2/3}$. This implies $y = 1.5^{1/3} a^{-1/3} (\log a)^{-1/3}$. Substituting y and z in (18) we get

$$
\begin{aligned}
P(a,b) &< 2 \cdot 1.5^{-1/3} a^{1/3} (\log a)^{1/3} + a \log(1 + 1.5^{2/3} a^{-2/3} (\log a)^{-2/3}) + \\
&\quad a \cdot 1.5^{2/3} a^{-2/3} (\log a)^{-2/3} \cdot \log\left(1 + \frac{a}{a \cdot 1.5^{2/3} a^{-2/3} (\log a)^{-2/3} - 1}\right) \\
&\quad + c_2 a^{33/100} \\
&= 2\sqrt[3]{2/3} \cdot a^{1/3} (\log a)^{1/3} + \Theta\left(a^{1/3} (\log a)^{-2/3}\right) + \\
&\quad \sqrt[3]{2/3} \cdot a^{1/3} (\log a)^{1/3} + o\left(a^{1/3} (\log a)^{1/3}\right) \\
&= 3\sqrt[3]{2/3} \cdot a^{1/3} (\log a)^{1/3} + o\left(a^{1/3} (\log a)^{1/3}\right) \\
&= \sqrt[3]{18} \cdot a^{1/3} (\log a)^{1/3} + o\left(a^{1/3} (\log a)^{1/3}\right) \\
&\approx 2.62074 \cdot a^{1/3} (\log a)^{1/3} + o\left(a^{1/3} (\log a)^{1/3}\right).
\end{aligned}
$$

Hence, we have our last upper bound.

Theorem 2

$$
P(a) < \sqrt[3]{18}\, a^{1/3} (\log a)^{1/3} + o(a^{1/3} (\log a)^{1/3})
$$

or

$$
P(a) = O(a^{1/3} (\log a)^{1/3}).
$$

4 Lower Bound

Let $r_1 < r_2 < \ldots < r_n$ be an increasing sequence of positive integers. Let us determine for which real numbers x $(0 < x < 1)$ $\{r_i\}$ is a starting subsequence of the sequence $\{q_i\}$ in the Pierce expansion of that number. Using observations from Section 2, we know that if q_n exists (>0) then

$$x_0 > x_1 > \ldots > x_{n-1} > 0$$

and

$$\begin{aligned} x_{i-1} &\in \left(\frac{1}{q_i+1}, \frac{1}{q_i}\right) \qquad \text{for } i = 1, 2 \ldots, n-1, \text{ and} \\ x_{n-1} &\in \left[\frac{1}{q_n+1}, \frac{1}{q_n}\right). \end{aligned}$$

We have

$$r_n = q_n \quad \Leftrightarrow \quad x_{n-1} \in \left[\frac{1}{r_n+1}, \frac{1}{r_n}\right).$$

Note that the interval $[1/(r_n+1), 1/r_n)$ has the length $1/(r_n(r_n+1))$. If we have $r_{n-1} = q_{n-1}$, besides having $r_n = q_n$, then an additional condition has to be satisfied:

$$x_{n-2} \in \left(\frac{1}{r_{n-1}+1}, \frac{1}{r_{n-1}}\right).$$

Since we know from (7) that

$$x_{n-1} = f(x_{n-2}) = 1 - r_{n-1}x_{n-2} \tag{19}$$

we get

$$q_{n-1} = r_{n-1} \wedge q_n = r_n \quad \Leftrightarrow \quad x_{n-2} \in f^{-1}\left(\left[\frac{1}{r_n+1}, \frac{1}{r_n}\right)\right),$$

where the function f is restricted to the formula (19). The set $f^{-1}\big([1/(r_n + 1), 1/r_n)\big)$ is an interval of the size

$$\frac{1}{r_{n-1}} \cdot \frac{1}{r_n(r_n+1)} = \frac{1}{r_{n-1}r_n(r_n+1)}.$$

Using previous argument as an inductive step, we can continue backwards and finally get

$$q_1 = r_1 \wedge q_2 = r_2 \wedge \ldots \wedge q_n = r_n \quad \Leftrightarrow \quad x_0 \in f^{-(n-1)}\left(\left[\frac{1}{r_n+1}, \frac{1}{r_n}\right)\right), \tag{20}$$

where the meaning of $f^{-(n-1)}$ should be understood in the "restricted" way as explained in the inductive step. The set $f^{-(n-1)}\big([1/(r_n+1), 1/r_n)\big)$ is an interval having the length

$$\frac{1}{r_1 r_2 \ldots r_{n-1} r_n (r_n+1)}.$$

We will denote that interval as I.

Let us fix $q_1 = 1$, $q_2 = 2$, ..., $q_n = n$. Then the interval I has the length $1/(n+1)!$. If we take a positive integer a such that

$$\frac{1}{a} < \frac{1}{(n+1)!} \tag{21}$$

then it is always possible to find a positive integer b ($b < a$) such that

$$\frac{b}{a} \in I.$$

For such a and b, the Pierce series of number a/b satisfies (20). Hence, we have

$$P(a,b) \geq n,$$

which implies

$$P(a) \geq n. \tag{22}$$

Because of Stirling's formula $n! \leq n(n/e)^n$, the inequality

$$a > n\left(\frac{n}{e}\right)^n (n+1) \tag{23}$$

implies (21). However, this is equivalent to

$$\log a > \log n + n \log n - n + \log(n+1). \tag{24}$$

Let a be an integer (large enough to have $\log\log\log a > 0$) and let $n = \lfloor \log a / \log\log a \rfloor$. Then

$$n \leq \frac{\log a}{\log\log a} \qquad \text{and} \qquad \log n \leq \log\log a - \log\log\log a < \log\log a.$$

These two inequalities imply

$$\log a > n \log n > n \log n - n + \log n + \log(n+1),$$

for n large enough. However, the last inequality is the same as (24) and, since we have

$$(24) \Rightarrow (23) \Rightarrow (21) \Rightarrow (22)$$

we get

$$P(a) \geq n = \left\lfloor \frac{\log a}{\log\log a} \right\rfloor.$$

We can state the following theorem:

Theorem 3 *For any real constant $1 - \epsilon < 1$ the inequality*

$$P(a) > (1-\epsilon) \cdot \frac{\log a}{\log\log a}$$

holds for numbers $a \in \mathbb{N}$ large enough. That inequality implies

$$P(a) = \Omega\left(\frac{\log a}{\log\log a}\right).$$

If n is any positive integer, and we choose $a = \mathrm{lcm}(2, 3, \ldots, n) - 1$ and $b = n$ then it is easy to see that

$$\begin{aligned}
b_0 &= b = n, \\
b_1 &= a \bmod b_0 = a \bmod n = n - 1, \\
b_2 &= a \bmod b_1 = a \bmod (n-1) = n - 2, \\
b_3 &= a \bmod b_2 = a \bmod (n-2) = n - 3, \\
\ldots & \\
b_n &= a \bmod b_{n-1} = a \bmod 1 = 0.
\end{aligned}$$

This gives

$$P(a, b) = n \;\Rightarrow\; P(a) \geq n.$$

Using the approximation $\varphi(x) < 1.03883x$ from [5] (Theorem 12) we get

$$\begin{aligned} \log a &= \log(\operatorname{lcm}(2, 3, \ldots, n)) \\ &= \log(\psi(n)) < 1.038838n \leq 1.038838P(a). \end{aligned}$$

Hence,

$$P(a) > 1.038838^{-1} \log a = 0.962614 \log a$$

for infinitely many a.

This relation is proven in [1].

5 Algorithms

Calculating $P(a, b)$
The algorithm for calculating $P(a, b)$ is the following:

Algorithm: $P(a, b)$

Input:	a, b	Two positive integers, $b \leq a$
Output:	$P(a, b)$	

1. $n \leftarrow 0$
2. **While** $b > 0$ **do**
3. $\quad$ | $b \leftarrow a \bmod b$
4. $\quad$ | $n \leftarrow n + 1$
5. **Return** n

Since steps 3 and 4 do not have bit complexity greater than $(\lg a)^2$, we get that the bit complexity of the algorithm above is $O(P(a)(\lg a)^2)$. Using Theorem 2, this gives upper bound on the running time $O(\sqrt[3]{a} \cdot (\lg a)^{7/3})$.

The upper bound given in Theorem 2 is very likely far from being tight, so the bit complexity of $O(\sqrt[3]{a} \cdot (\lg a)^{7/3})$ does not necessarily reflect the true behavior of the algorithm above.

Calculating $P(a)$
Note: The functions $P(a,b)$ and $P(a)$ should be differentiated by the number of arguments.

When calculating $P(a)$ we are interested also in the least value of b for which the maximum $P(a) = P(a,b)$ is reached. A simple way of calculating $P(a)$ is the following:

Algorithm: $P_s(a)$

Input:	a	A positive integer
Output:	$P(a)$, b	$P(a) = P(a,b)$

1. $n \leftarrow 1$
2. $b \leftarrow 1$
3. **For** $i \leftarrow 1$ **to** a **do**
4.   **If** $P(a,i) > n$ **then**
5. $n \leftarrow P(a,i)$
6. $b \leftarrow i$
7. **Return** (n, b)

The bit complexity is

$$a \times \text{bit complexity of } P(a,b) = O(aP(a)(\lg a)^2) = O(a^{4/3}(\lg a)^{7/3}),$$

so the subscript s means a simple, but also a slow, algorithm.

A faster algorithm with running time $O(a(\lg a)^2)$ uses the recursive relation

$$P(a,b) = P(a, a \bmod b) + 1.$$

Algorithm: $P_f(a)$

Input:	a	A positive integer
Output:	$P(a)$, b	$P(a) = P(a,b)$

1. $p_0 \leftarrow 0$
2. $n \leftarrow 0$

3. $b \leftarrow 0$
4. **For** $i \leftarrow 1$ **to** a **do**
5. $\quad$ $p_i \leftarrow p_{a \bmod i} + 1$
6. $\quad$ **If** $p_i > n$ **then**
7. $\quad\quad$ $n \leftarrow p_i$
8. $\quad\quad$ $b \leftarrow i$
9. **Return** (n, b)

The step 5 takes at most $(\lg a)^2$ running time so the algorithm's running time is

$$a(\lg a)^2.$$

The drawback is the large amount of memory which the algorithm requires: the array p_i has $a + 1$ entries, so the memory requirement is

$$a \lg P(a) + O(1) = O(a \lg a).$$

We can use the inequality $b_i < 1/(i+1)$ to overcome this potential problem. Thus, the algorithm can be modified so that it uses less memory but the running time increases. Since access to the elements of a large array does not have to be very a fast operation (e.g. because of paging) the modified algorithm which uses less memory could be in practice even faster than the algorithm above. The modified algorithm with a parameter $k \in \{1, 2, \ldots, a\}$ represents, actually, the whole spectrum of algorithms between the algorithms P_f and P_s: $k = 1$ gives the algorithm P_f and $k = a$ gives P_s.

Algorithm: $P_m(a, k)$

Input:	a, k	$k \in \{1, 2, \ldots, a\}$ parameter
Output:	$P(a)$, b	$P(a) = P(a, b)$

1. $p_0 \leftarrow 0$
2. $n \leftarrow 0$
3. $b \leftarrow 0$
4. **For** $i \leftarrow 1$ **to** $\lfloor a/k \rfloor$ **do**
5. $\quad$ $p_i \leftarrow p_{a \bmod i} + 1$
6. $\quad$ **If** $p_i > n$ **then**

```
7.          n ← p_i
8.          b ← i
9.  For i ← ⌊a/k⌋ + 1 to a do
10.     p ← 1
11.     j ← a mod i
12.     While j > ⌊a/k⌋ do
13.         p ← p + 1
14.         j ← a mod j
15.     p ← p + p_j
16.     If p > n then
17.         n ← p
18.         b ← j
19. Return (n, b)
```

The running time of the algorithm is

$$\underbrace{\frac{a}{k}(\lg a)^2}_{\text{loop 4–8}} + \underbrace{\left(a - \frac{a}{k}\right)\left(\underbrace{(k-2)(\lg a)^2}_{\text{loop 12–14}} + \underbrace{(\lg a)^2}_{\text{step 11}}\right)}_{\text{loop 9–18}} + o(a(\lg a)^2) =$$

$$\left(k - 2 + \frac{2}{k}\right) a(\lg a)^2 + o(a(\lg a)^2).$$

Notice that the running time for $k = 1$ and $k = 2$ is the same. This means that the choice $k = 2$ is better even when we are primarily interested in achieving a good running time and not concerned about memory. The memory requirement for the algorithm P_m is

$$\frac{a}{k} \lg P(a) + O(1).$$

According to Theorem 2, this gives the memory usage of

$$\frac{a}{3k} \lg a + O(1).$$

6 Numerical Results

Table 1 contains the longest cases for Pierce Expansions; i.e. for all values $n = 1, 2, \ldots, 49$ the values of a

$$a = \min\{a \ : \ P(a) = n\},$$

and of b

$$b = \min\{b \ : \ P(a, b) = n\},$$

are given. $P(a)$ is calculated for all values of a up to 3600000. Table 2 gives the shortest cases for Pierce expansions, i.e. for all $n \in \{1, 2, \ldots, 49\}$ and for $a \in \{1, 2, \ldots, 3600000\}$ the column a is defined to be

$$a = \max\{a \ : \ P(a) = n\},$$

and b is

$$b = \min\{b \ : \ P(a, b) = n\}.$$

This tables changes with each new calculation of $P(a)$. After calculating $P(a)$ for $a > 3600000$, we can expect that only the entries $P(a) < 15$ in the table will remain the same.

Figure 2 gives a graph showing a grey area which includes the graph of function P. The lower and upper bounds obtained are also presented. We can note that the lower bound doesn't seem so bad while the upper bound is really loose. The dotted lines present some speculations about bounds: the lower one has the formula $2 \log a / \log\log a$ and the upper one has the formula $0.25 \cdot (\log a)^2$.

n	b	a	n	b	a
1	1	1	26	3749	5879
2	2	3	27	6546	17747
3	3	5	28	11201	17747
4	4	11	29	2159	23399
5	7	11	30	2360	23399
6	12	19	31	5186	23399
7	22	35	32	6071	23399
8	30	47	33	8664	23399
9	32	53	34	14735	23399
10	61	95	35	59745	93596
11	65	103	36	68482	186479
12	115	179	37	117997	186479
13	161	251	38	175672	278387
14	189	299	39	268618	442679
15	296	503	40	135585	493919
16	470	743	41	178909	493919
17	598	1019	42	314752	493919
18	841	1319	43	490652	830939
19	904	1439	44	76800	1371719
20	1856	2939	45	116789	1371719
21	2158	3359	46	125493	1371719
22	2416	3959	47	290641	1371719
23	1925	5387	48	540539	1371719
24	3462	5387	49	831180	1371719
25	2130	5879			3600000

Table 1: The Longest Cases for Pierce Expansions

n	b	a	n	b	a
1	1	2	26	2173029	3599980
2	4	6	27	2266788	3599995
3	13	24	28	2310242	3599990
4	41	72	29	2060744	3599982
5	146	240	30	2276141	3599992
6	407	720	31	2273176	3599994
7	1537	2880	32	2273313	3599996
8	3667	6720	33	2271838	3599984
9	10291	20160	34	2197792	3599979
10	31261	60480	35	2271841	3599969
11	126223	241920	36	2173023	3599999
12	259591	483840	37	2298936	3599998
13	501953	950400	38	2268946	3599879
14	895247	1647360	39	2653511	3598558
15	2117833	3507840	40	2273868	3597299
16	2004599	3598560	41	2294962	3596207
17	2283651	3595200	42	2294608	3595649
18	2107085	3598848	43	2269649	3590997
19	2114425	3599640	44	2535257	3576382
20	2069477	3600000	45	2217162	3477599
21	2034242	3599856	46	2182157	3427199
22	2173221	3599960	47		Not found
23	2123394	3599872	48		Not found
24	2272080	3599988	49	1662360	2743438
25	2119295	3599946			3600000

Table 2: The Shortest Cases for Pierce Expansions

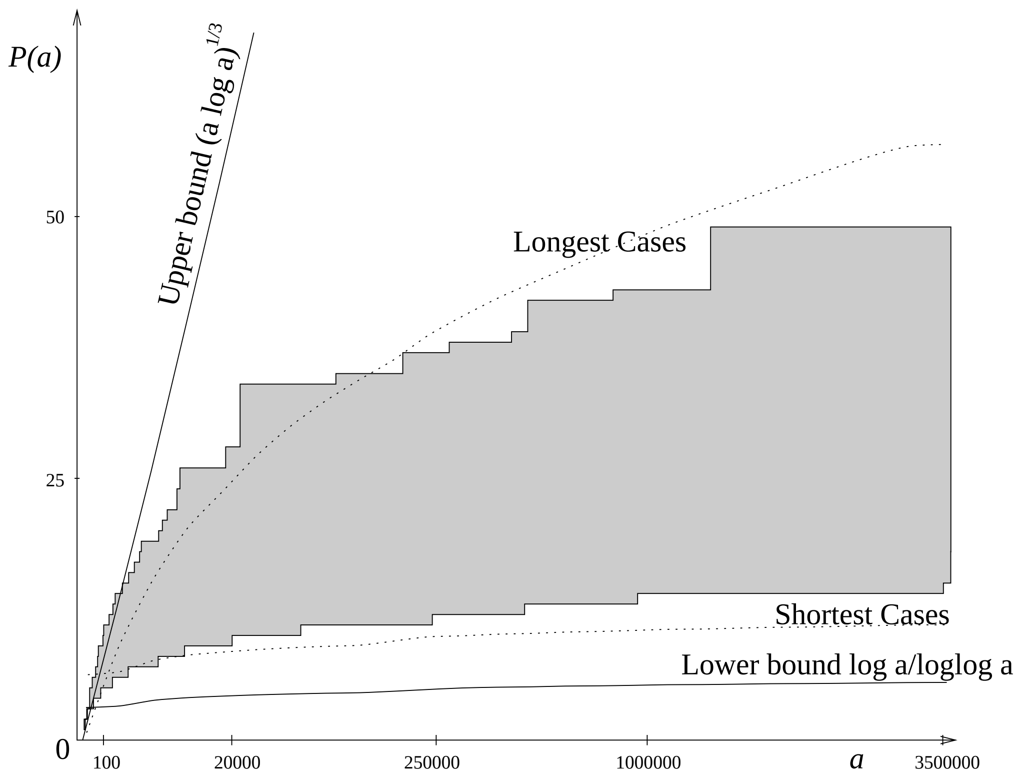

Figure 2: The upper and lower bound

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