NP-Hardness

Textbook Reading

Chapter 34

Overview

- Computational (in)tractability
- Decision problems and optimization problems
- Decision problems and formal languages
- The class P
- Decision and verification
- The class NP
- NP hardness and NP completeness
- Polynomial-time reductions

NP-complete problems:

- Satisfiability
- Vertex cover
- Hamiltonian cycle
- Subset sum

Computational (In)Tractability

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A decision problem asks a yes/no question:

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To turn an optimization problem into a decision problem, we provide a threshold for the cost/weight/… of the solution.

Decision Is No Harder Than Optimization

Yes/no answers usually aren't that useful in practice.

However, if we can provide evidence that the decision version of an optimization problem is intractable, then so is the optimization problem itself, by the following lemma:

Lemma. If an optimization problem can be solved in polynomial time, then so can its decision version.

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Decision algorithm:

- Solve the optimization problem.
- Compare the value of its solution to the given threshold.

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Consider the transformations

- Problem $P \rightarrow$ language $L \rightarrow$ problem P'
- Language $L \rightarrow$ problem $P \rightarrow$ language L'

Then $P = P'$ and $L = L'$.

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Informally, P is the set of all tractable decision problems, since

- We observed that decision problems and formal languages are the same thing and
- We consider a problem tractable if it can be solved in polynomial time.

Verification

Consider an algorithm V that decides a language $L' \subseteq \Sigma^* \times \Sigma^*$, that is, its input is a pair (x, y) such that $x, y \in \Sigma^*$.

Algorithm V is said to verify a language L if

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Thus, given an input (x, y) consisting of an element $x \in L$ and an appropriate "proof" $y \in \Sigma^*$ that shows that $x \in L$, V answers yes.

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V does not decide whether $x \in L$. V may answer no even if $x \in L$ if the provided proof of its membership in L is incorrect.

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Verifying L can be done in constant time!

- Let $L' = \{(S, (i, j)) \mid x_i = x_j, i \neq j\}$
- Given some pair $(S, (i, j))$, we can decide in constant time whether $(S, (i, j)) \in L'$ by comparing x_i and x_j.
- This algorithm verifies L because $x \in L$ if and only if there exists a pair (i, j) such that $(S, (i, j)) \in L'$.

The Complexity Class NP

The complexity class NP is the set of all languages that can be verified in polynomial time.

Formally, a language L belongs to NP if and only if there exists a language $L' \in P$ and a constant c such that $x \in L$ if and only if $(x, y) \in L'$ for some $y \in \Sigma^*$, $|y| \leq |x|^c$.

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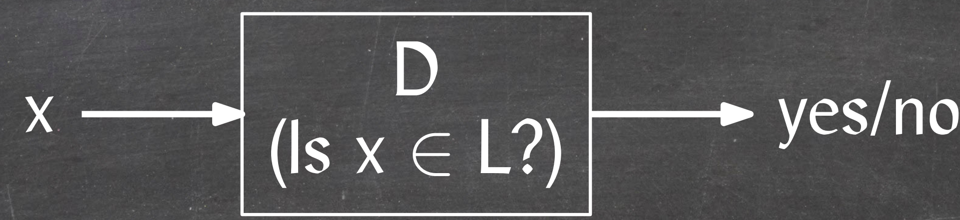

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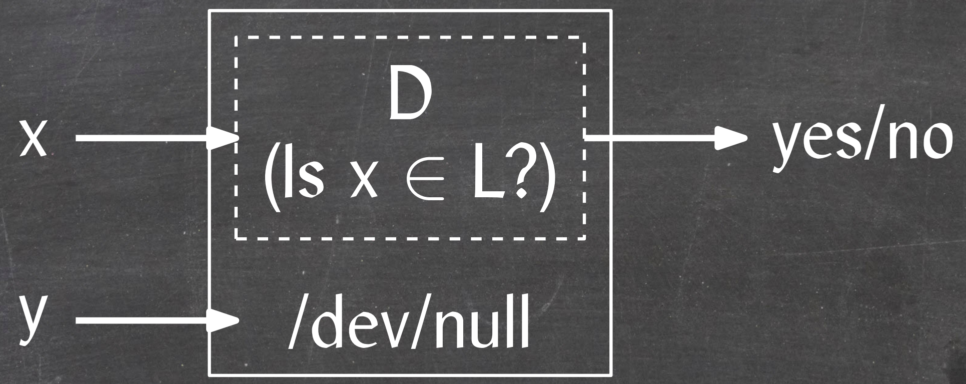

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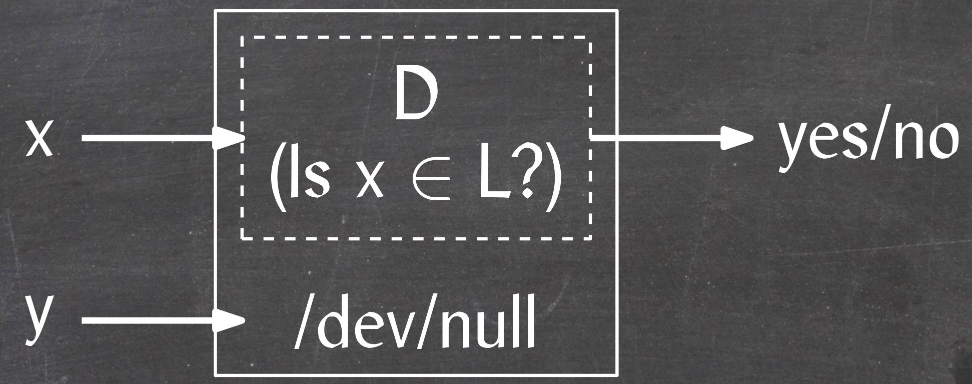

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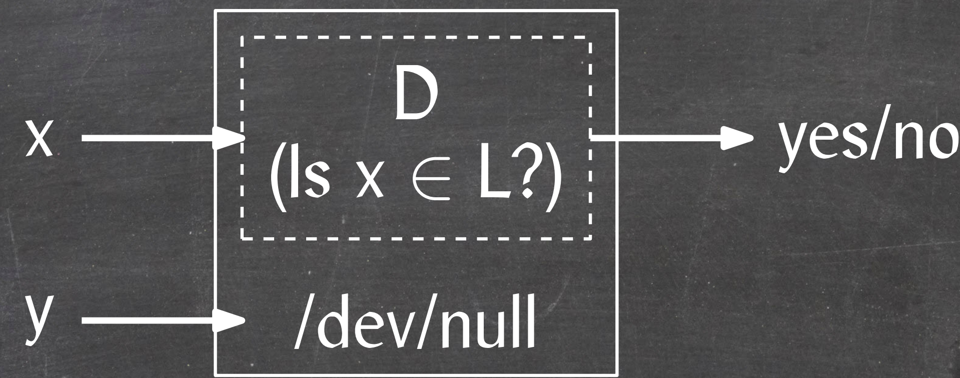

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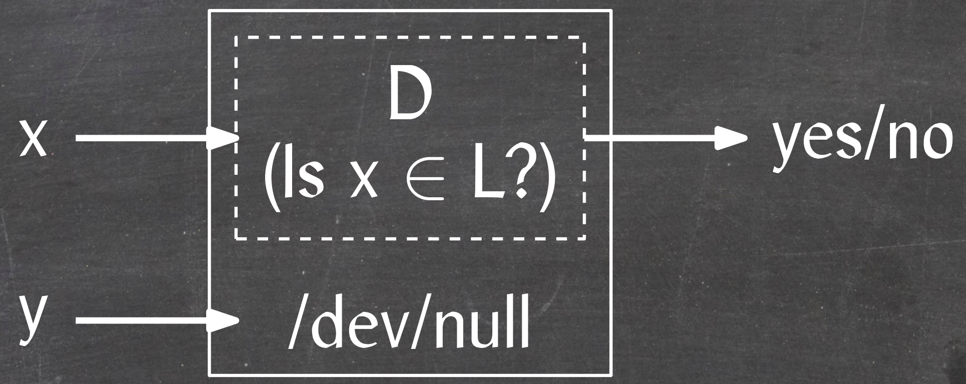

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We will show that there exist languages that cannot be decided (decision problems that cannot be solved) in polynomial time unless $P = NP$!

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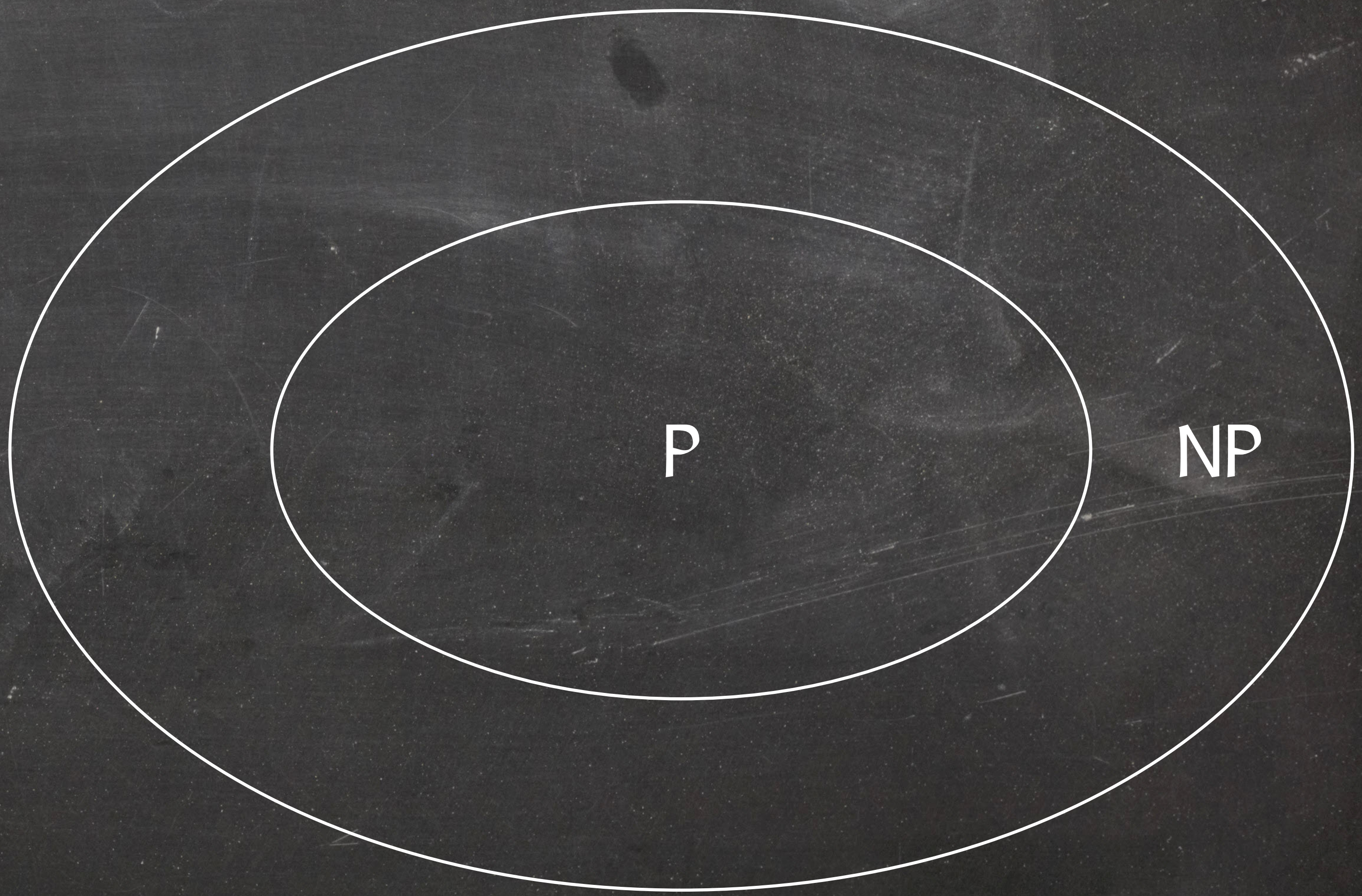

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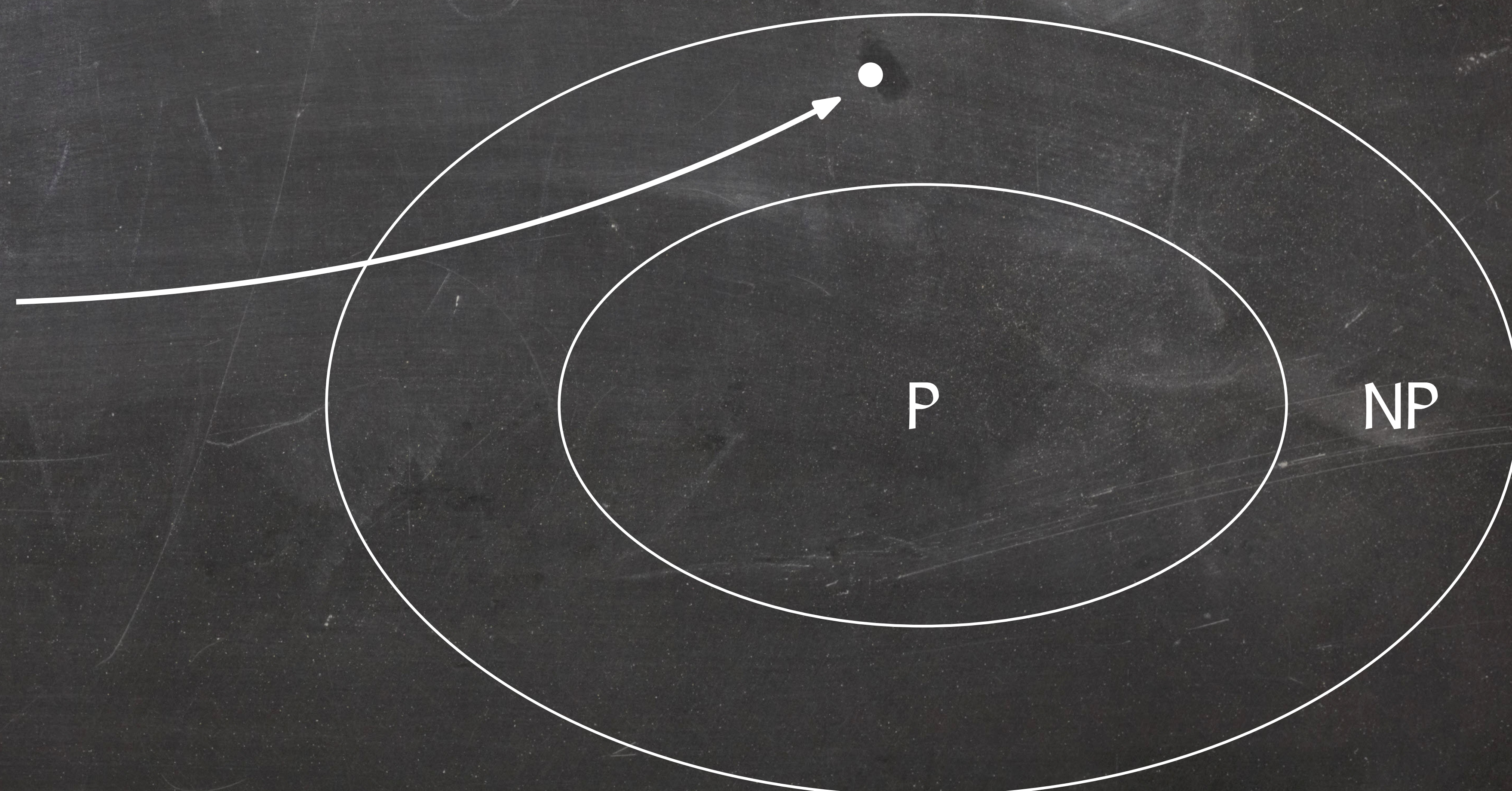

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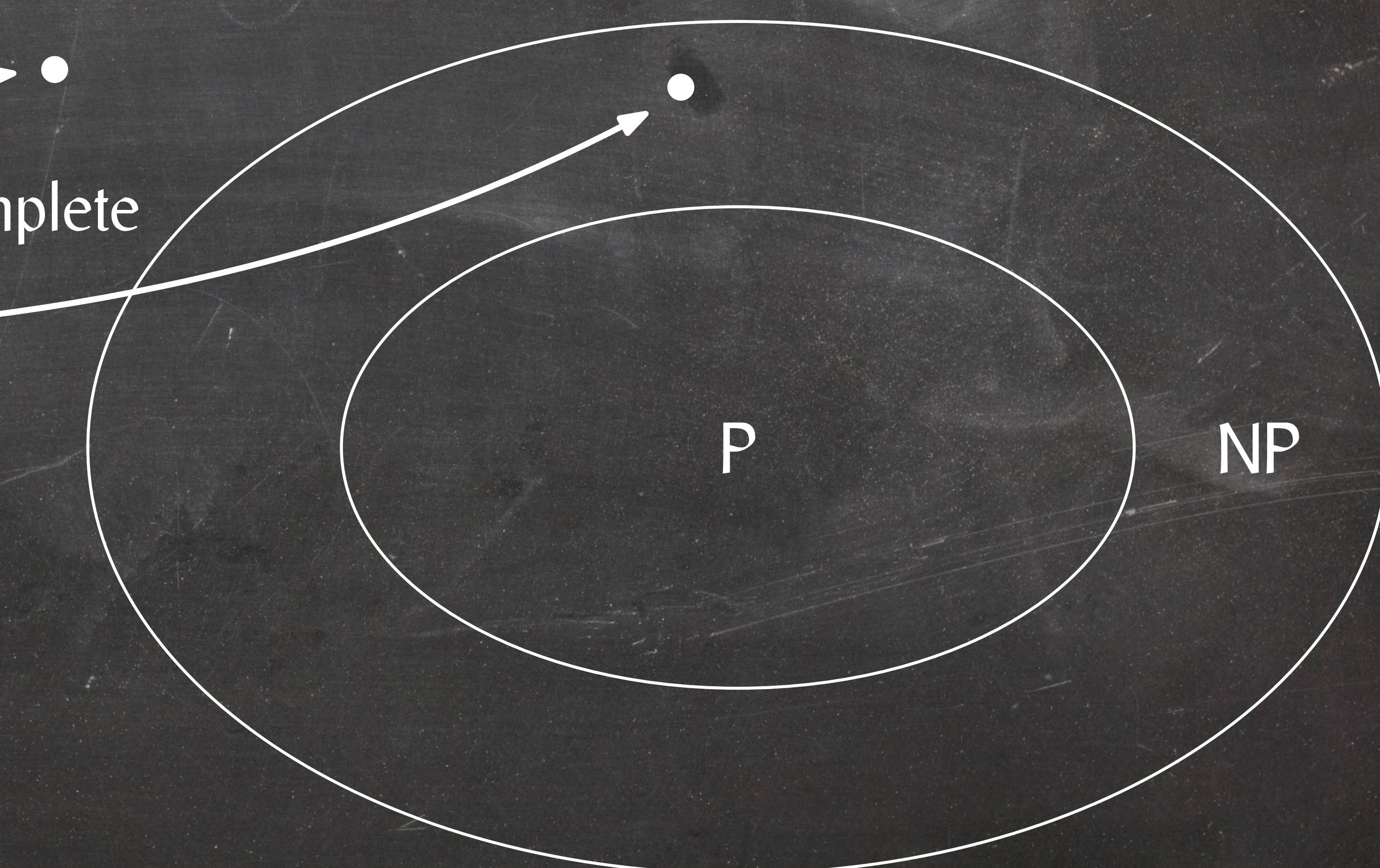

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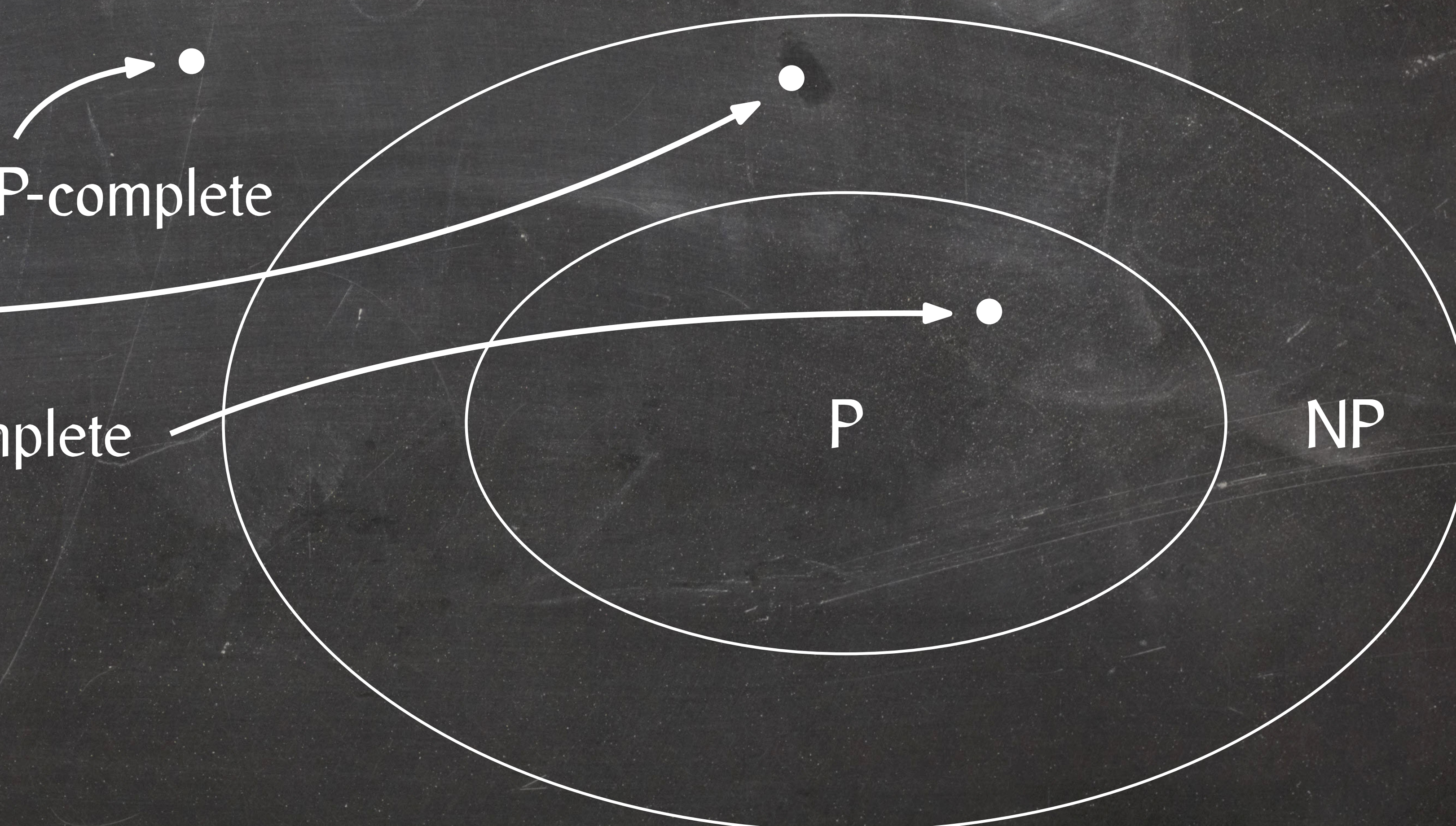

Polynomial-Time Reductions

An algorithm R reduces a language $L_1 \subseteq \Sigma^*$ to a language $L_2 \subseteq \Sigma^*$ if, for all $x \in \Sigma^*$,

$$x \in L_1 \Leftrightarrow R(x) \in L_2.$$

$$x \longrightarrow \boxed{\quad R \quad} \longrightarrow R(x)$$

R is a polynomial-time reduction if its running time is polynomial in $|x|$.

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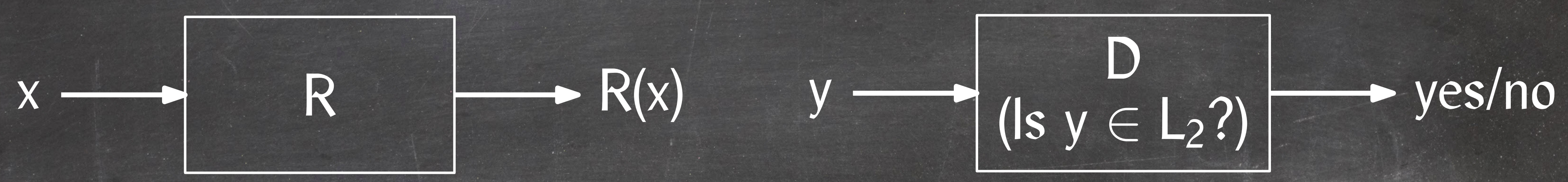

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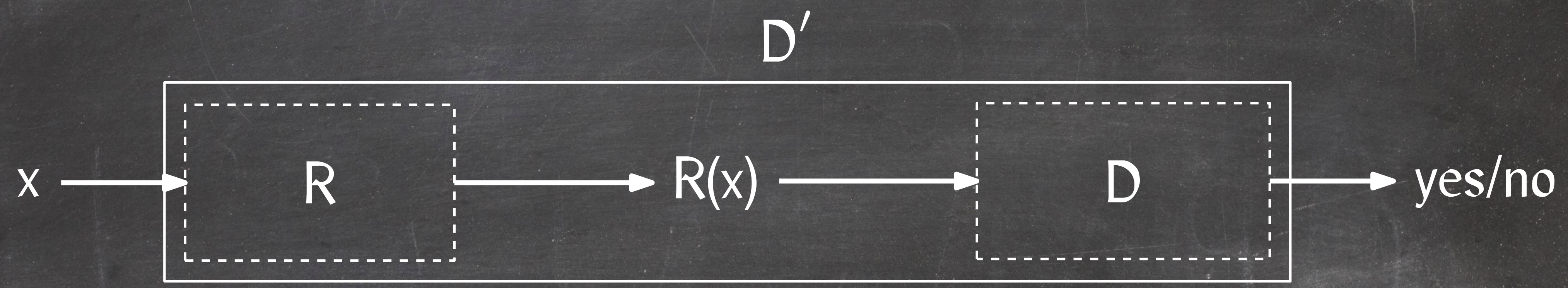

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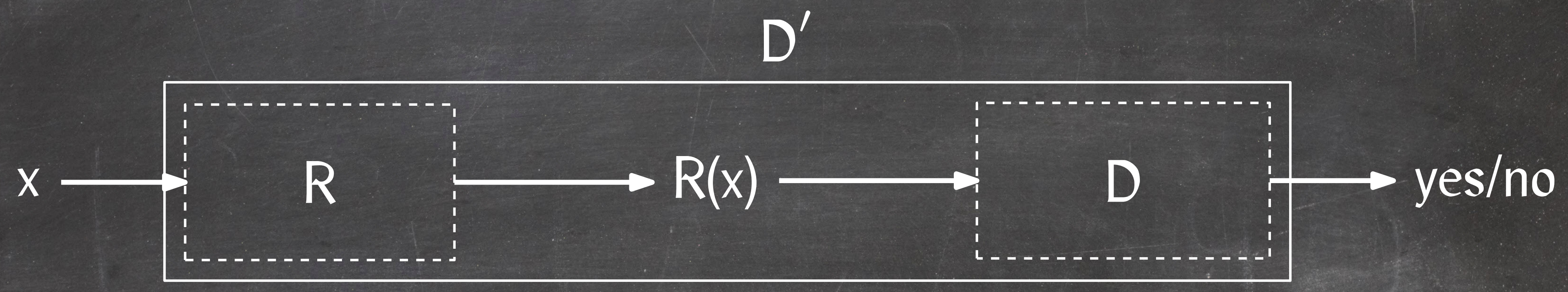

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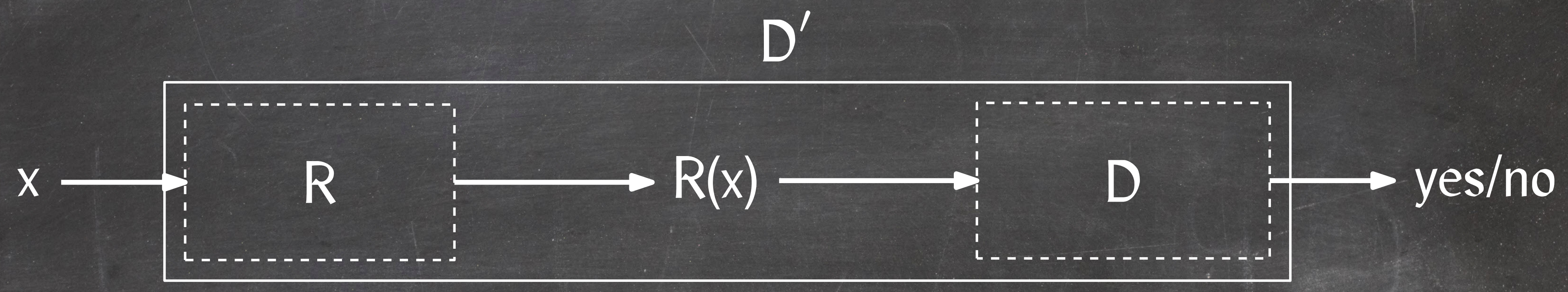

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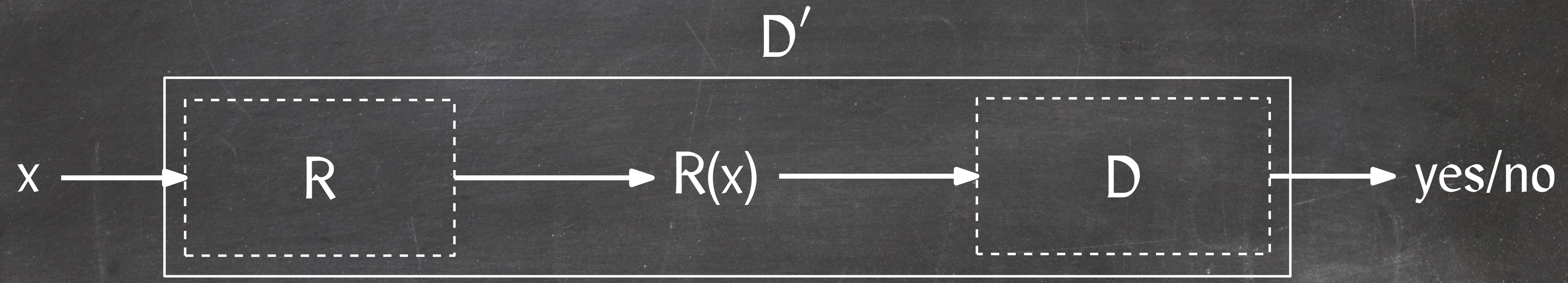

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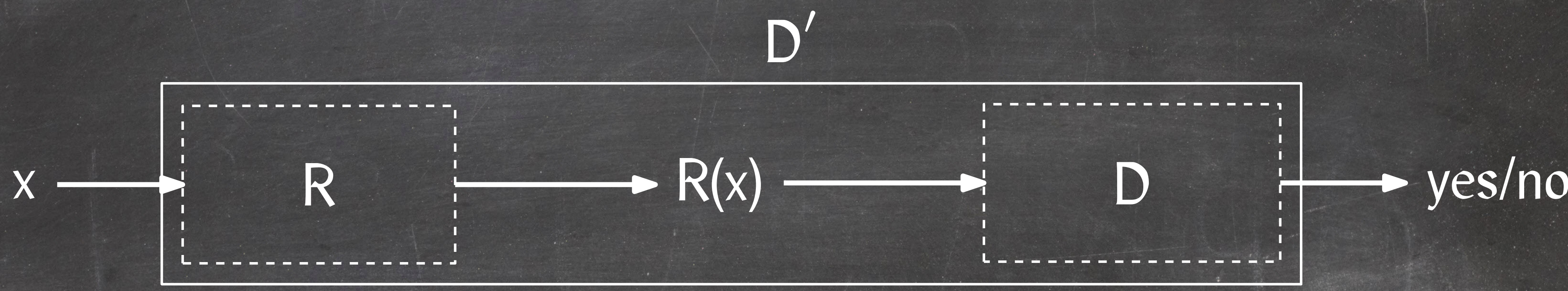

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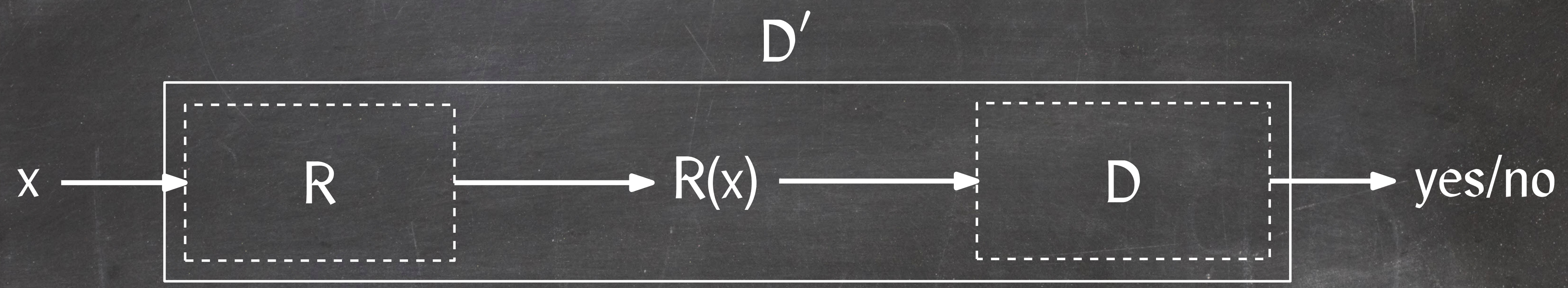

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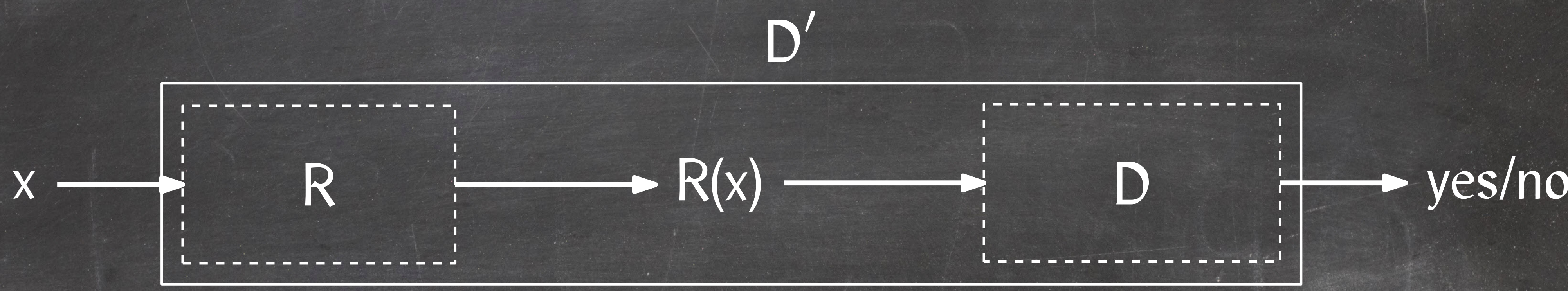

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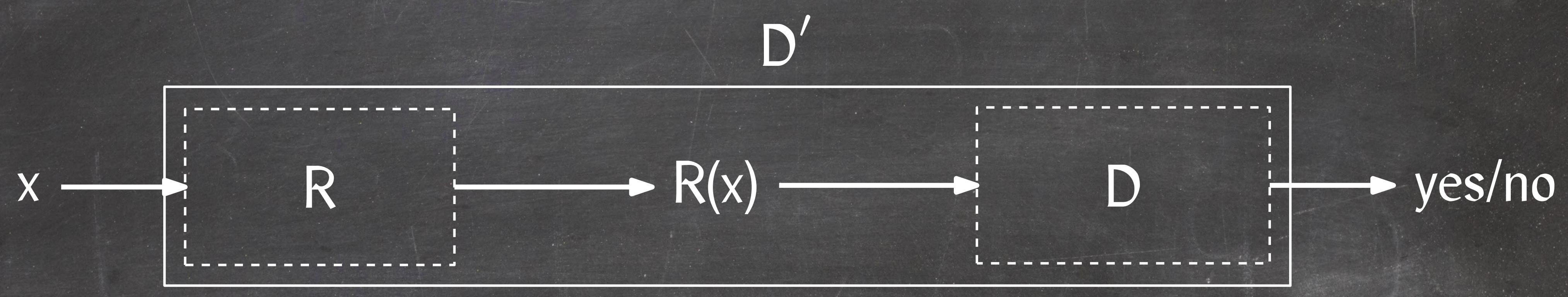

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$\Rightarrow \ T_{D'}(|x|) = T_R(|x|) + T_D(|R(x)|) \leq c|x|^a + c'(c|x|^a)^{a'} \in O(|x|^{aa'})$.

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Corollary: If there exists a polynomial-time reduction from an NP-hard language L_1 to a language L_2, then L_2 is also NP-hard.

$$\begin{tikzcd}
L_2 \in P \arrow[r, "\text{Polynomial-time reduction}"] \arrow[rr, bend right, "L_2 \text{ is NP-hard.}"'] & L_1 \in P \arrow[r, "\text{NP-hardness of } L_1"] & P = NP
\end{tikzcd}$$

Where Do We Get Our First NP-Hard Problem From?

To prove that a language L is NP-hard, we need an NP-hard language L' that we can reduce to L in polynomial time.

How do we prove a language L is NP-hard when we haven't shown any other language to be NP-hard yet?

Enter Satisfiability, the mother of all NP-hard problems …

Satisfiability (SAT)

A Boolean formula:

$$F = (x_1 \vee (x_2 \wedge \bar{x}_3)) \wedge (\bar{x}_1 \vee x_4)$$

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The satisfiability problem (SAT): Given a Boolean formula F, decide whether F is satisfiable.

3-SAT

A formula is in conjunctive normal form (CNF) if it is a conjunction of disjuctions.

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

The disjunctions are also called clauses.

A formula is in 3-CNF if each of its clauses consists of three literals.

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3)$$

3-SAT: Decide whether a given formula in 3-CNF is satisfiable.

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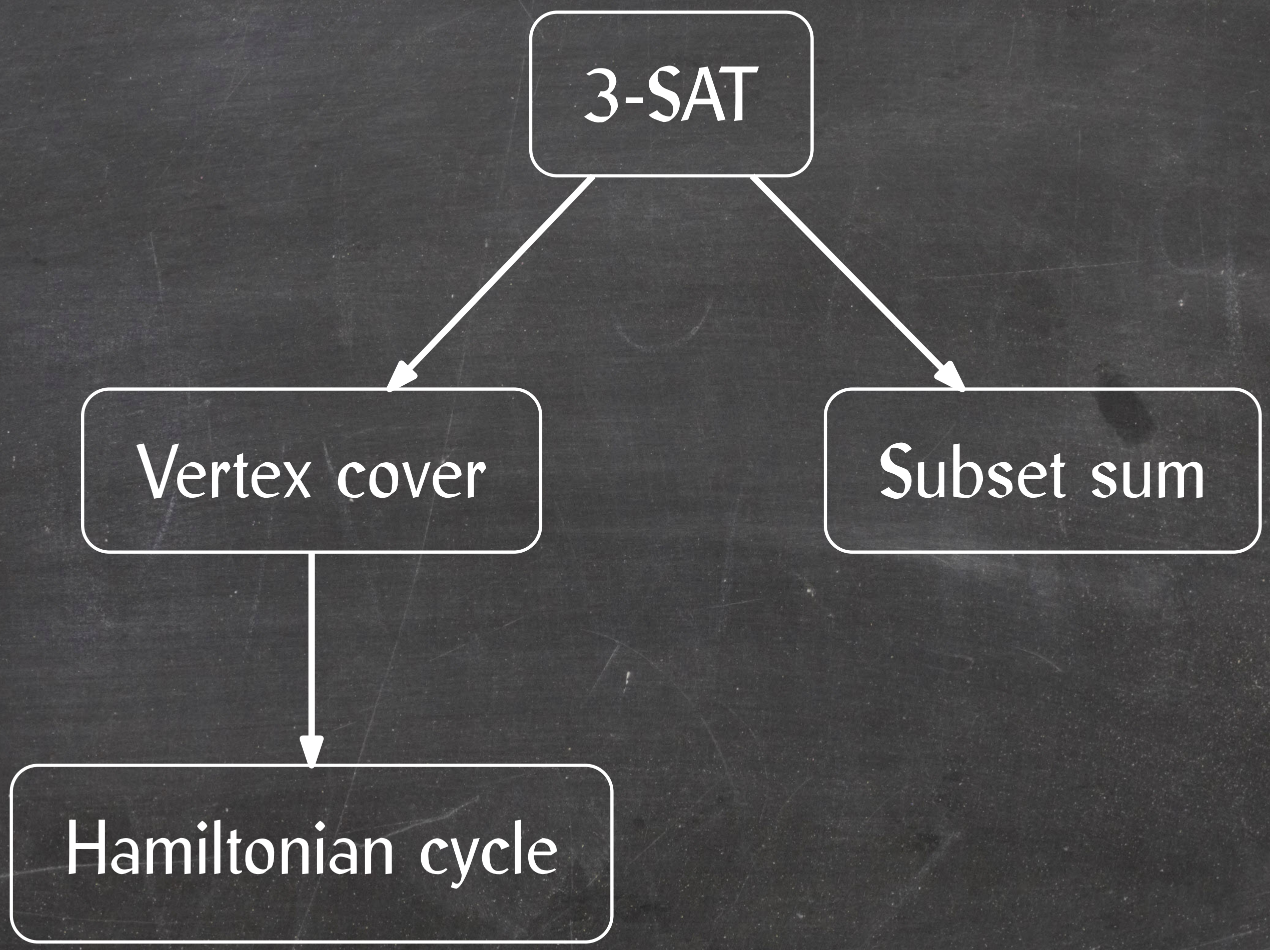
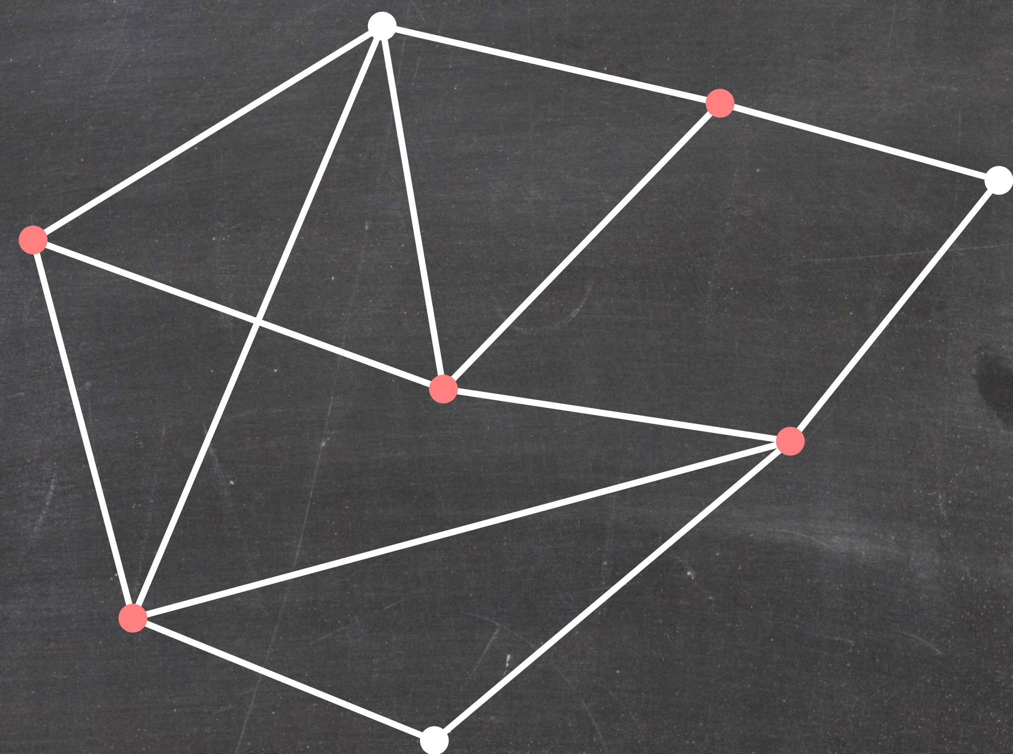
Examples of Polynomial-Time Reductions



Vertex Cover

A vertex cover of a graph $G = (V, E)$ is a subset $S \subseteq V$ such that every edge in E has at least one endpoint in S.



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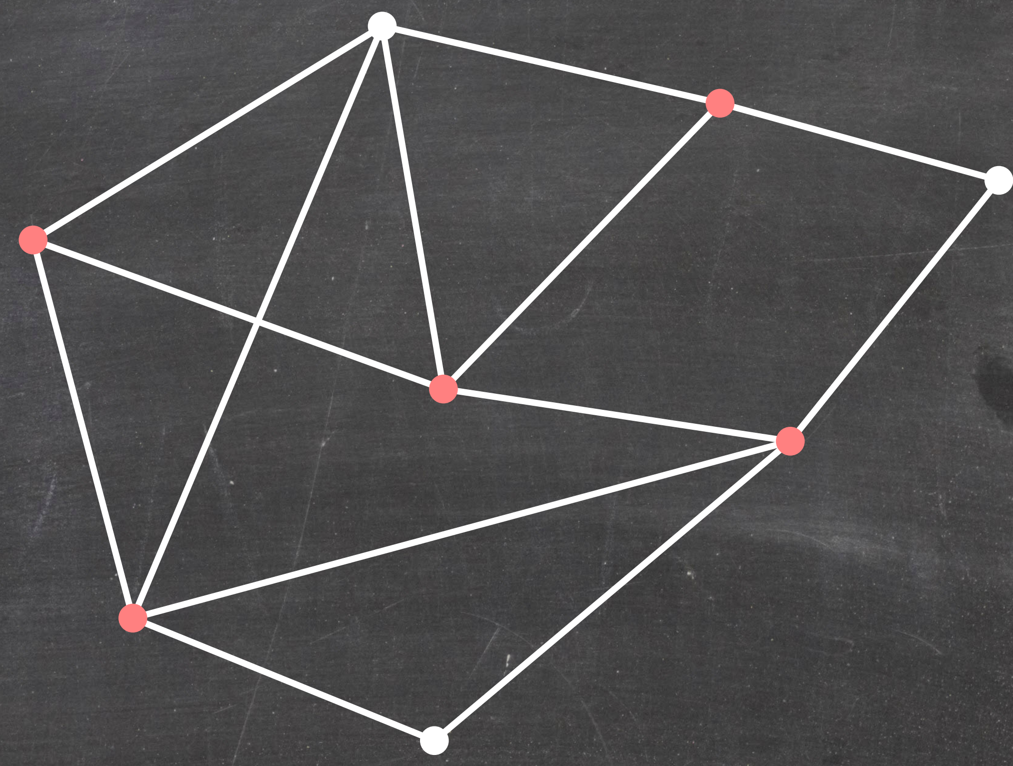

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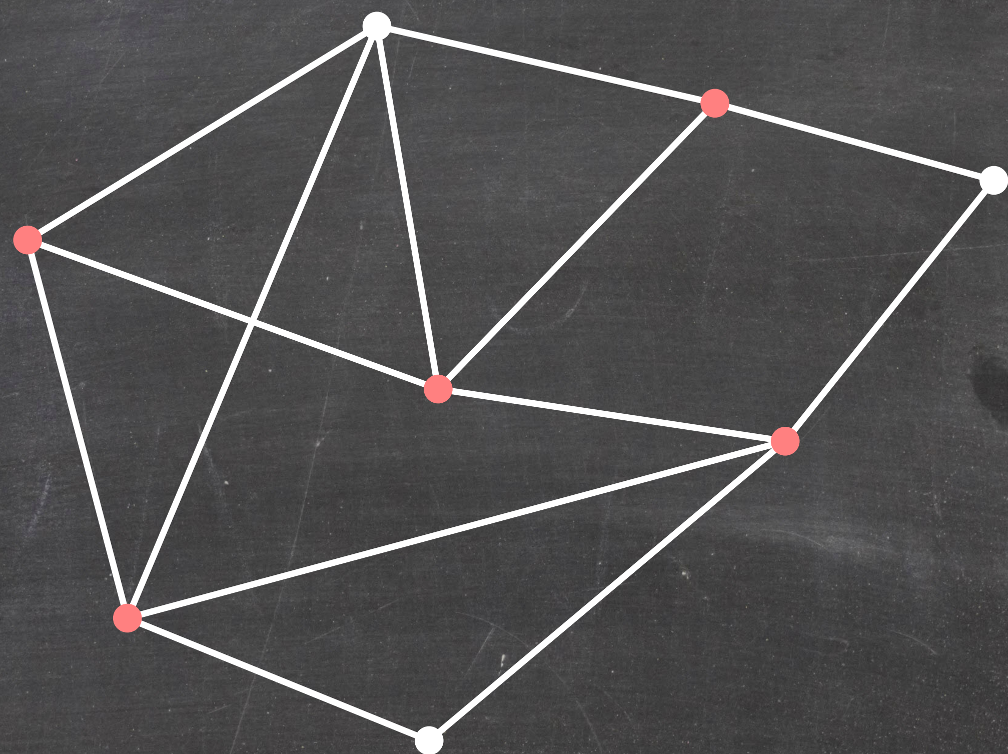

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Decision problem: Given a graph G and an integer k, decide whether G has a vertex cover of size k.

Vertex Cover is NP-Hard

Reduction from 3-SAT:

- Given any formula F, we build a graph G_F that has a small vertex cover if and only if F is satisfiable.
- G_F will be built from subgraphs, called widgets, that guarantee certain properties of G_F.
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Clause widget for clause C_j:

- Three literal vertices $\lambda_{j,1}$, $\lambda_{j,2}$, and $\lambda_{j,3}$
- Three edges $(\lambda_{j,1}, \lambda_{j,2})$, $(\lambda_{j,2}, \lambda_{j,3})$, and $(\lambda_{j,3}, \lambda_{j,1})$

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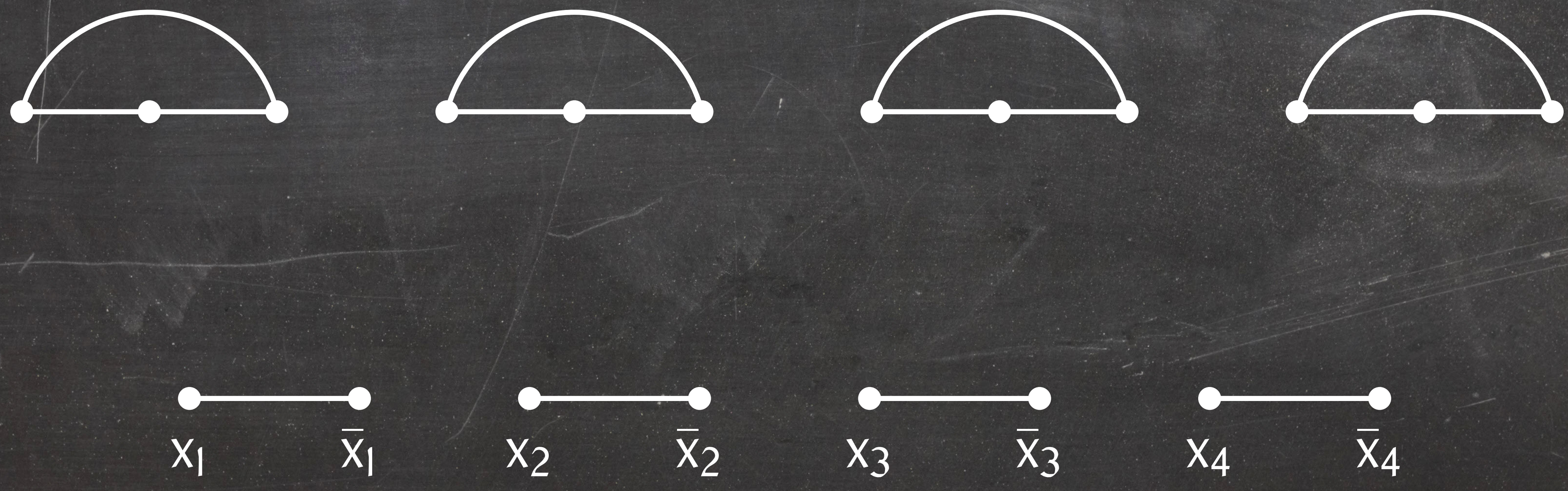

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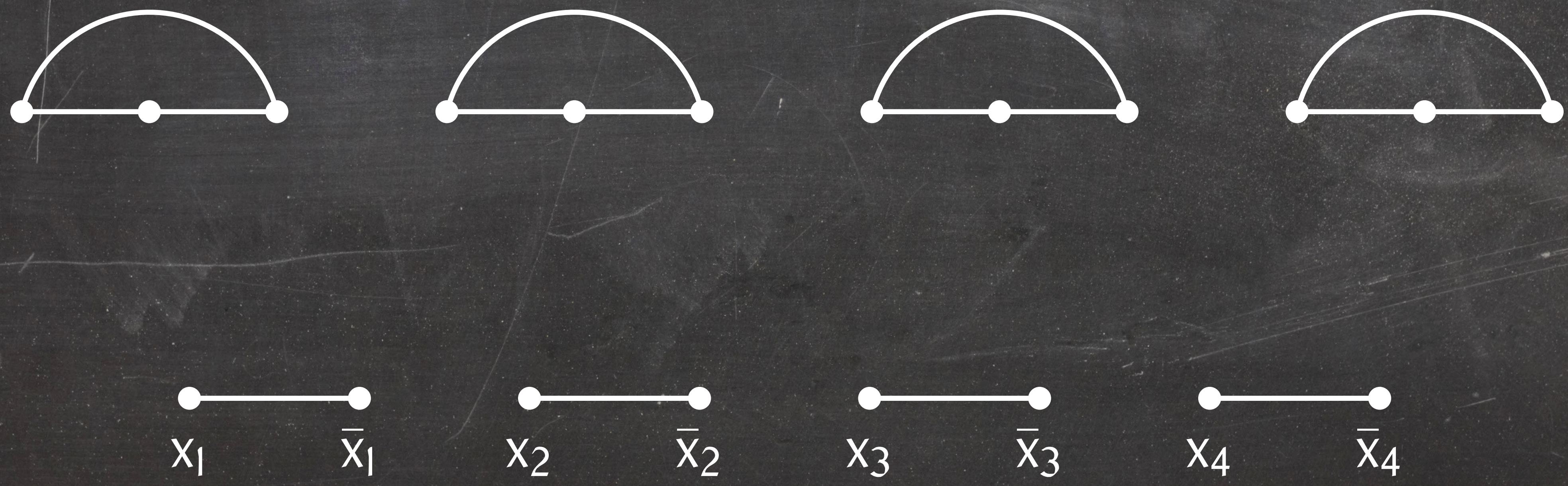

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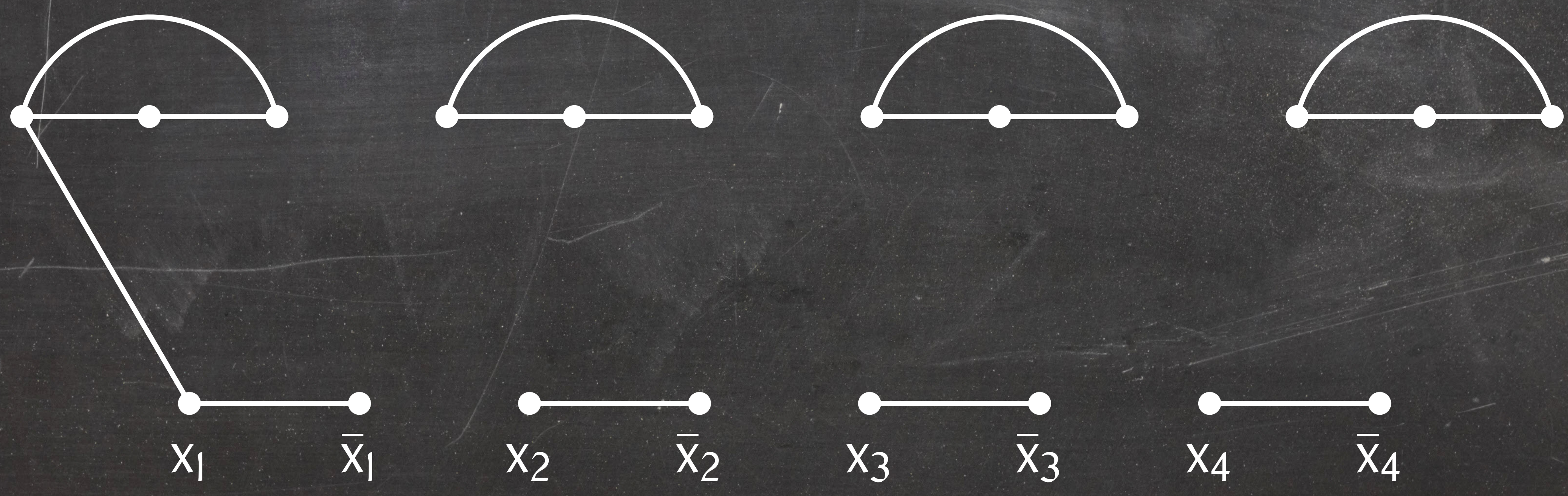

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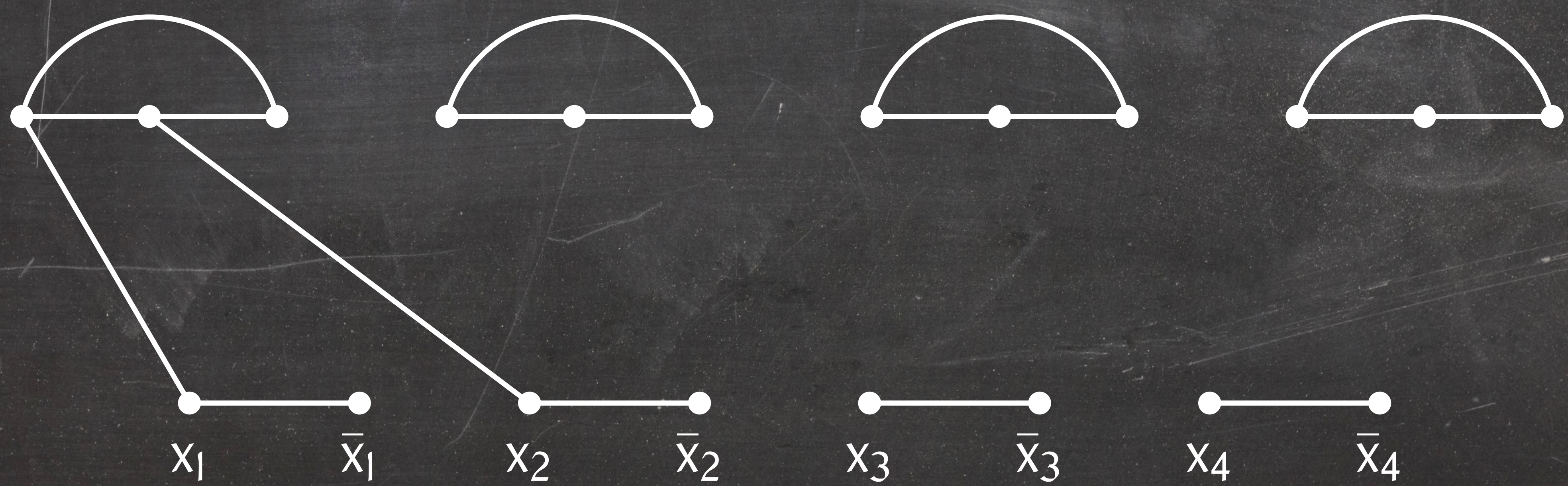

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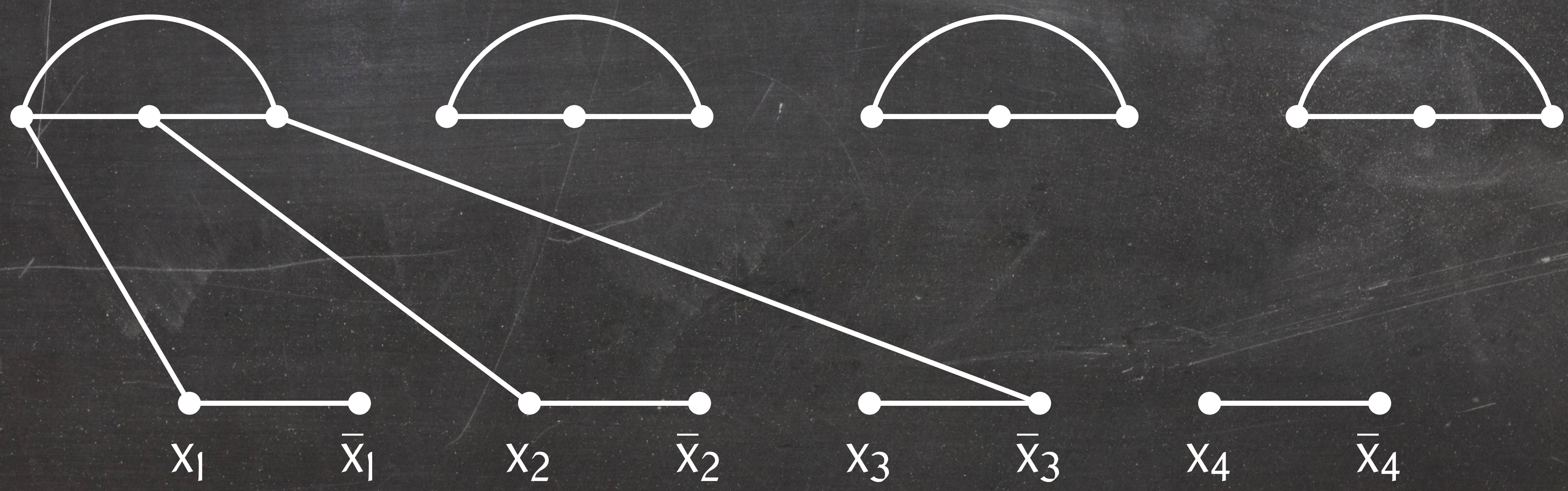

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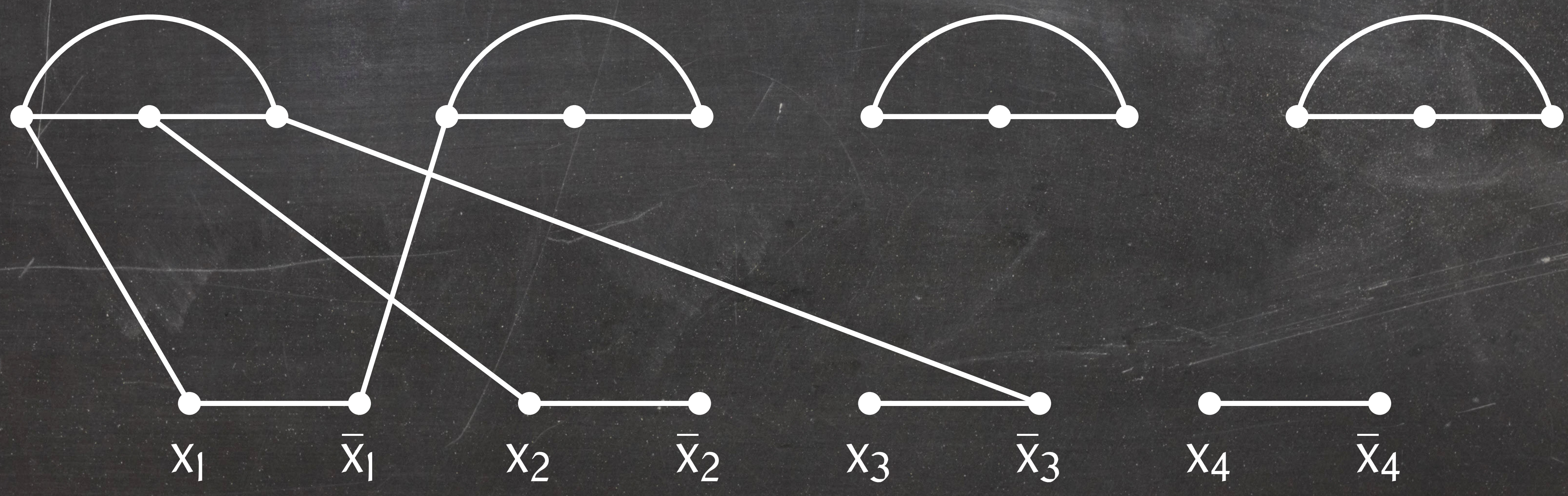

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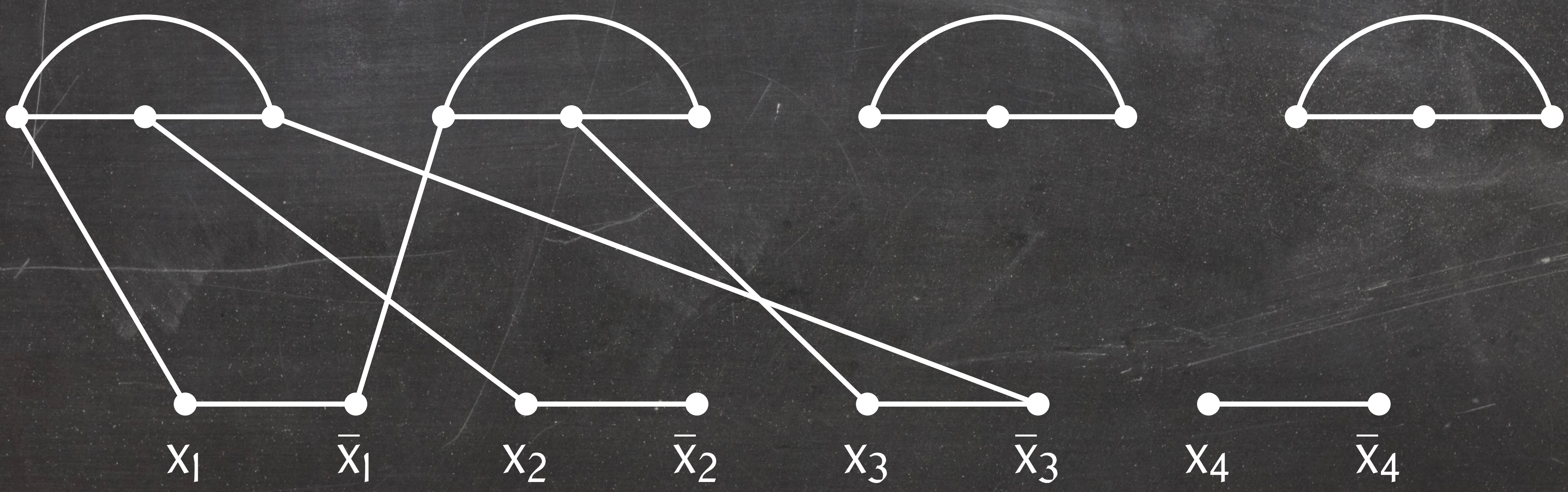

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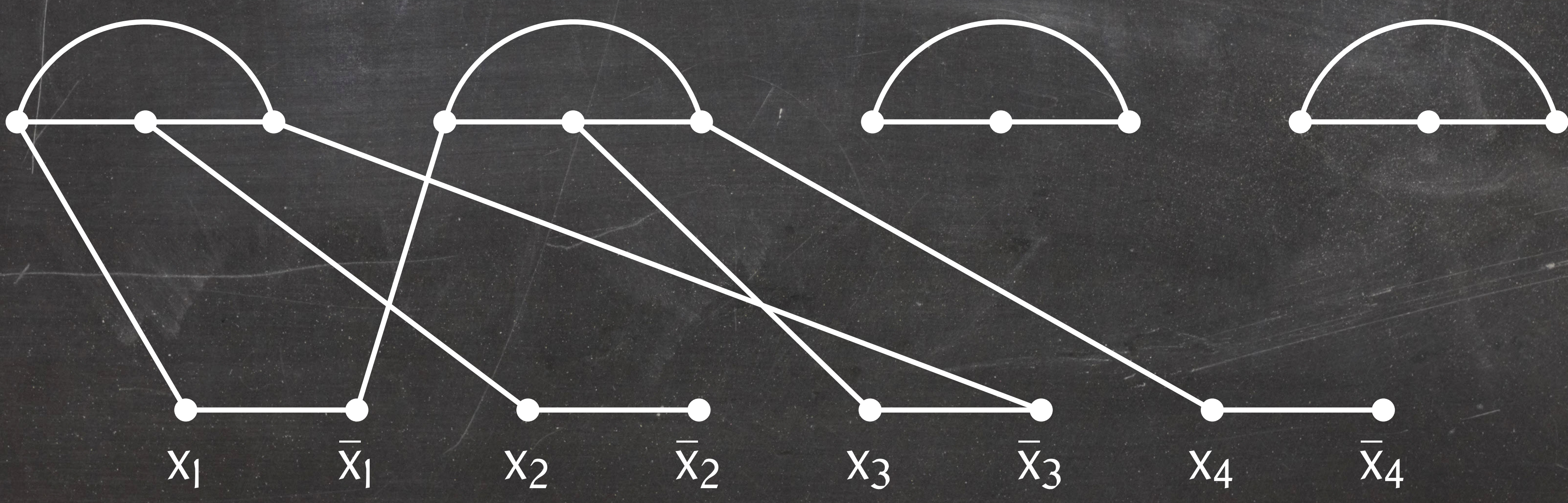

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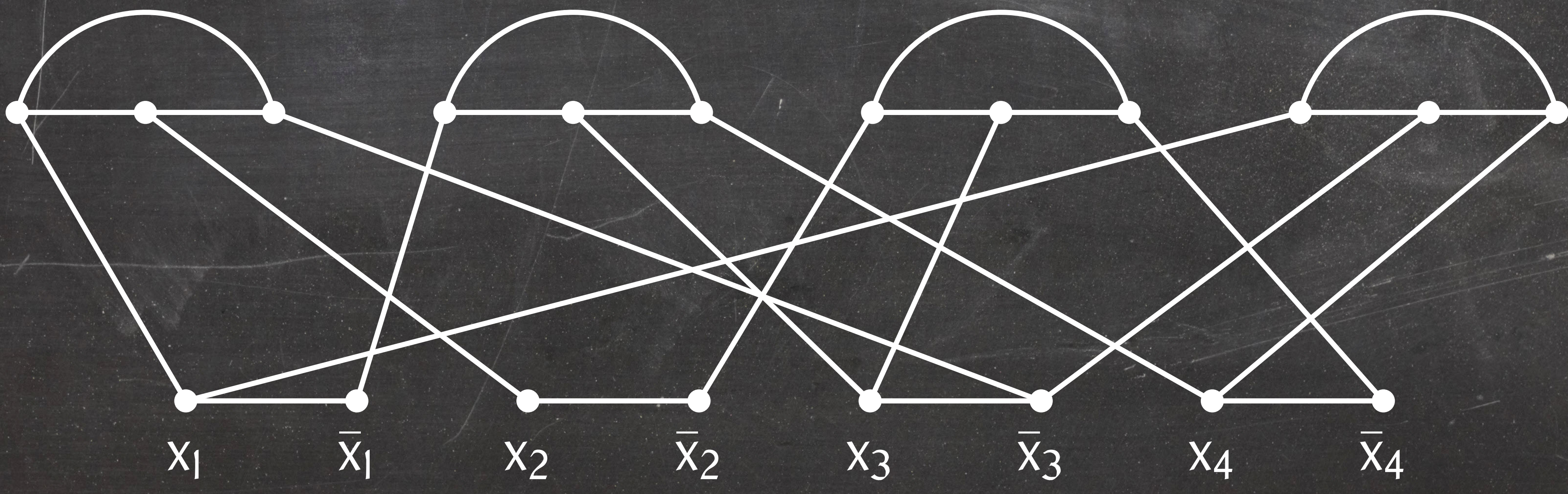

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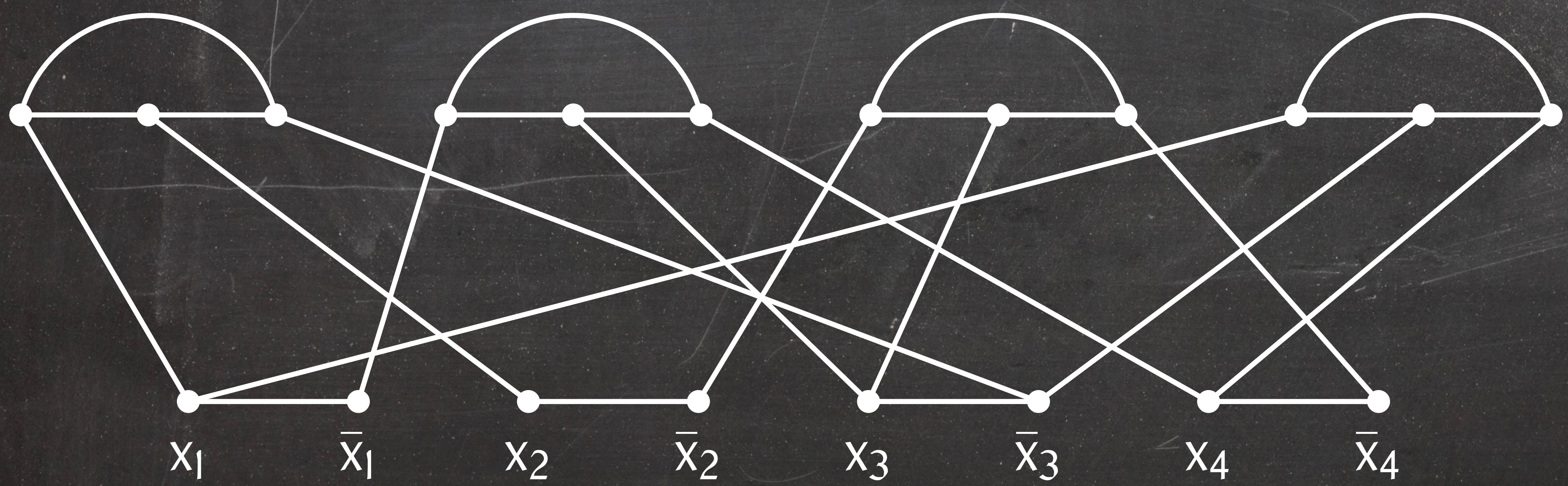

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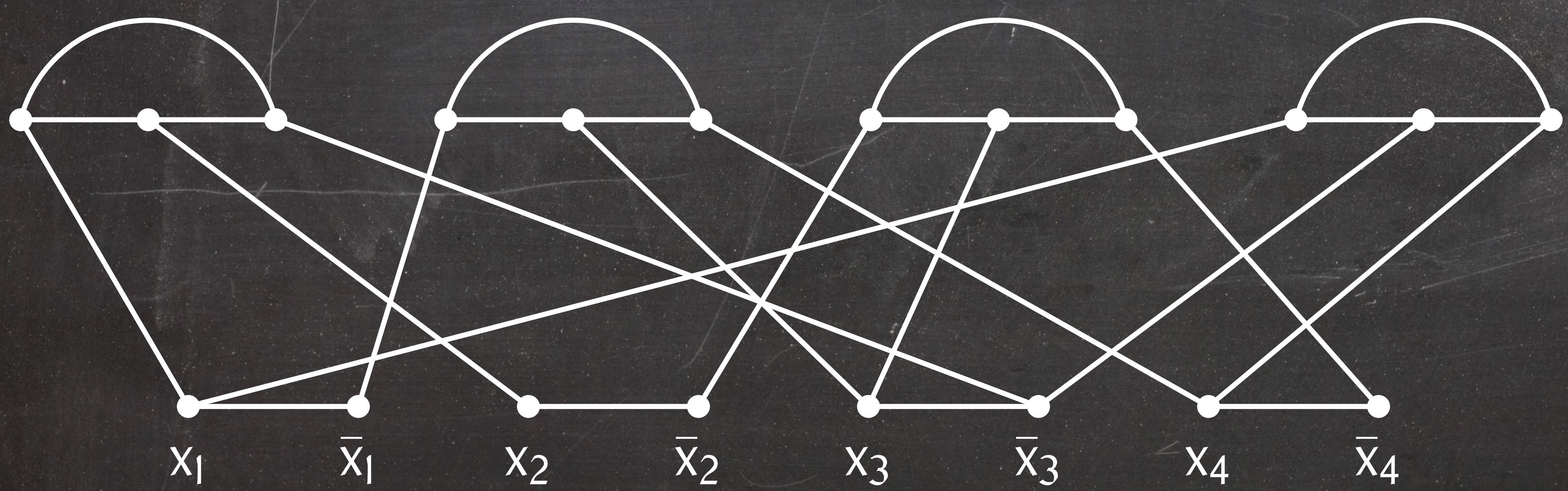

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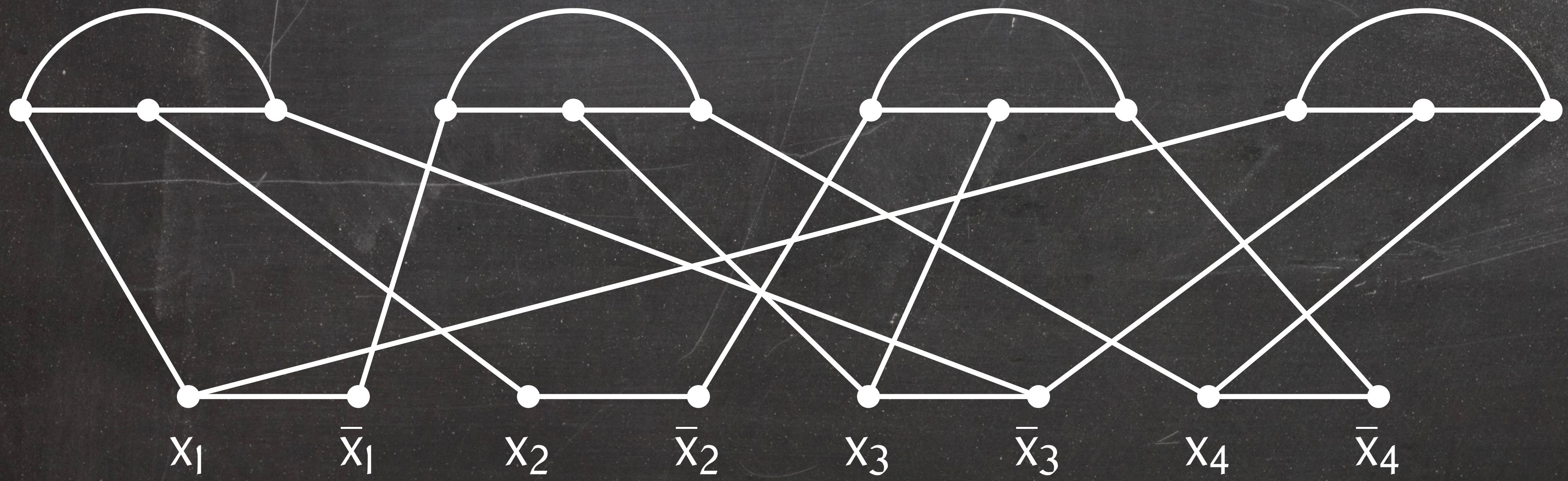

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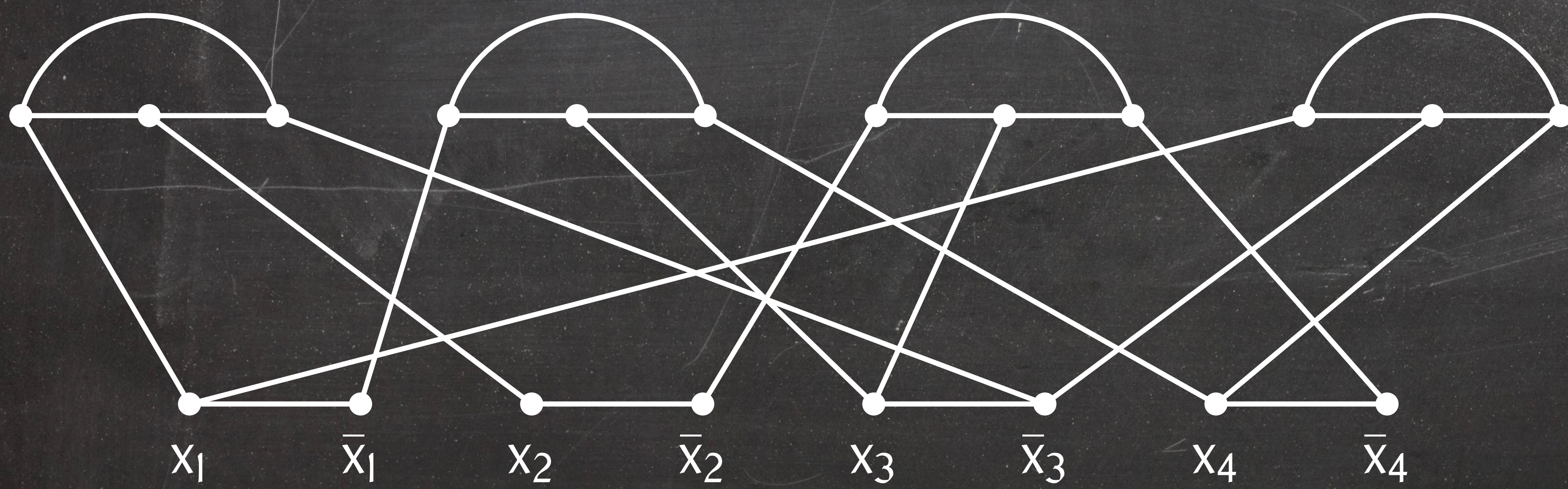

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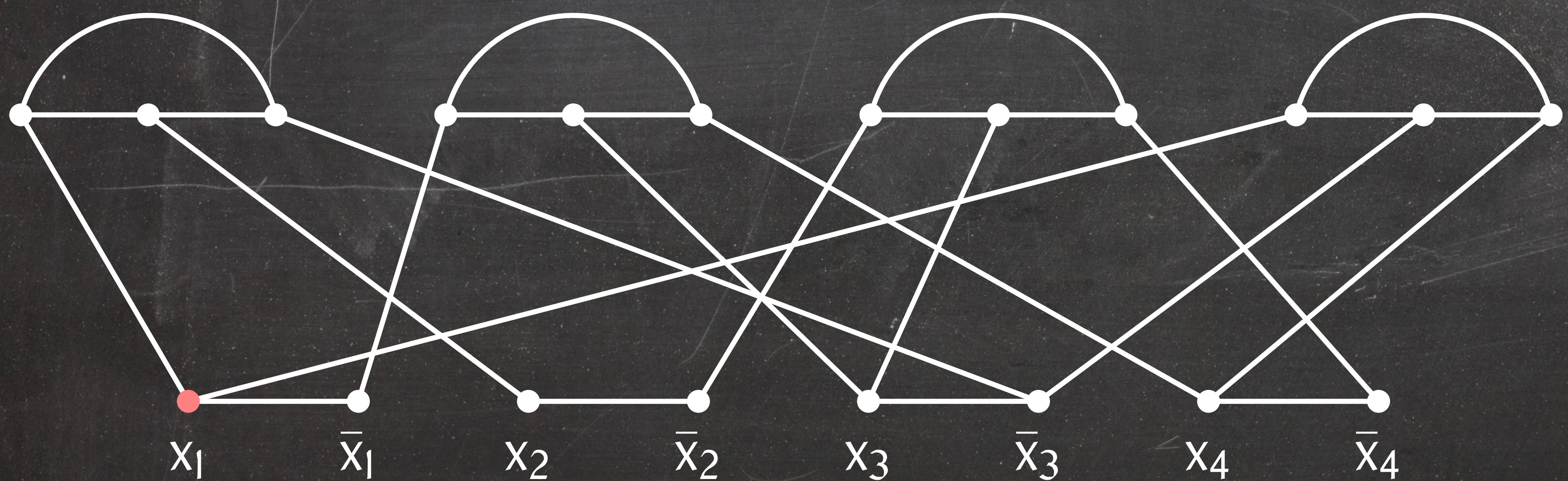

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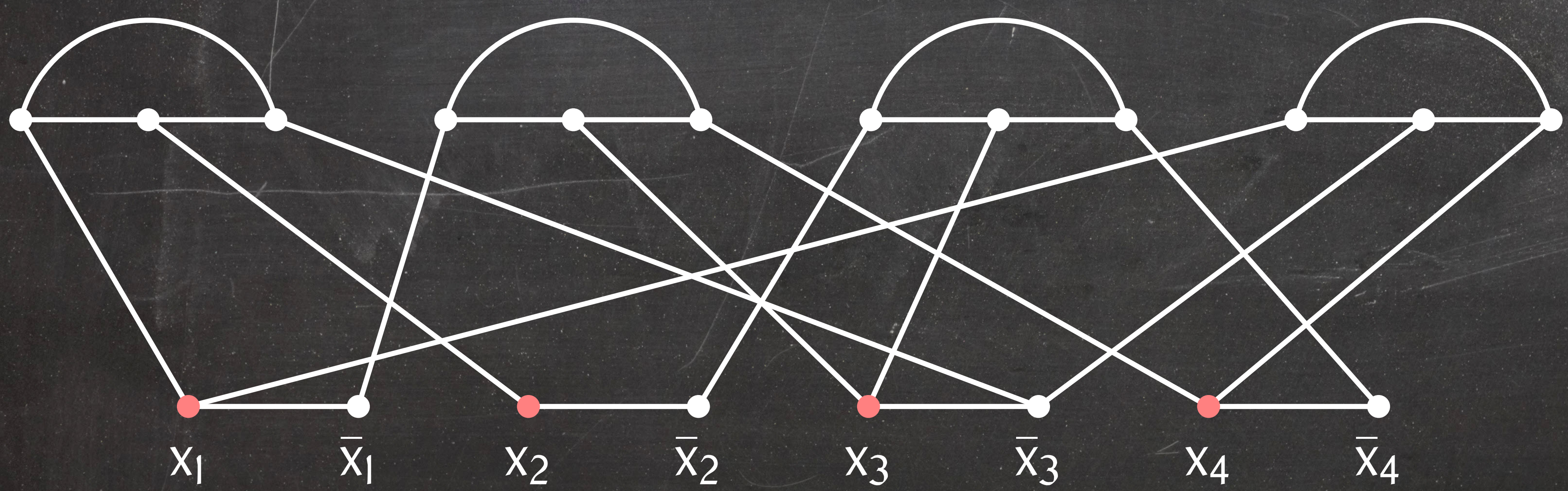

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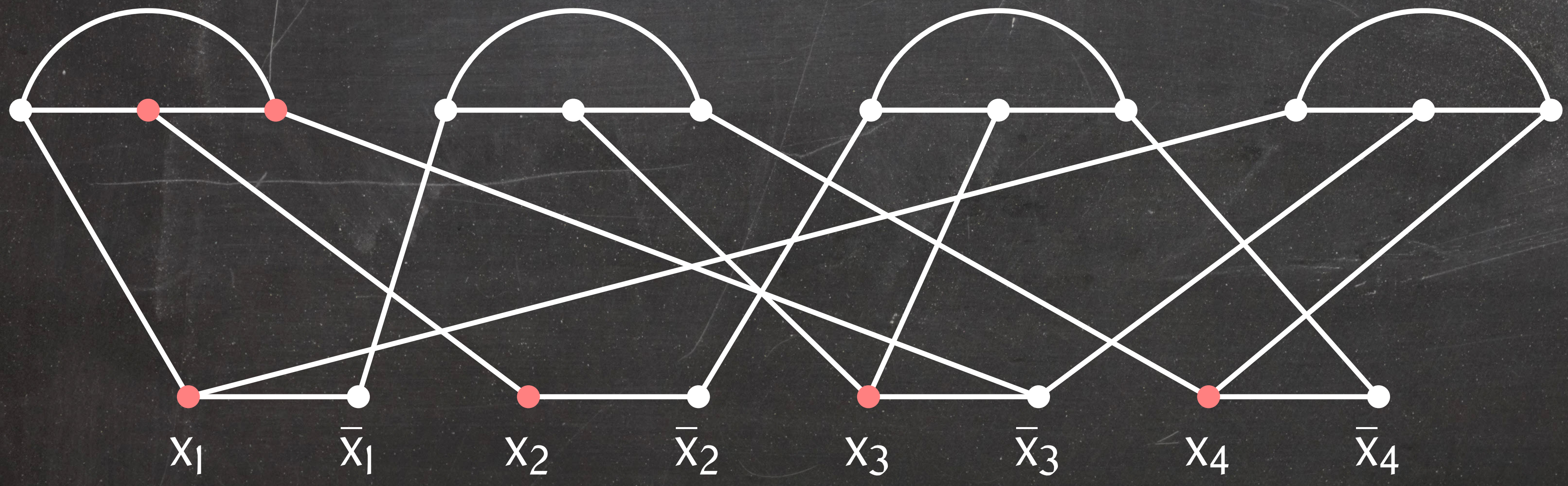

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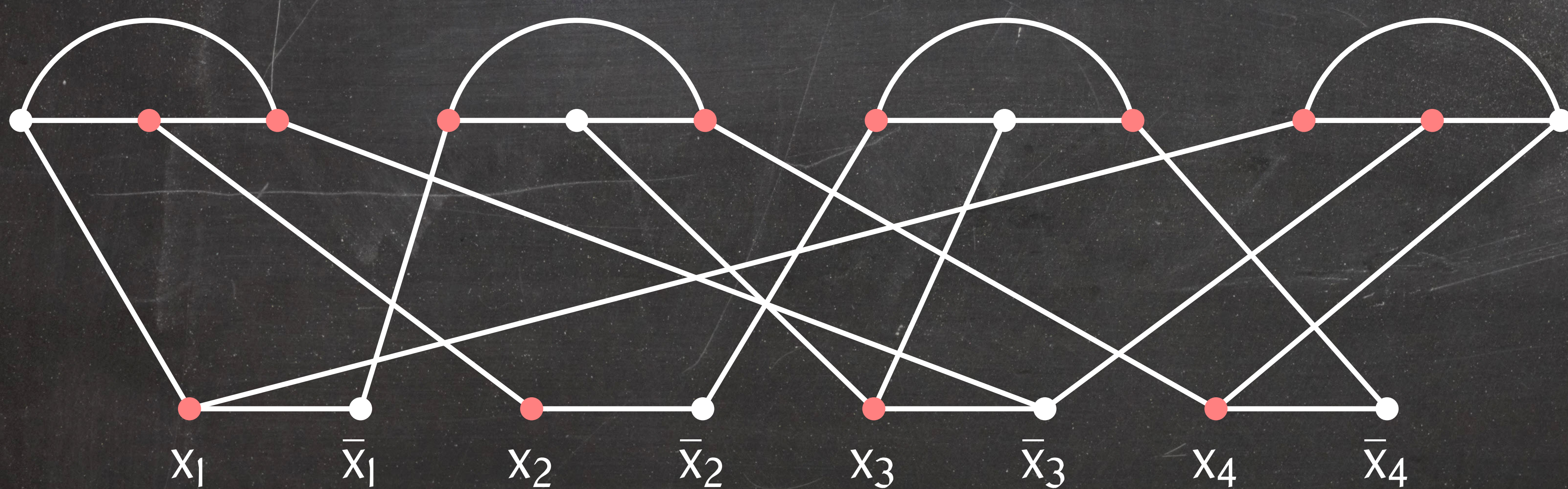

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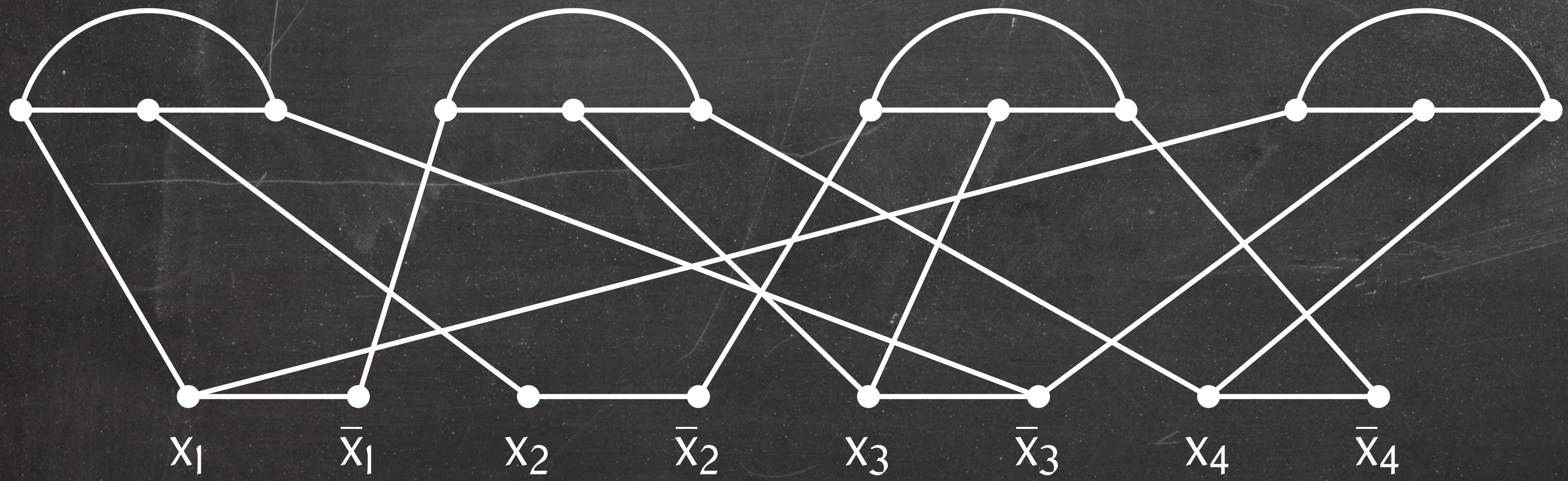

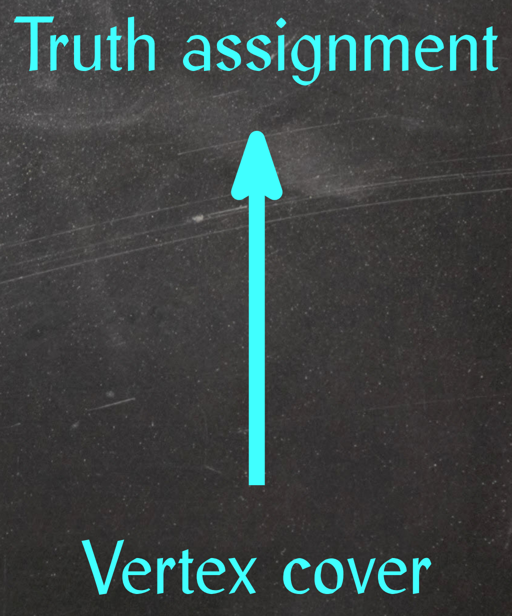

Vertex Cover is NP-Hard

n = number of variables $\quad$ m = number of clauses

Observation: Any vertex cover of G_F of size $n + 2m$ contains one vertex per variable widget and two vertices per clause widget.

Lemma: F is satisfiable if and only if G_F has a vertex cover of size $n + 2m$.

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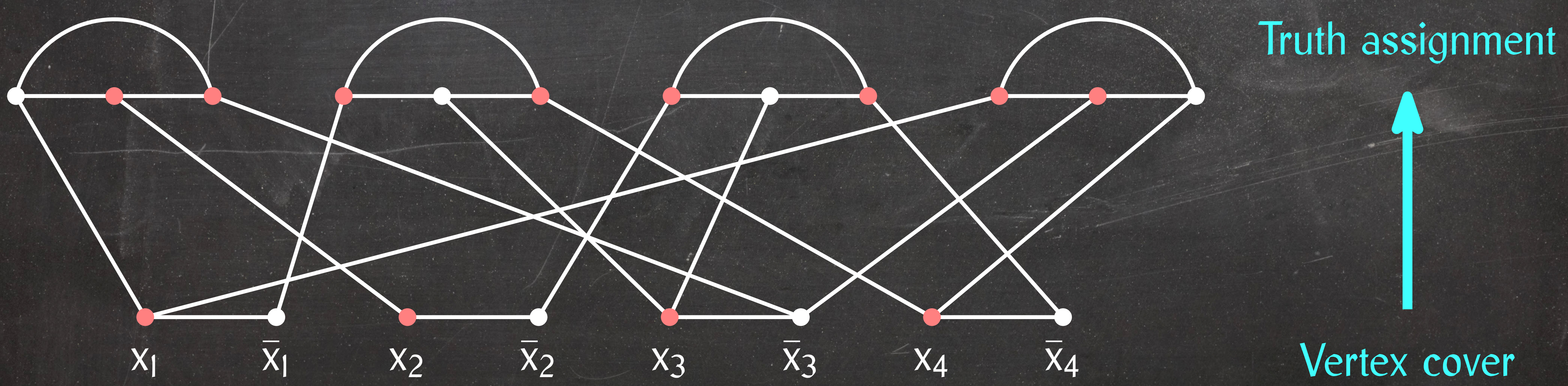

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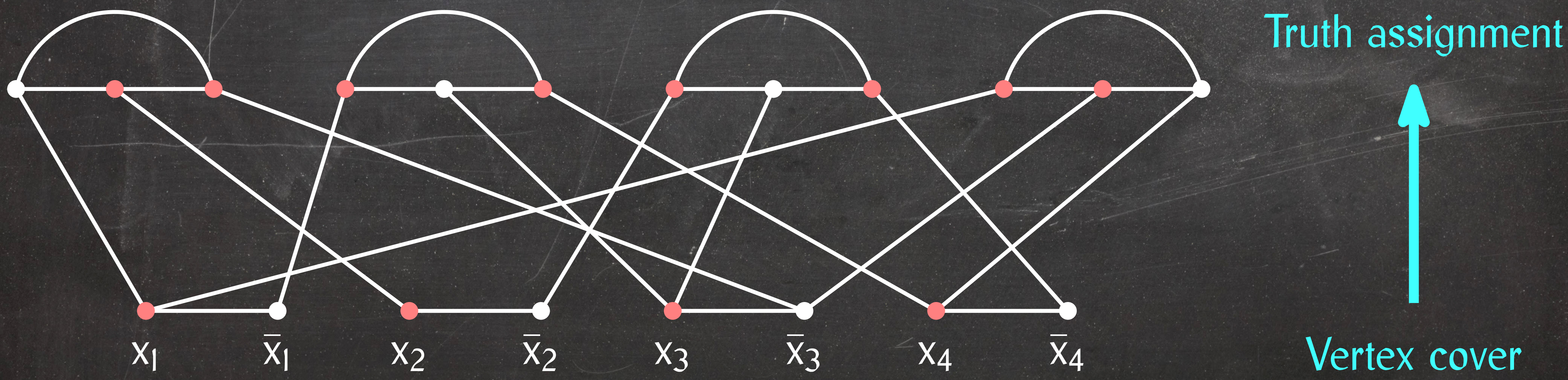

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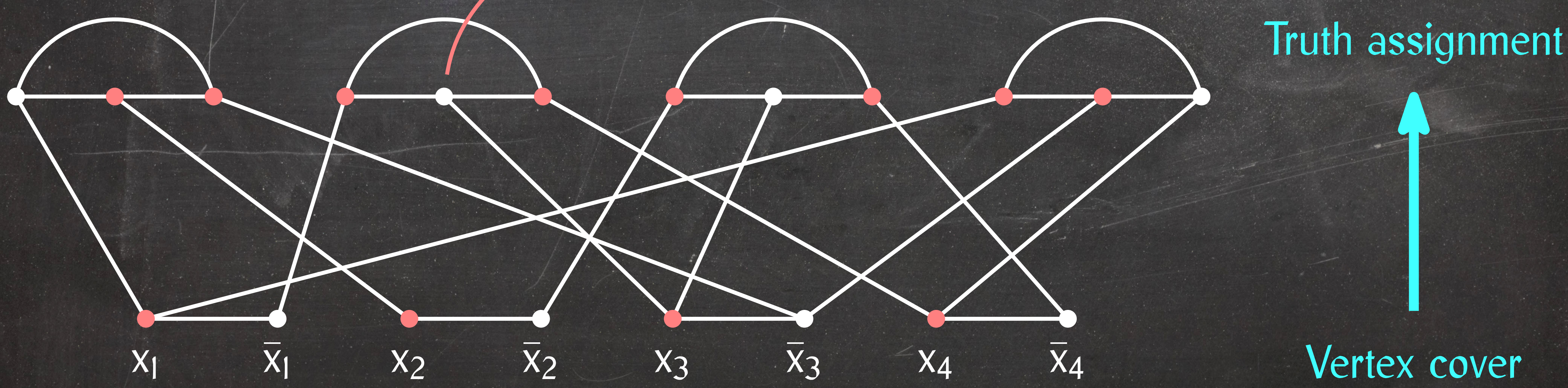

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To prove that $VC \in NP$, we have to prove that there exists a language $VC' \in P$ such that $(G, k) \in VC$ if and only if $(G, k, y) \in VC'$ for some y with $|y| \in O(|(G, k)|^c)$.

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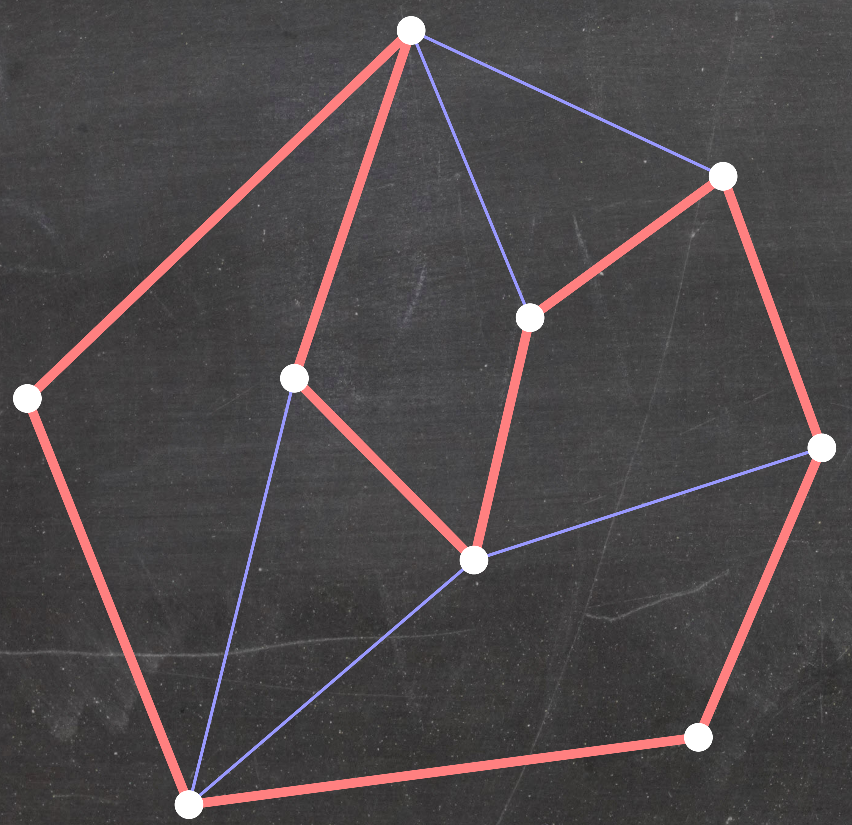
Let $VC' = \{(G, k, C) \mid C \text{ is a vertex cover of } G \text{ of size } k\}$.

$VC' \in P$:

- We can test in polynomial time whether every vertex in C belongs to G.
- We can test in polynomial time whether $|C| = k$.
- We can test in polynomial time whether every edge of G has at least one endpoint in C.

Hamiltonian Cycle

A Hamiltonian cycle of a graph G is a simple cycle that contains all vertices of G and whose edges are edges of G.



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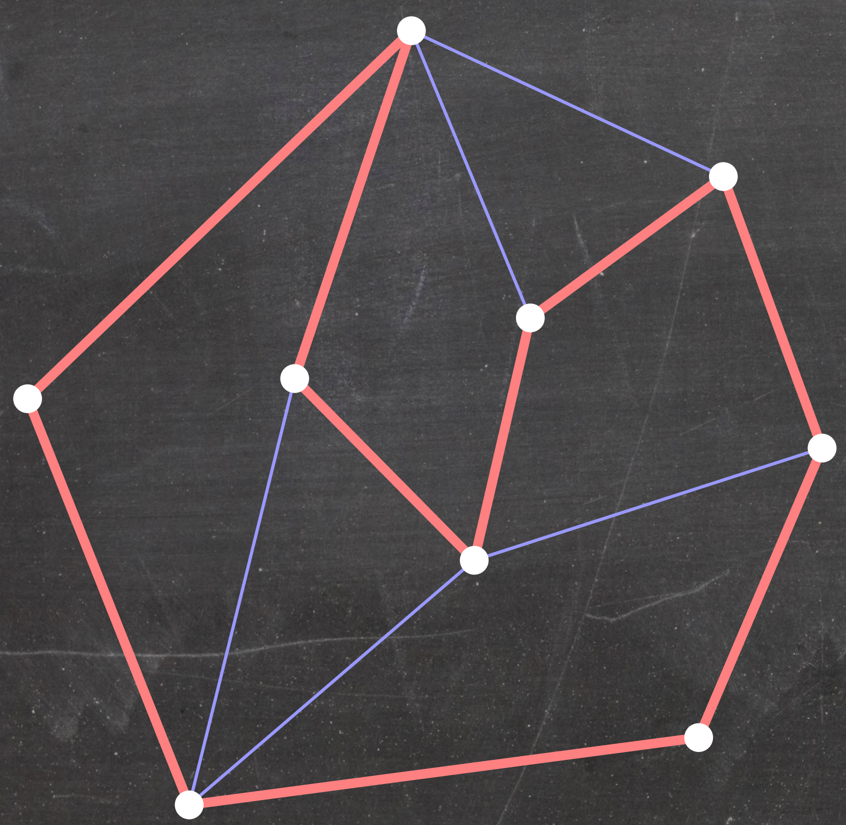

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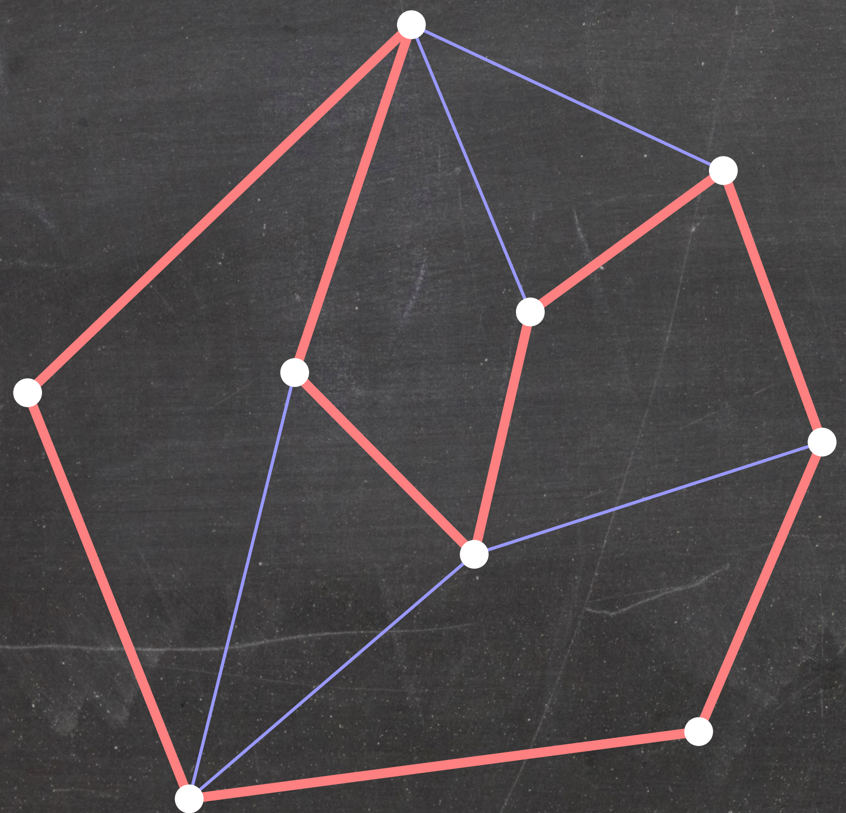

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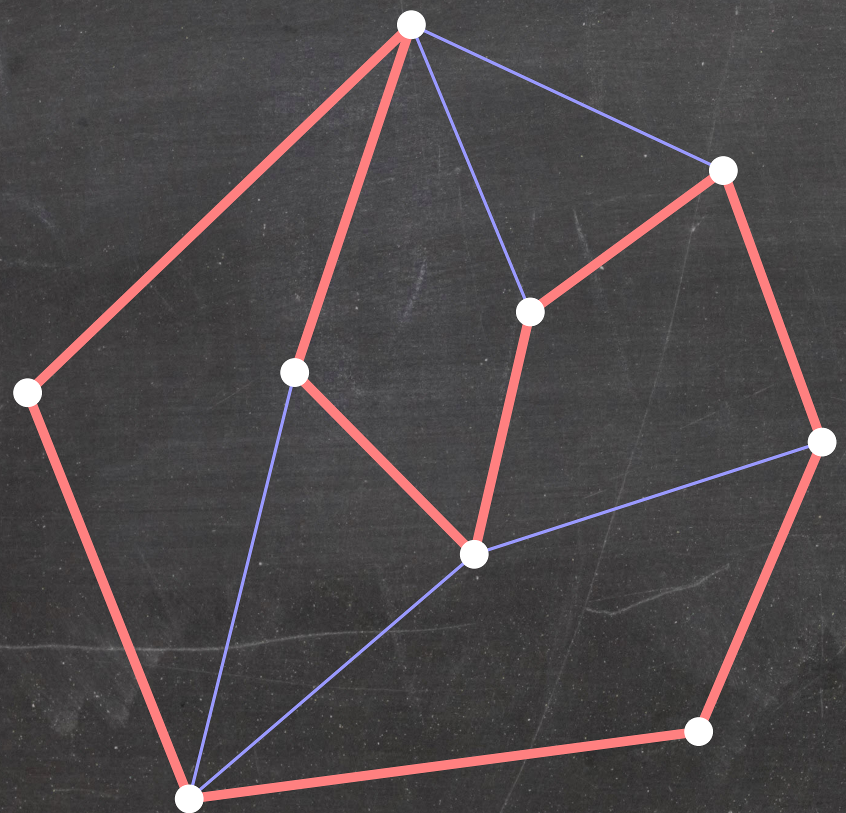

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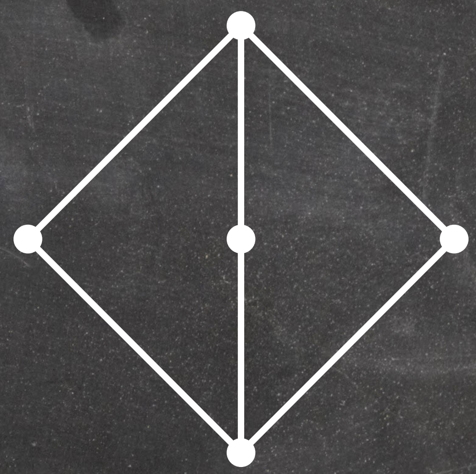

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Hamiltonian Cycle Problem: Decide whether a given graph G is Hamiltonian.

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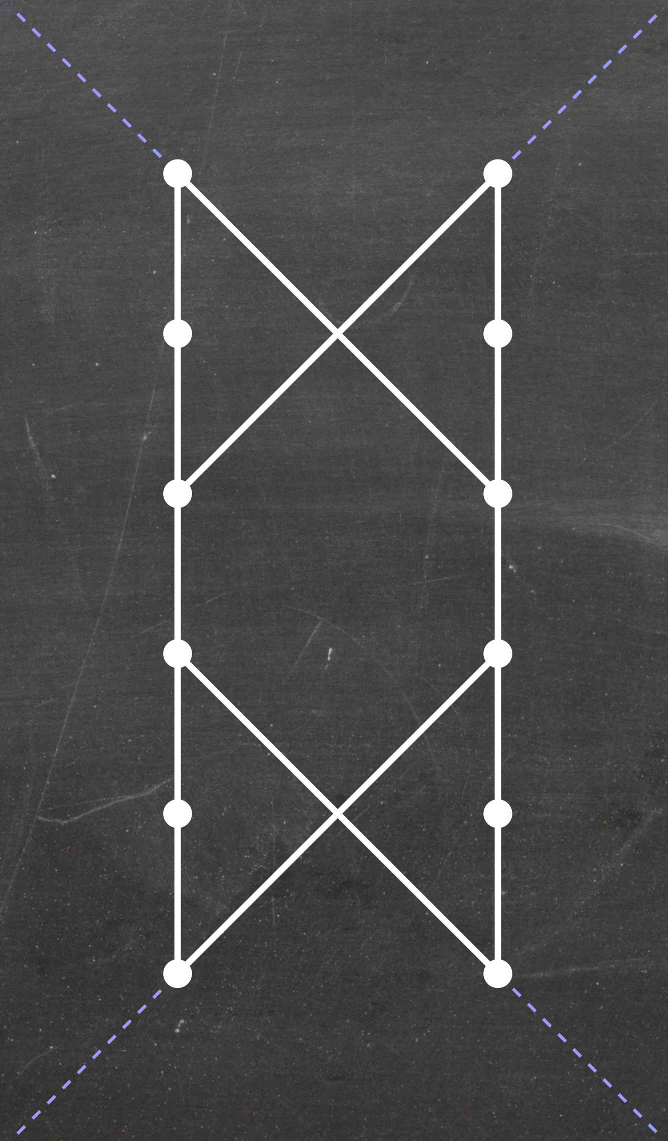
Again, it is trivial to verify that the construction takes polynomial time.

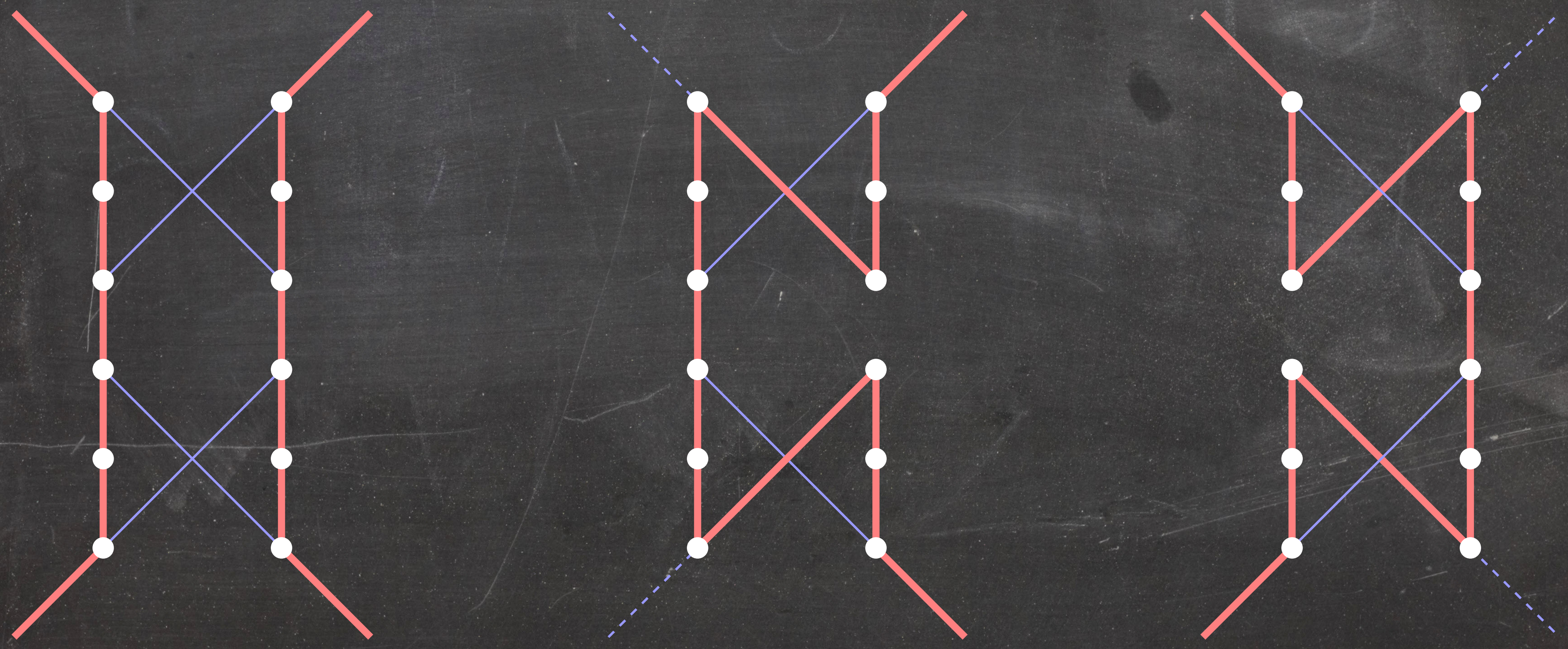
Edge Widgets

We build G' from edge widgets.



Edge Widgets

Observation: Any Hamiltonian cycle of a graph built from edge widgets traverses every edge widget in one of three ways.



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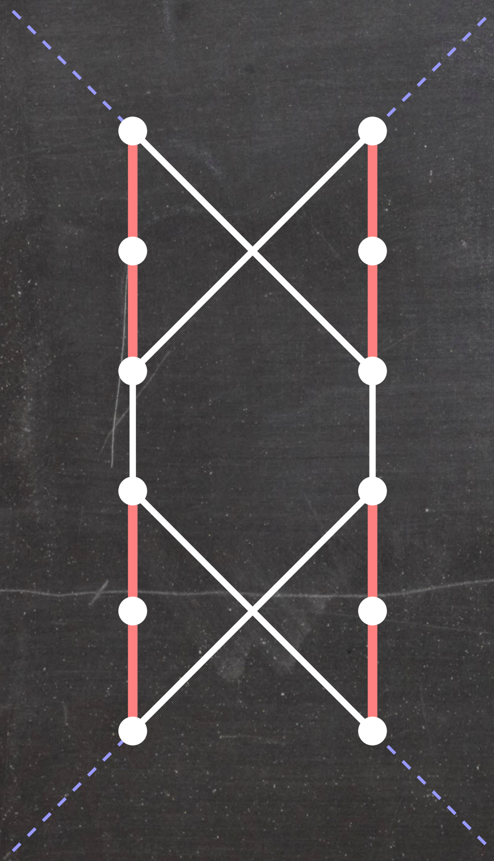

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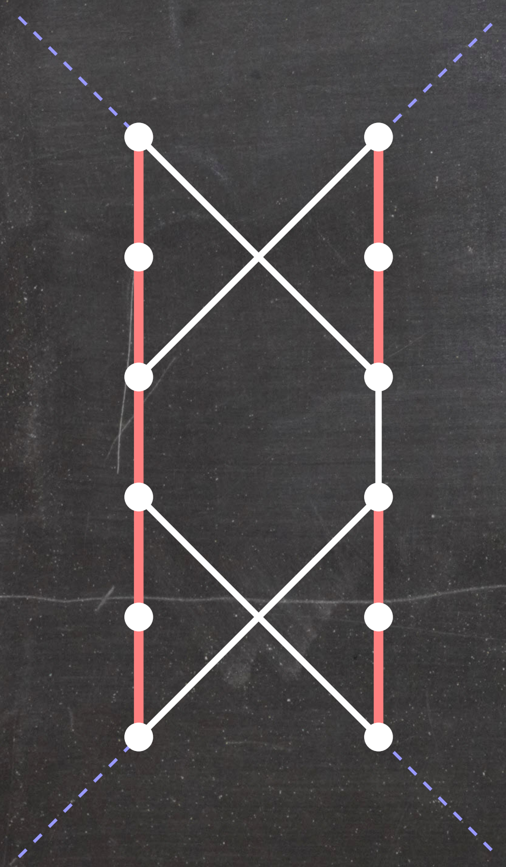

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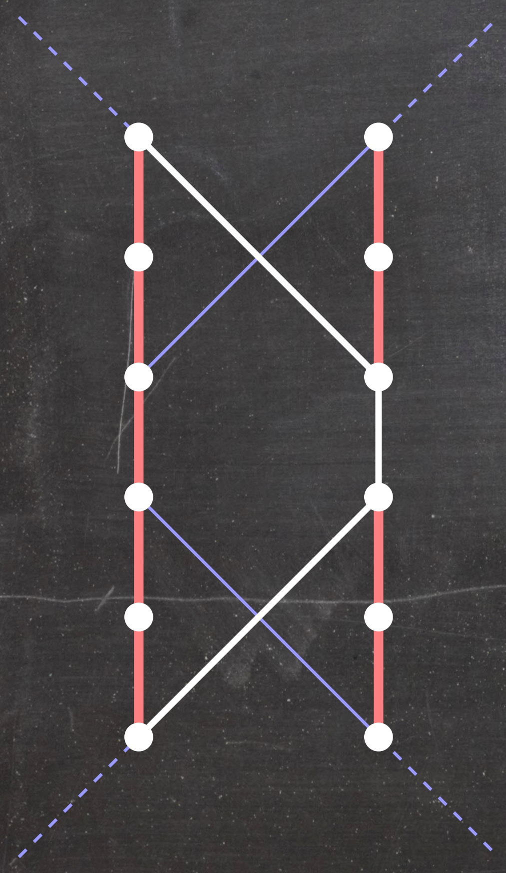

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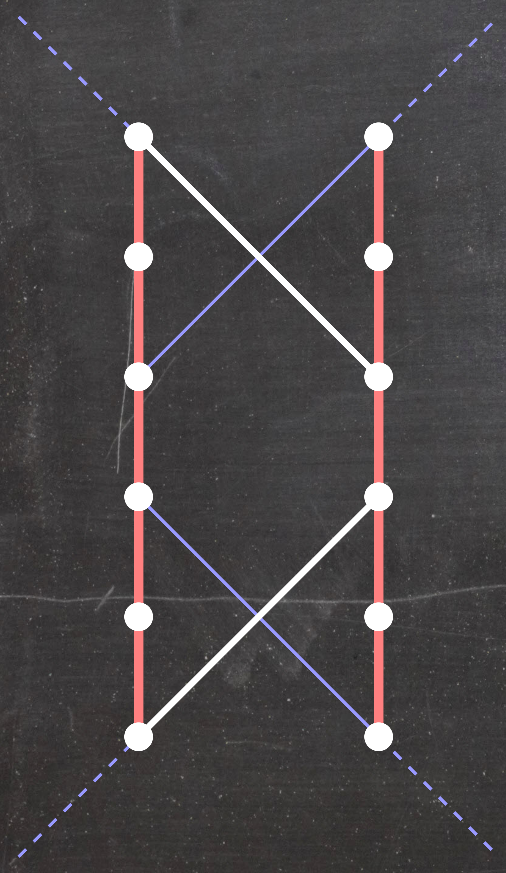

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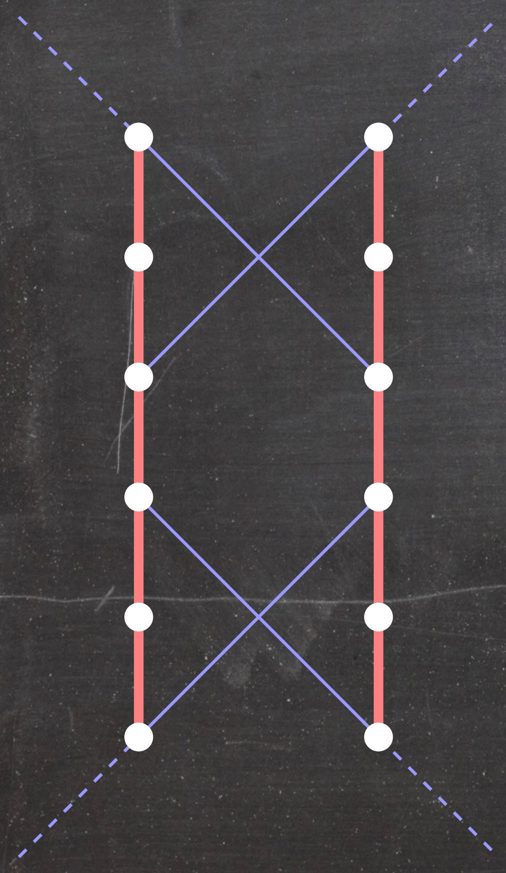

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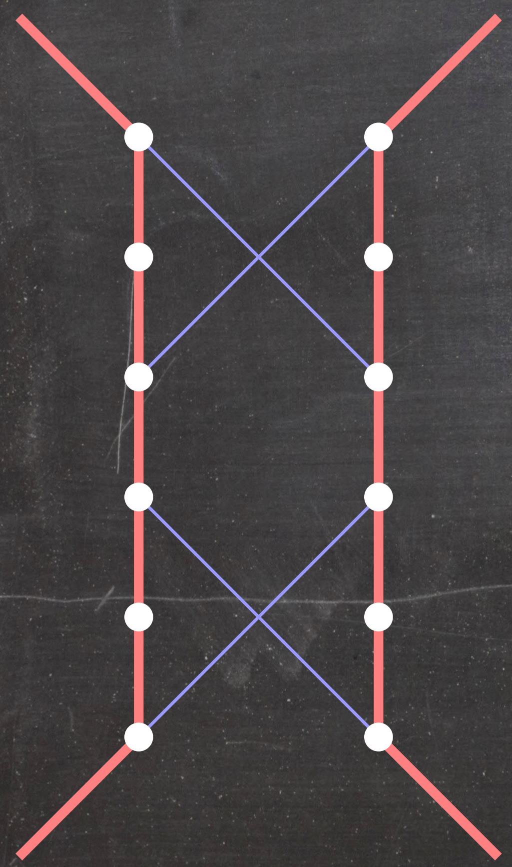

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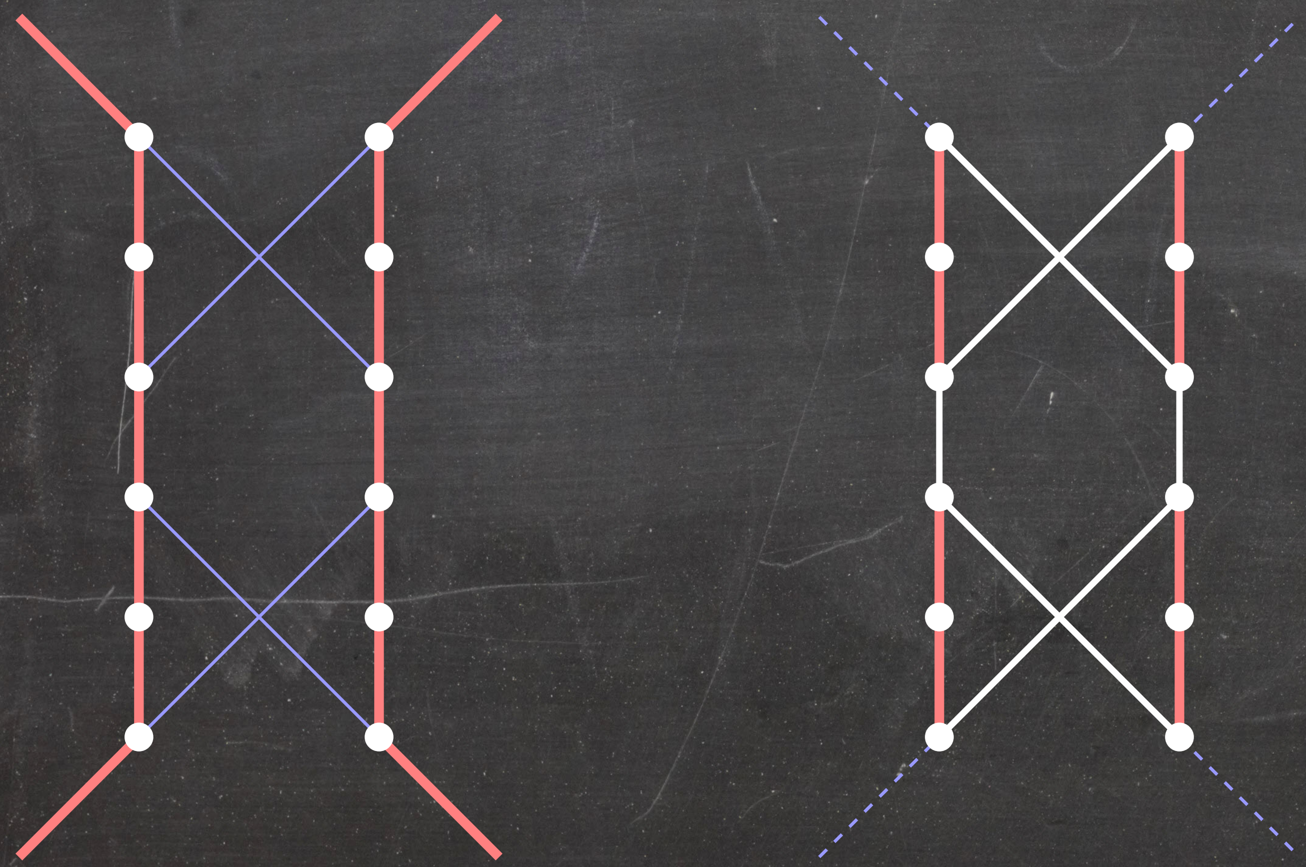

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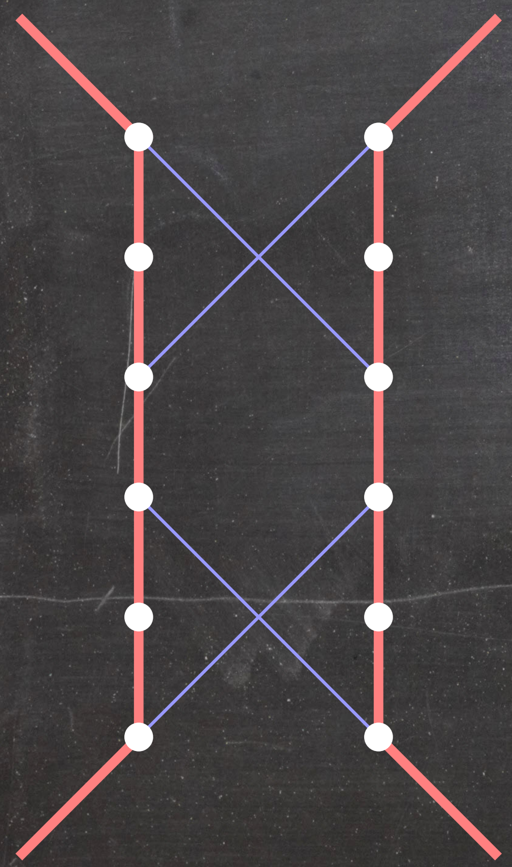

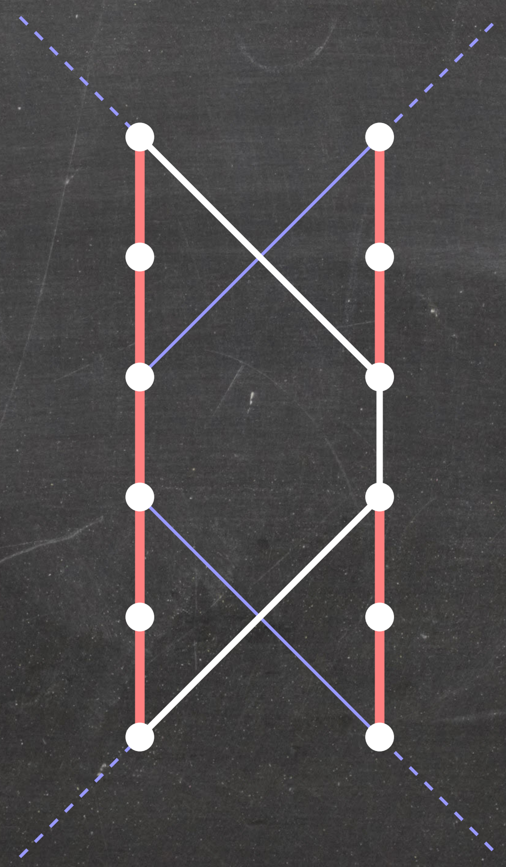

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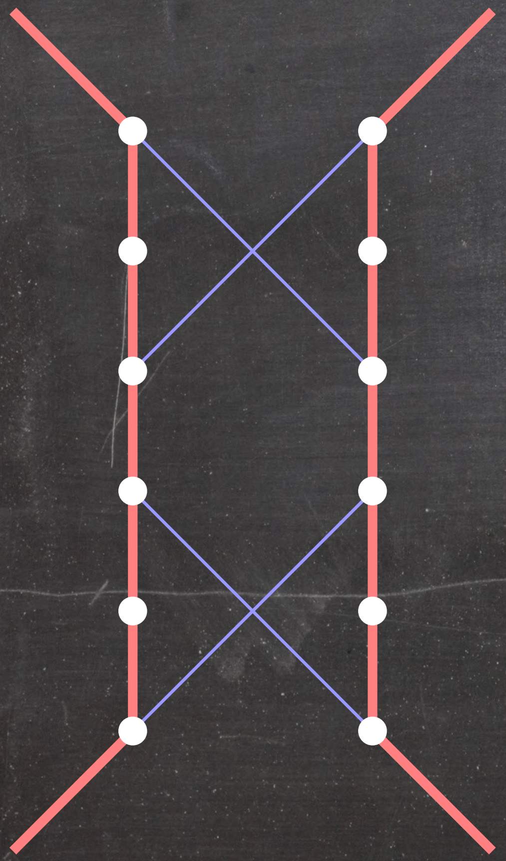

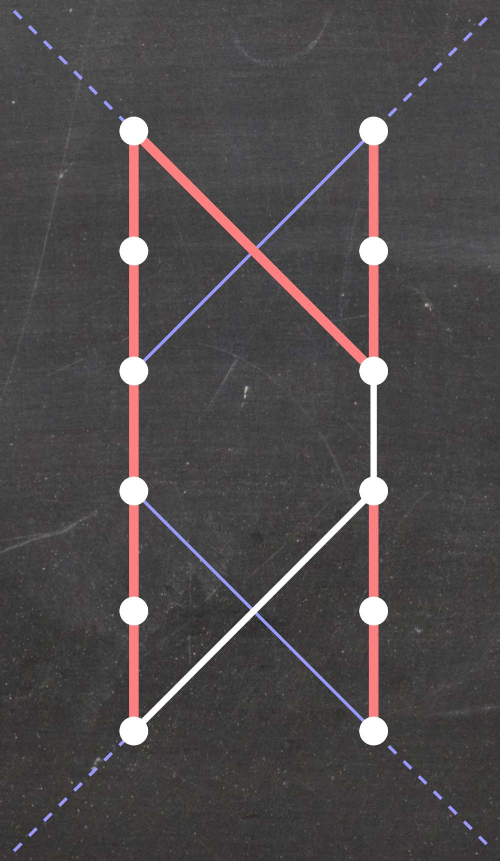

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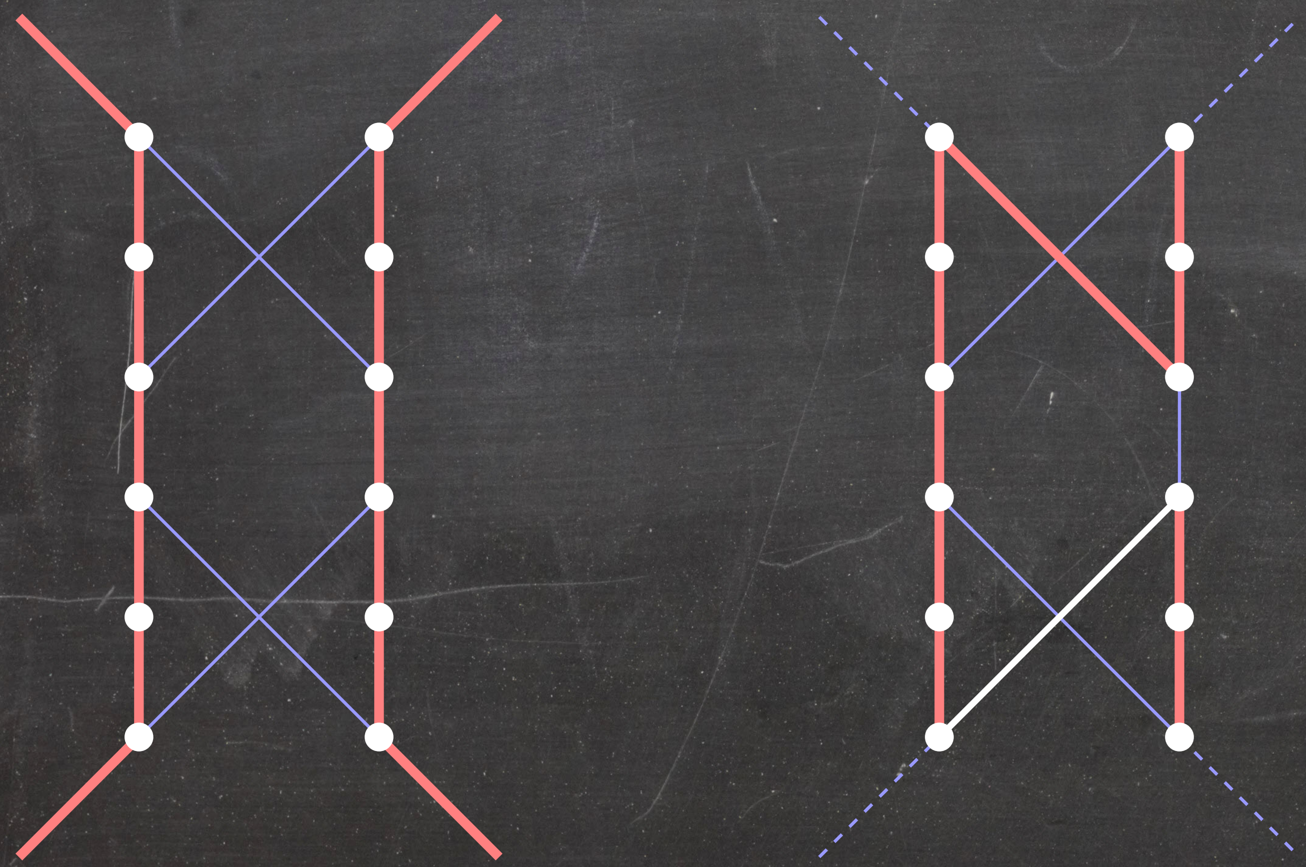

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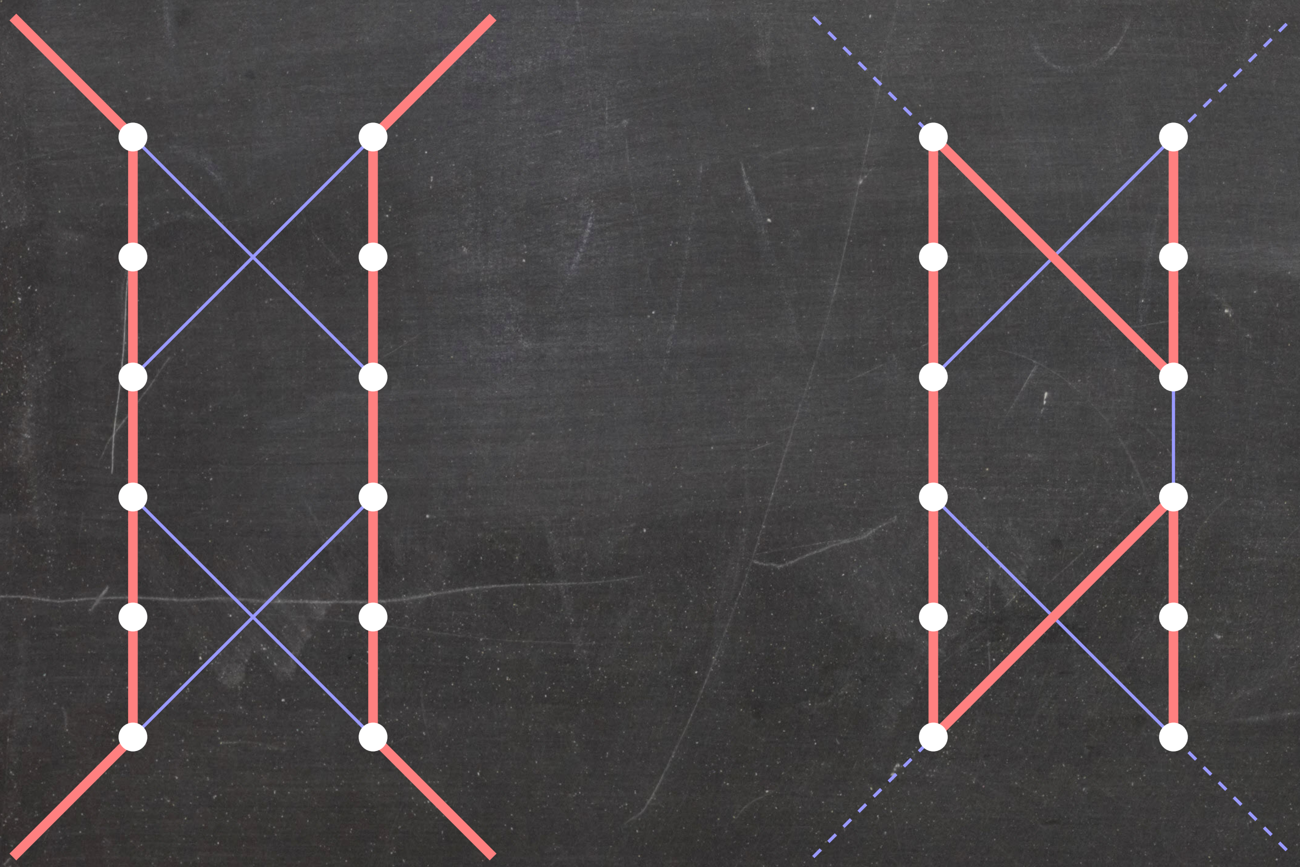

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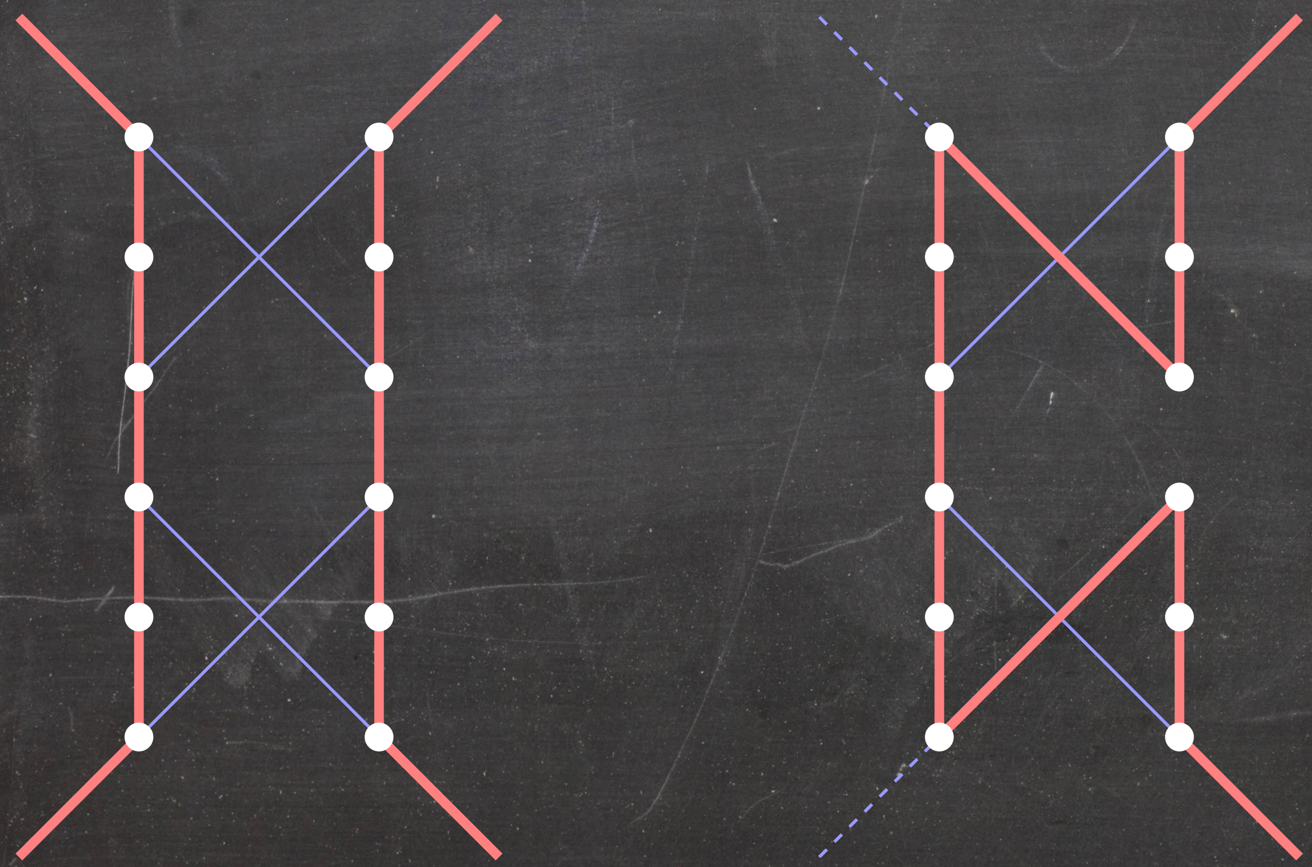

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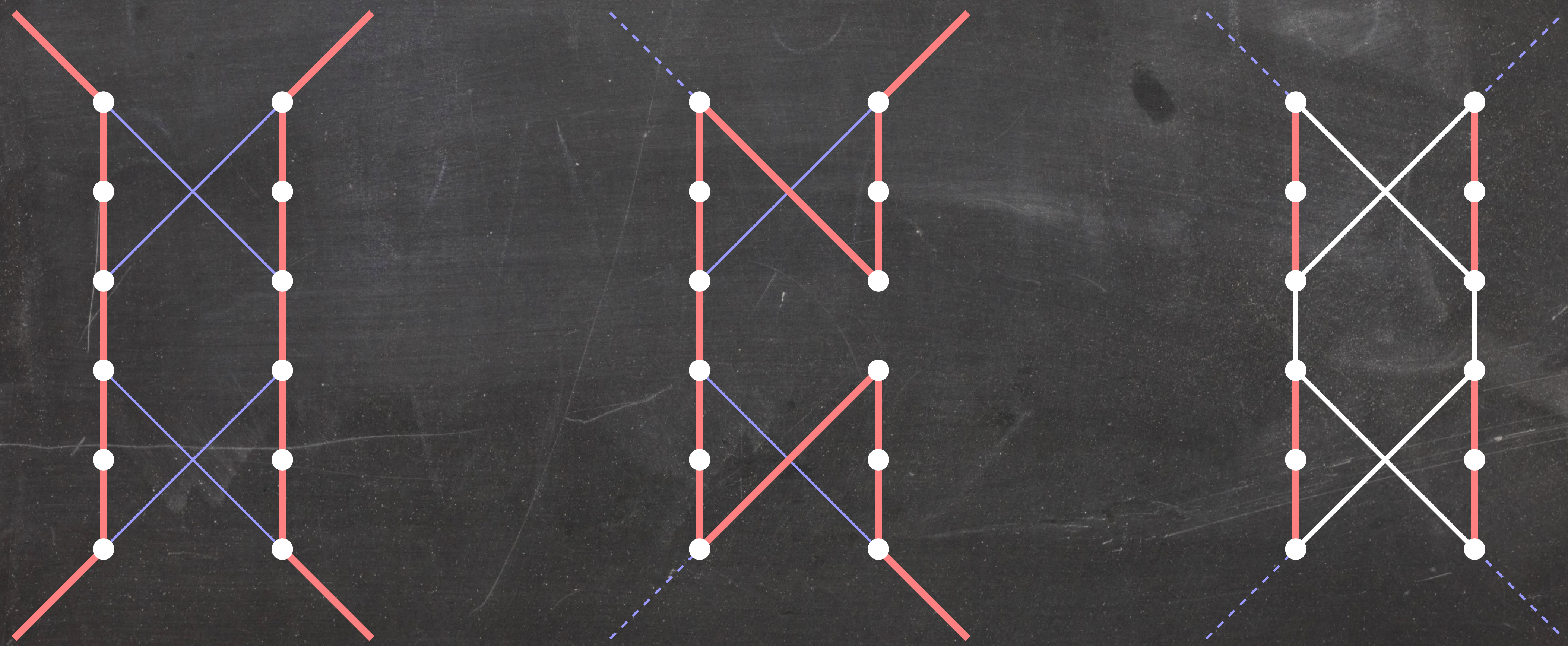

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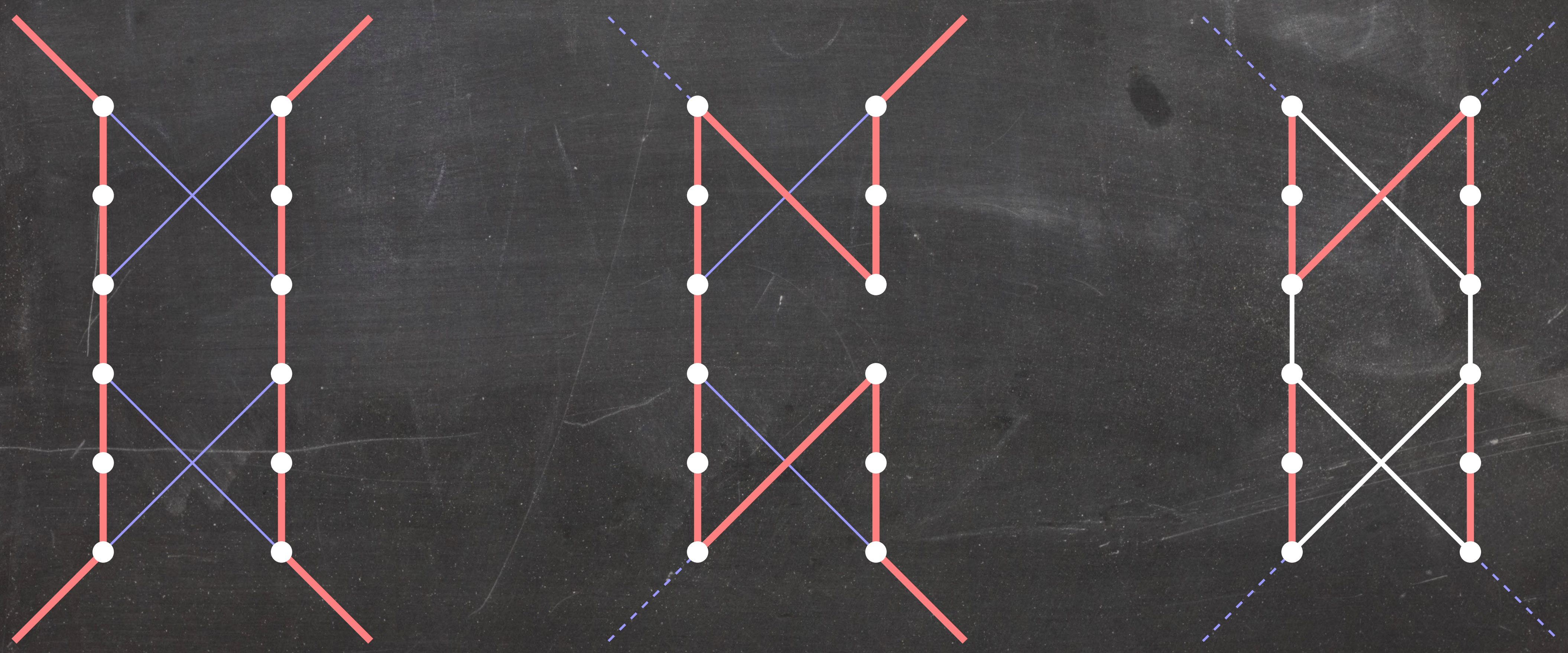

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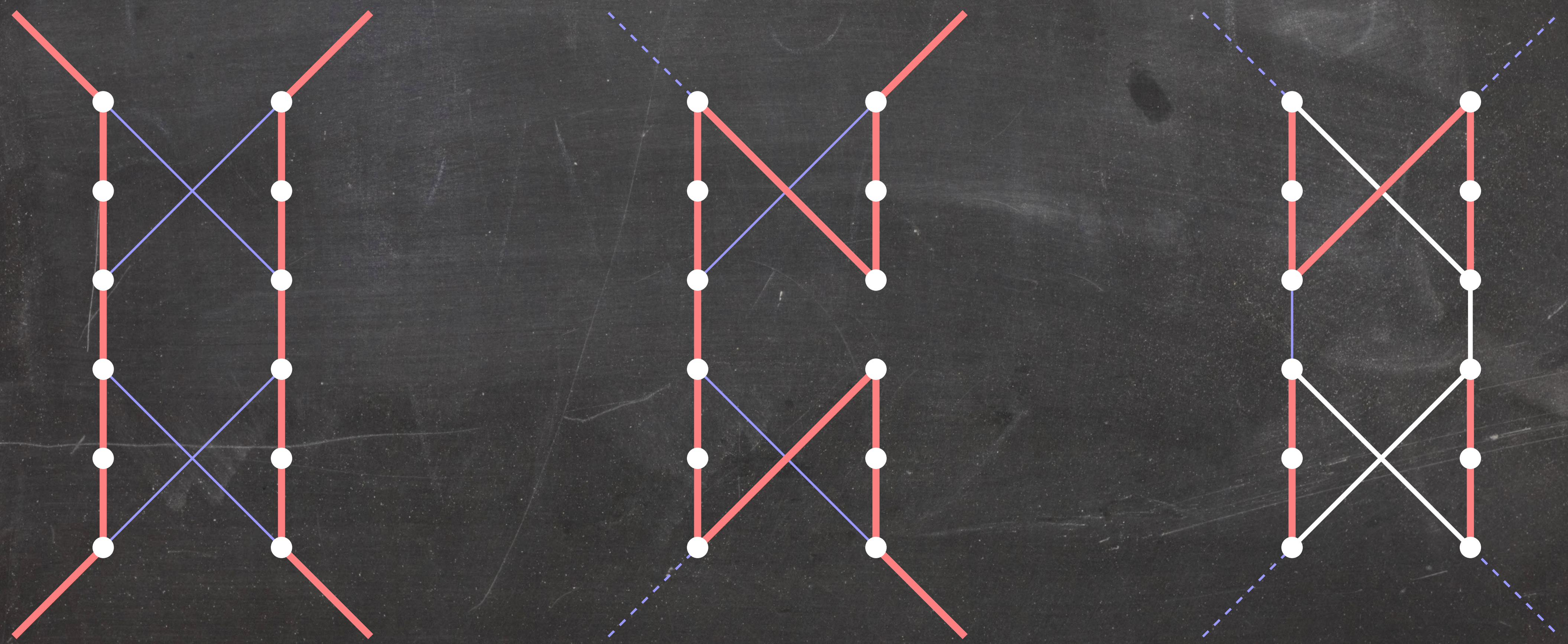

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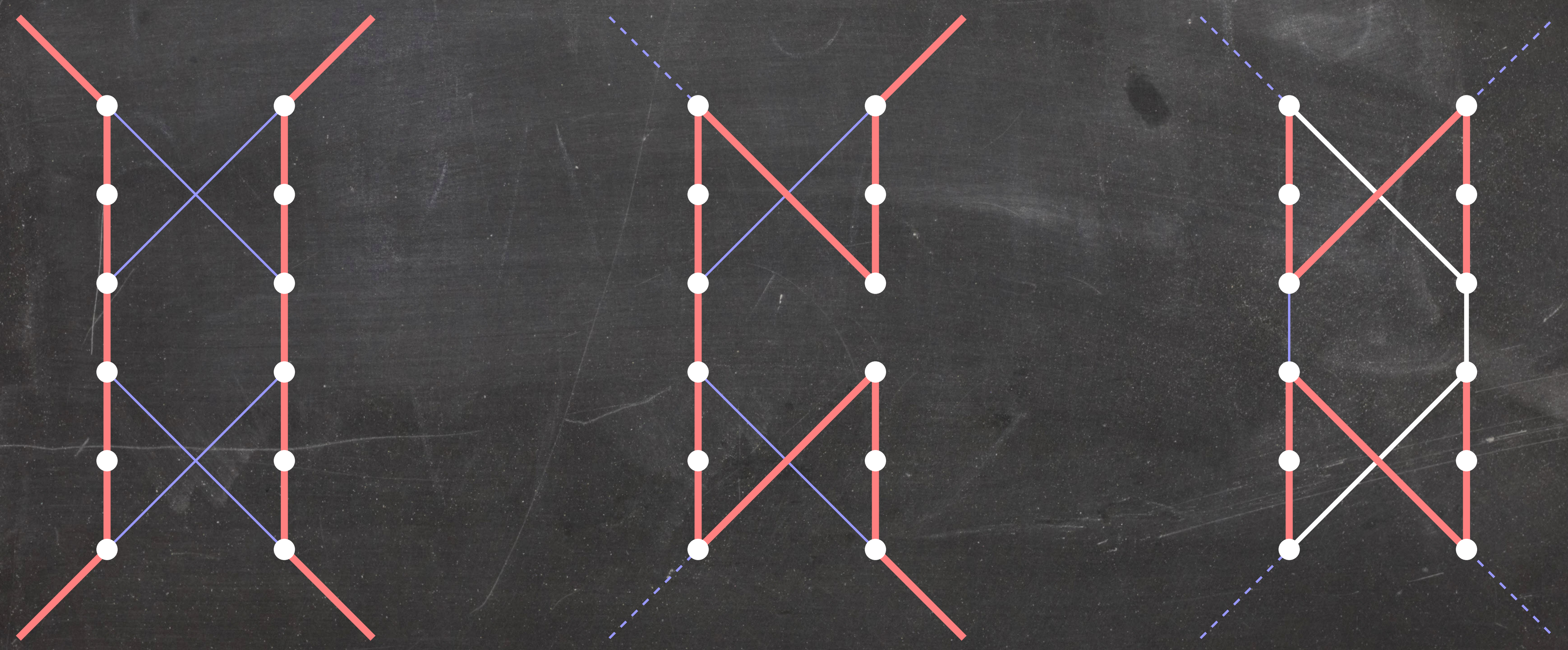

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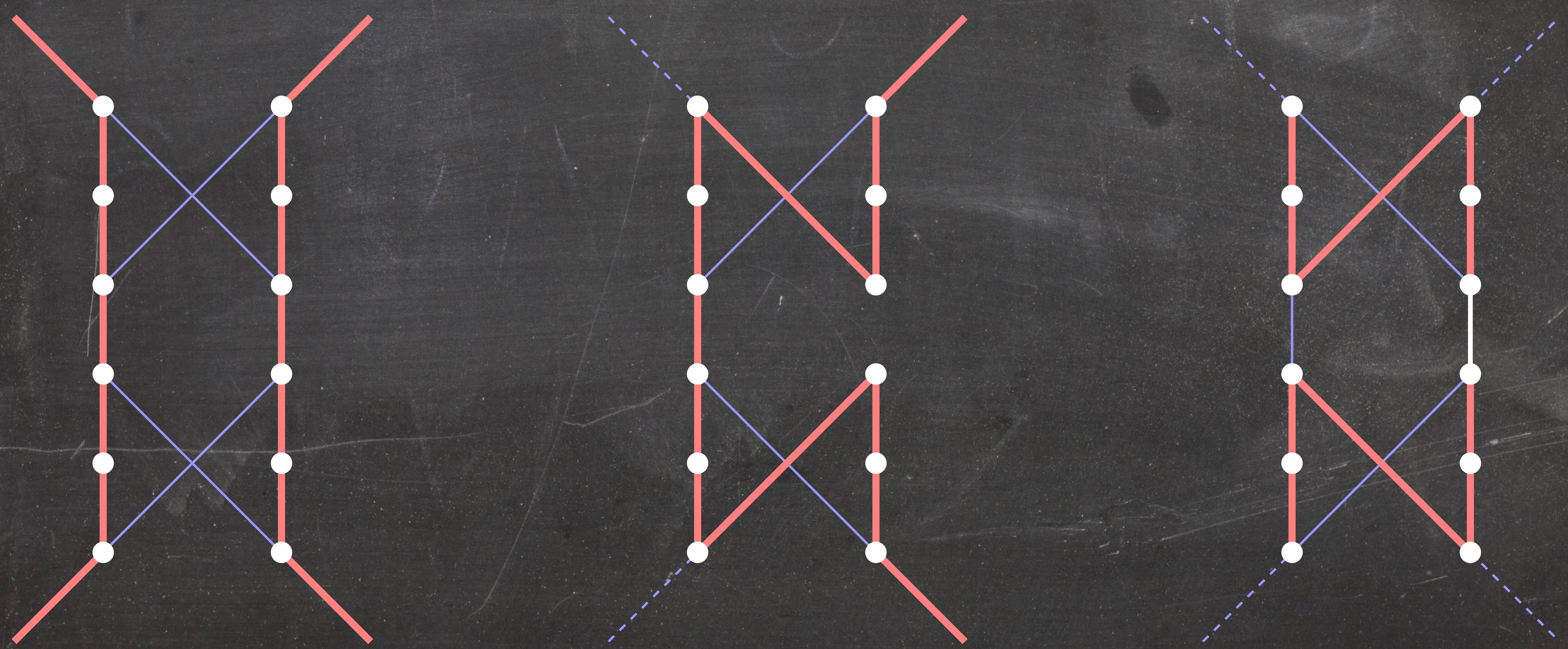

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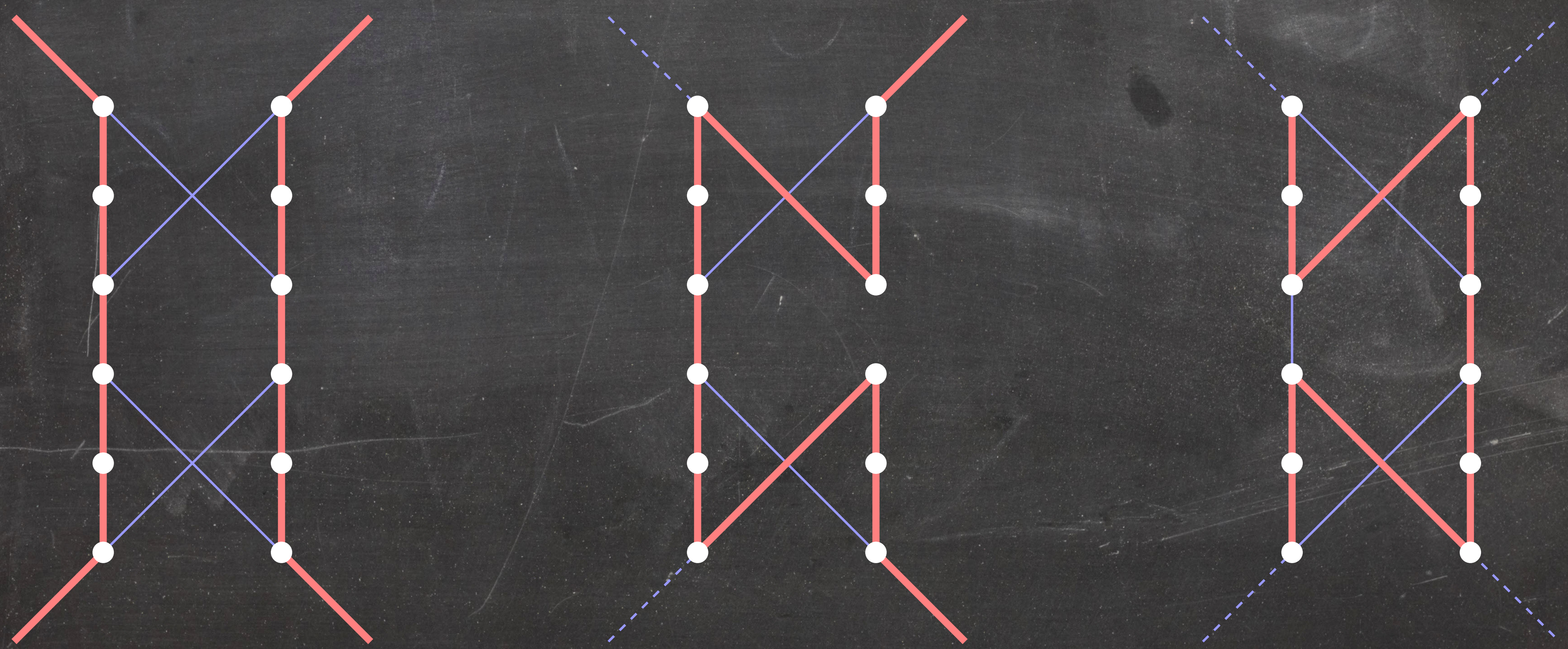

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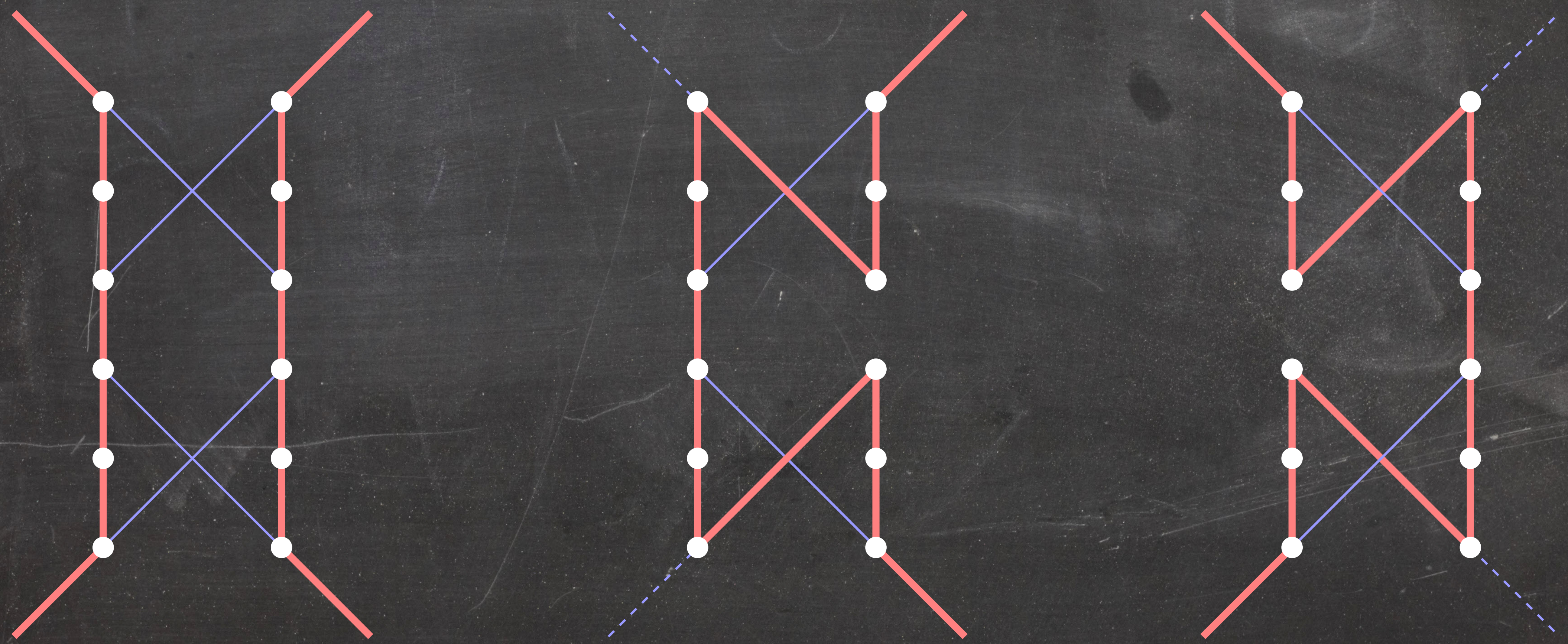
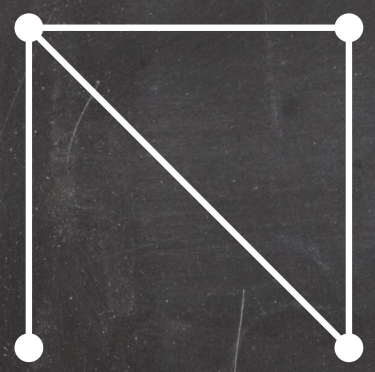


Hamiltonian Cycle is NP-Complete



G

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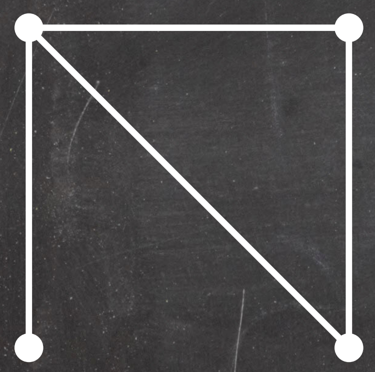

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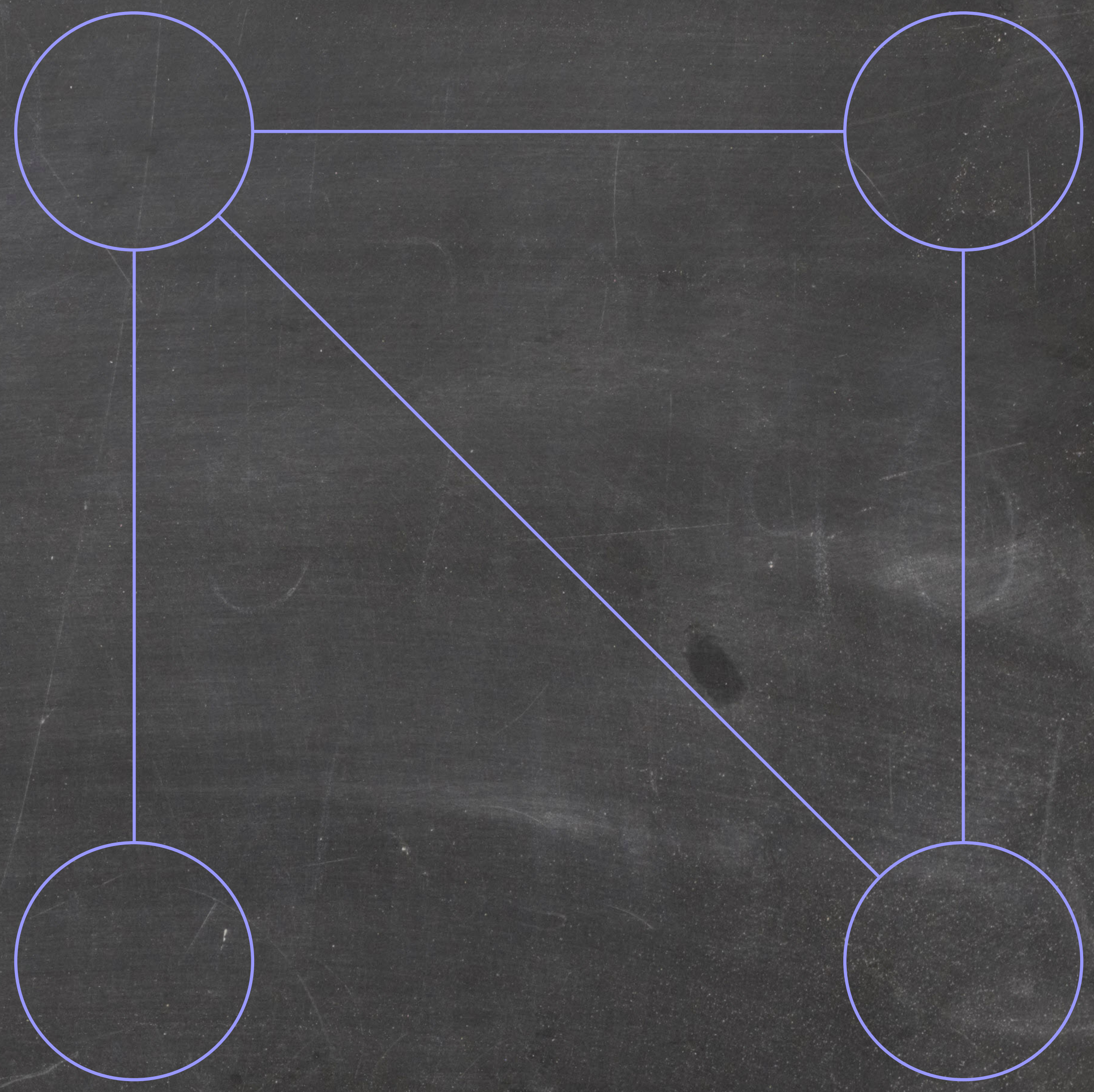

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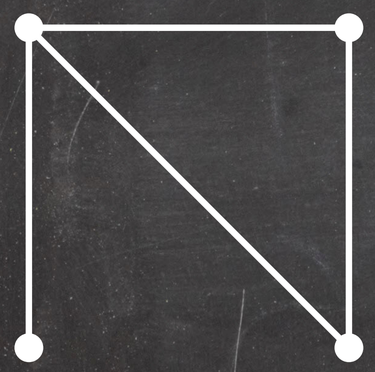

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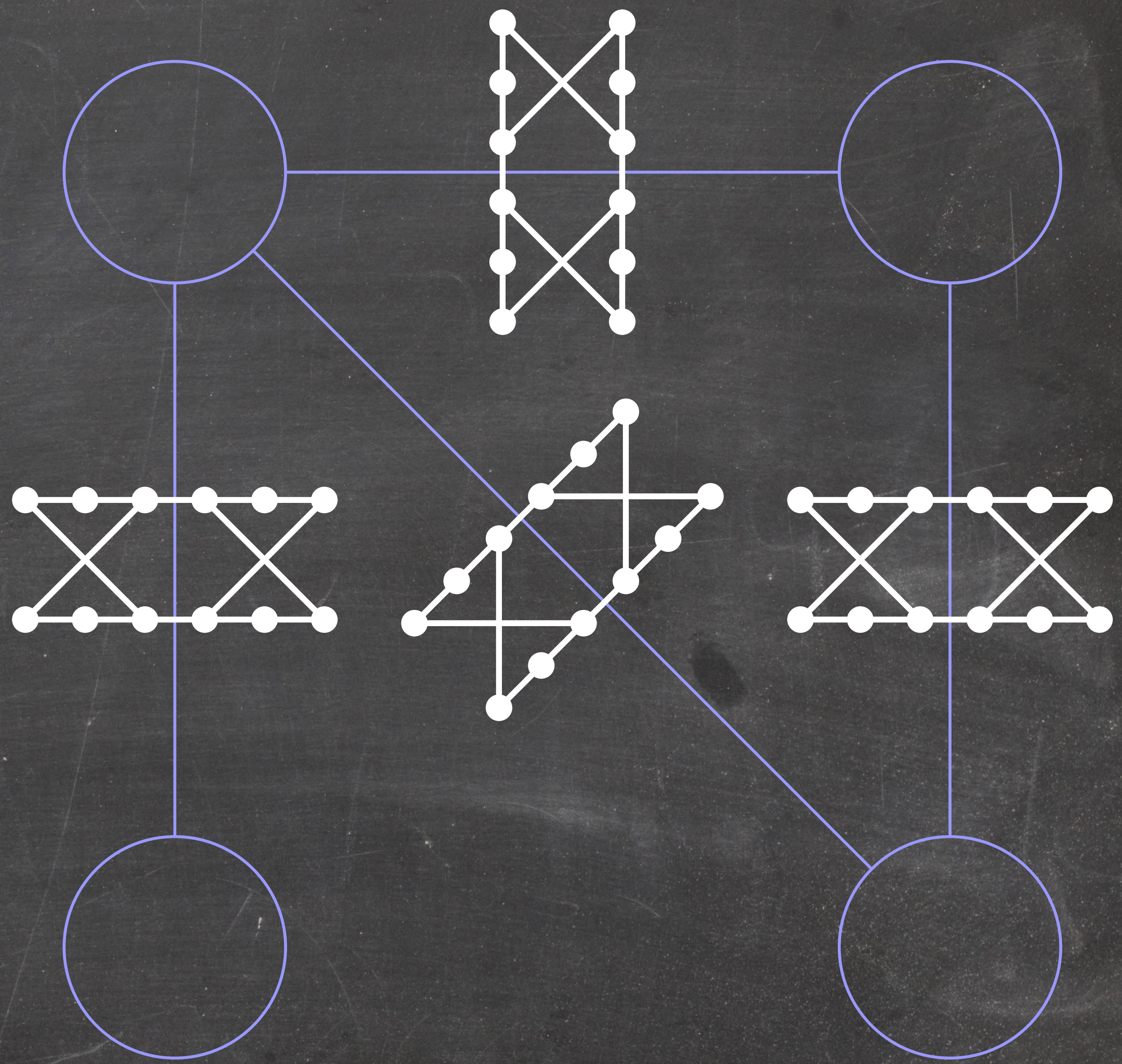

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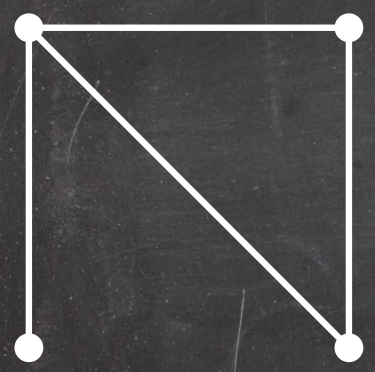

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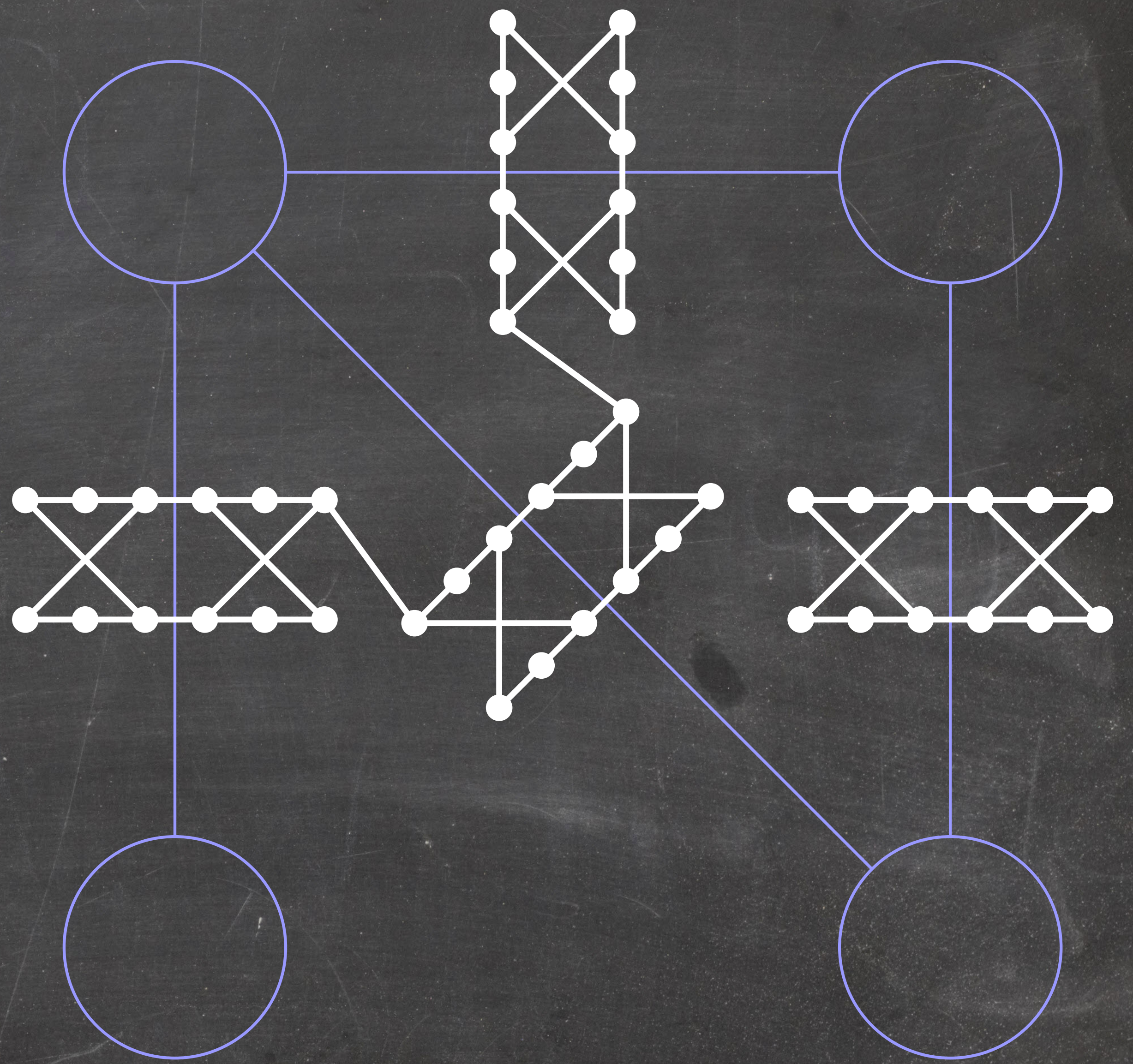

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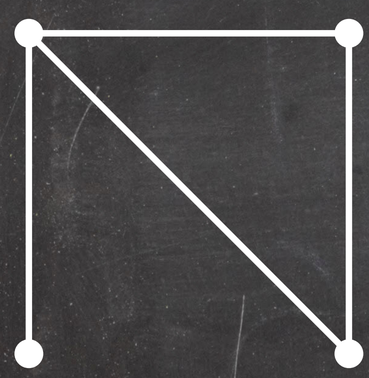

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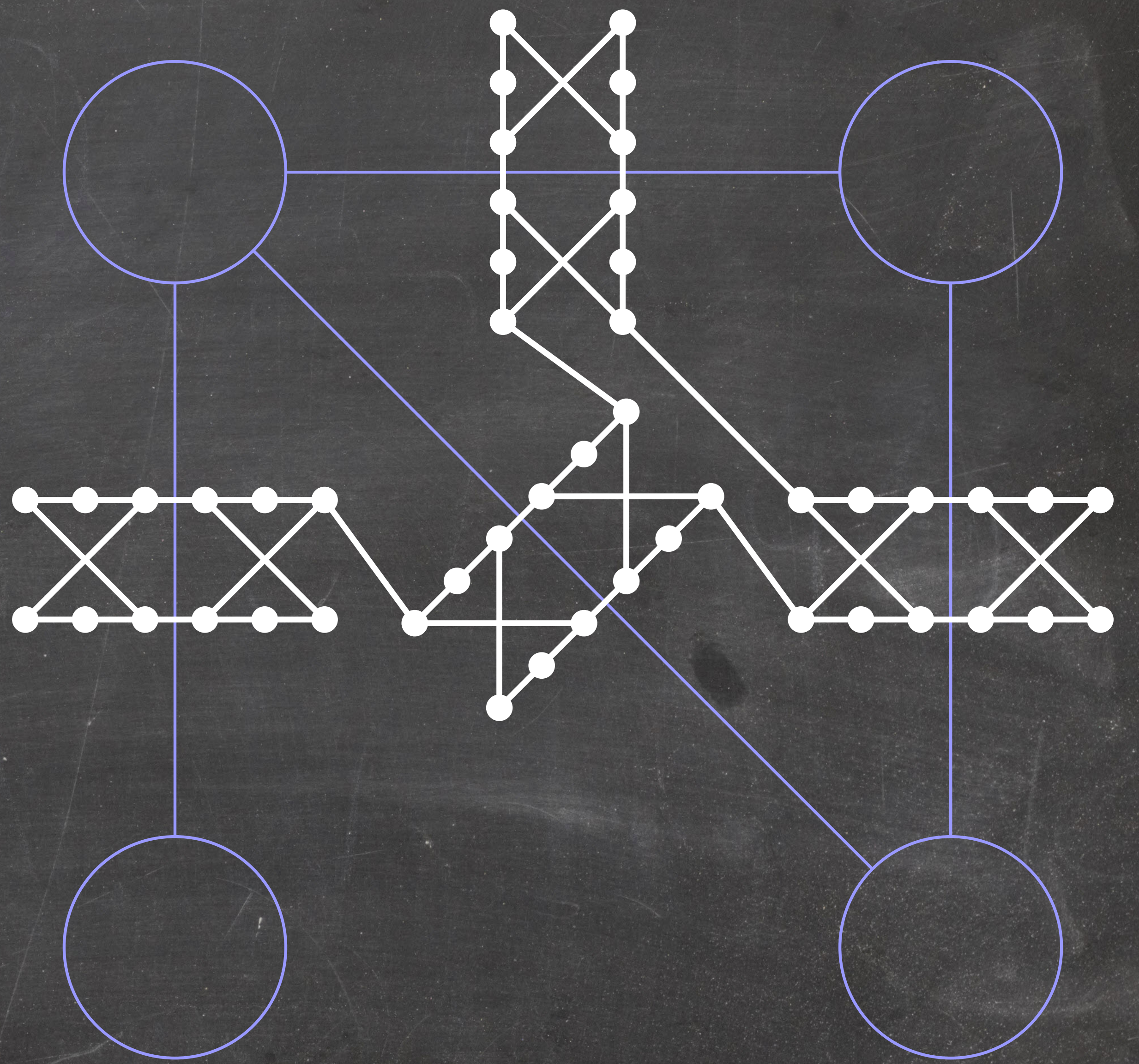

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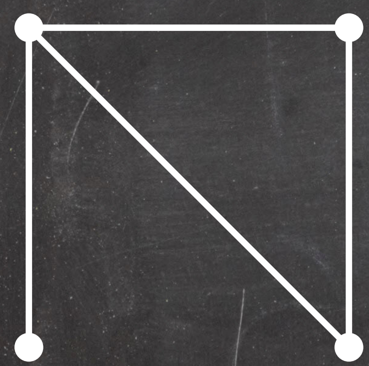

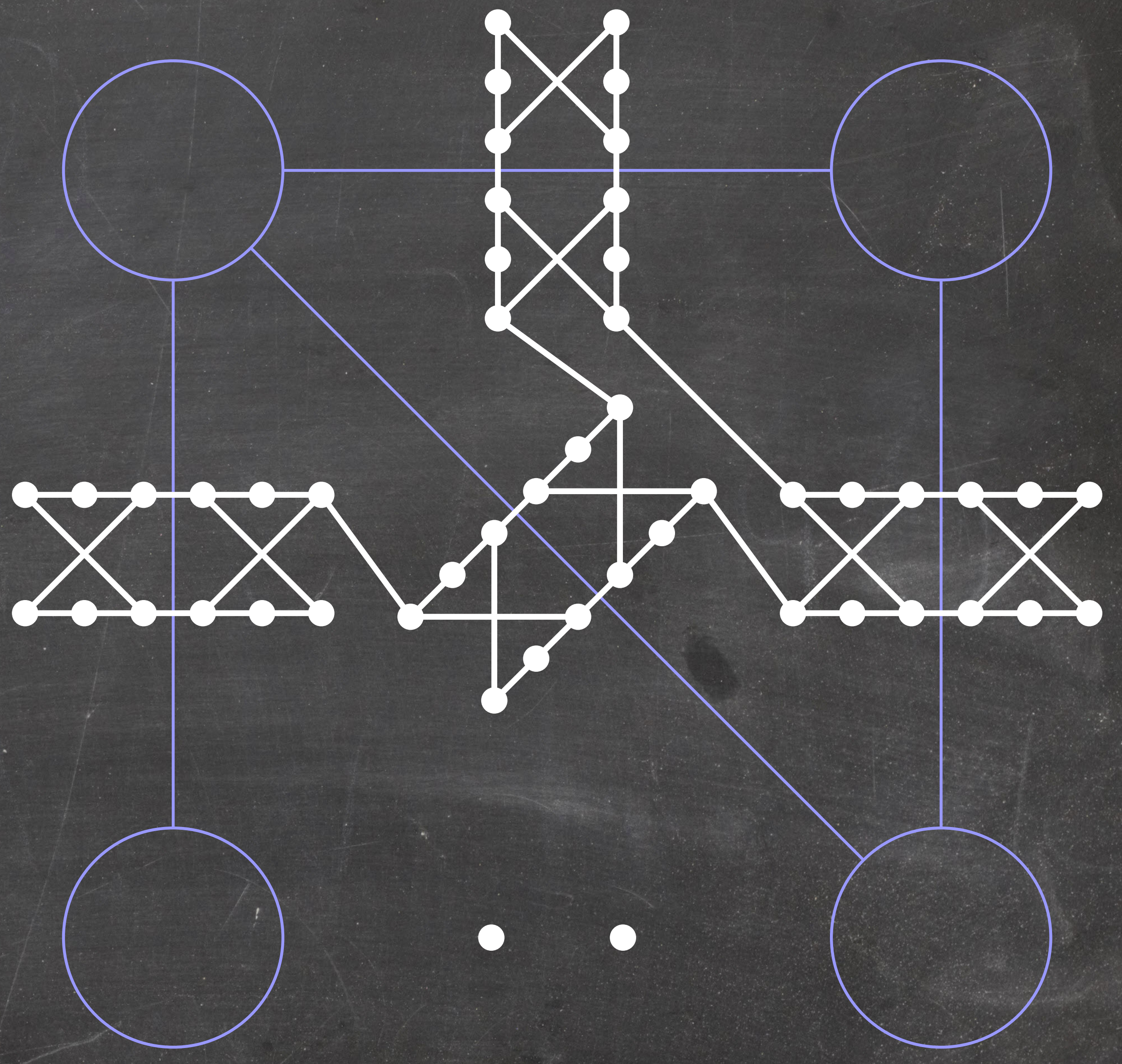

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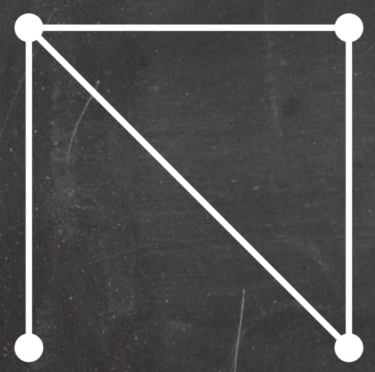

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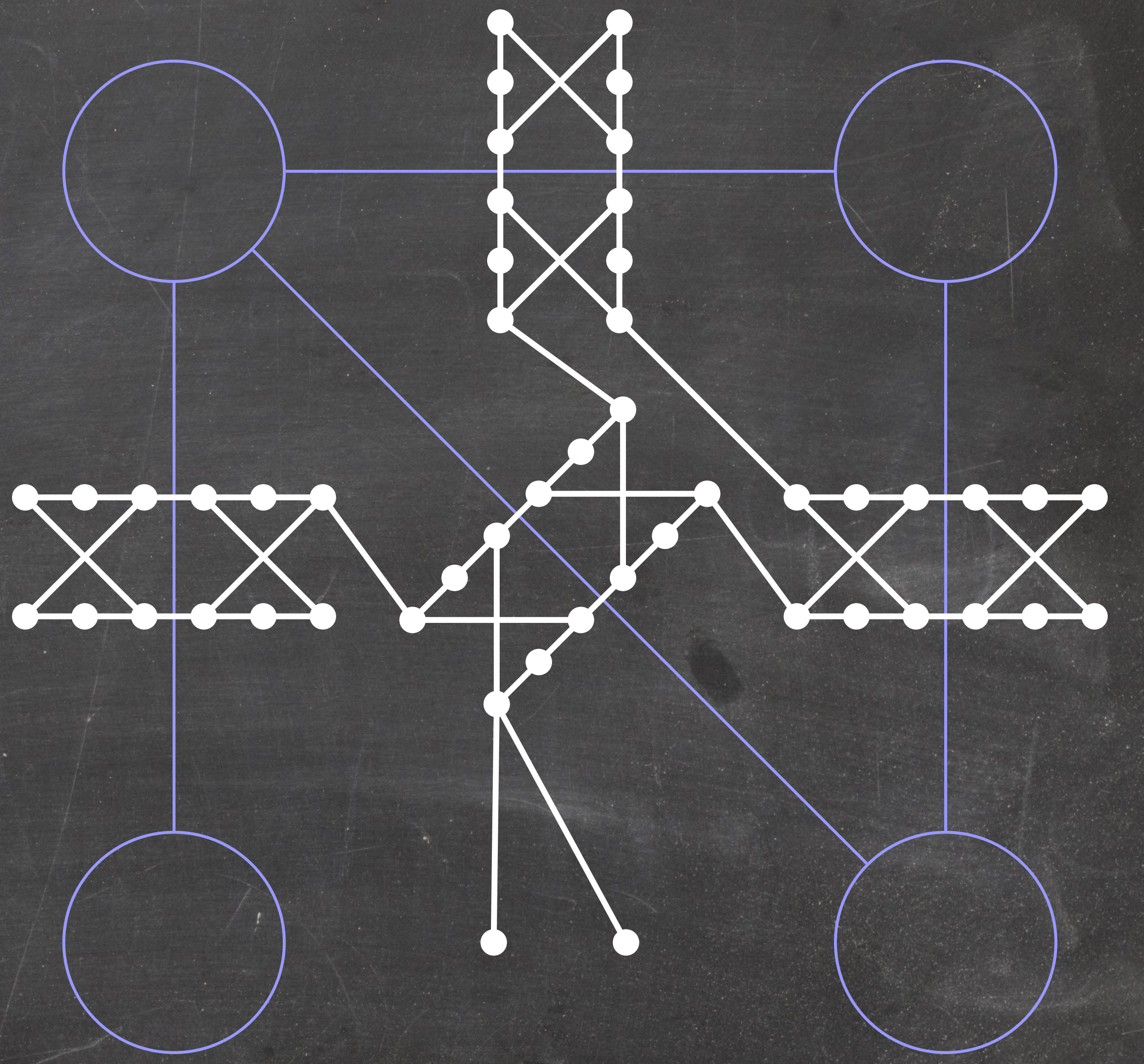

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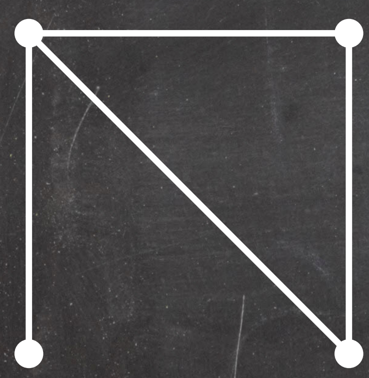

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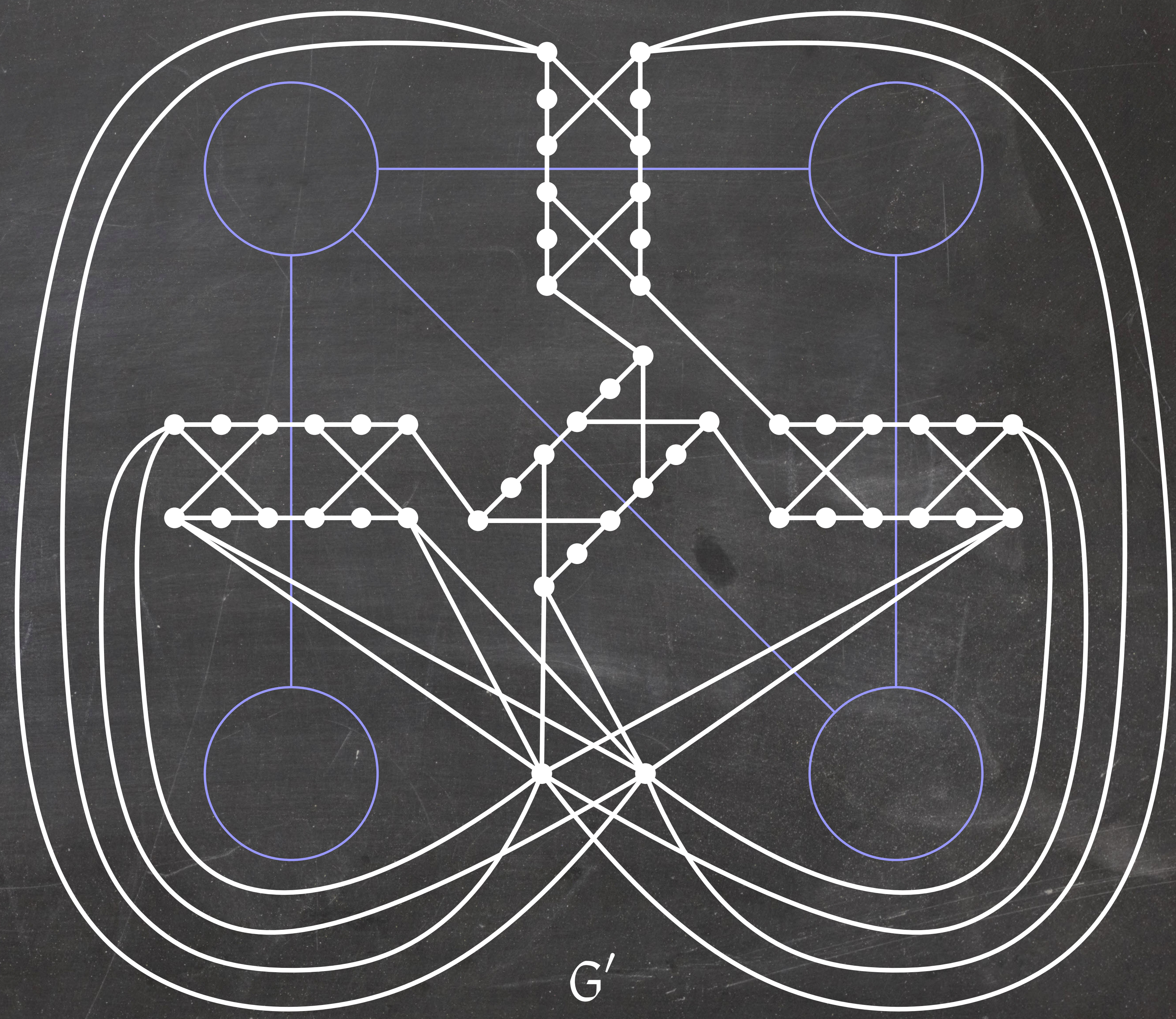

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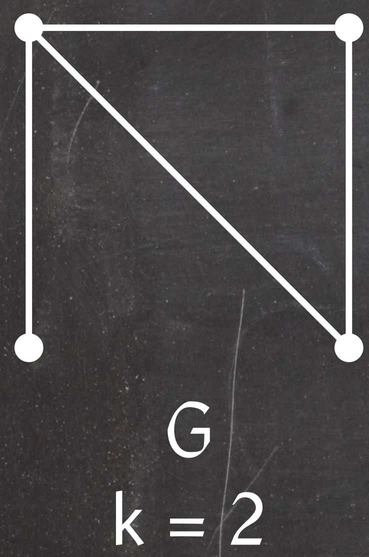

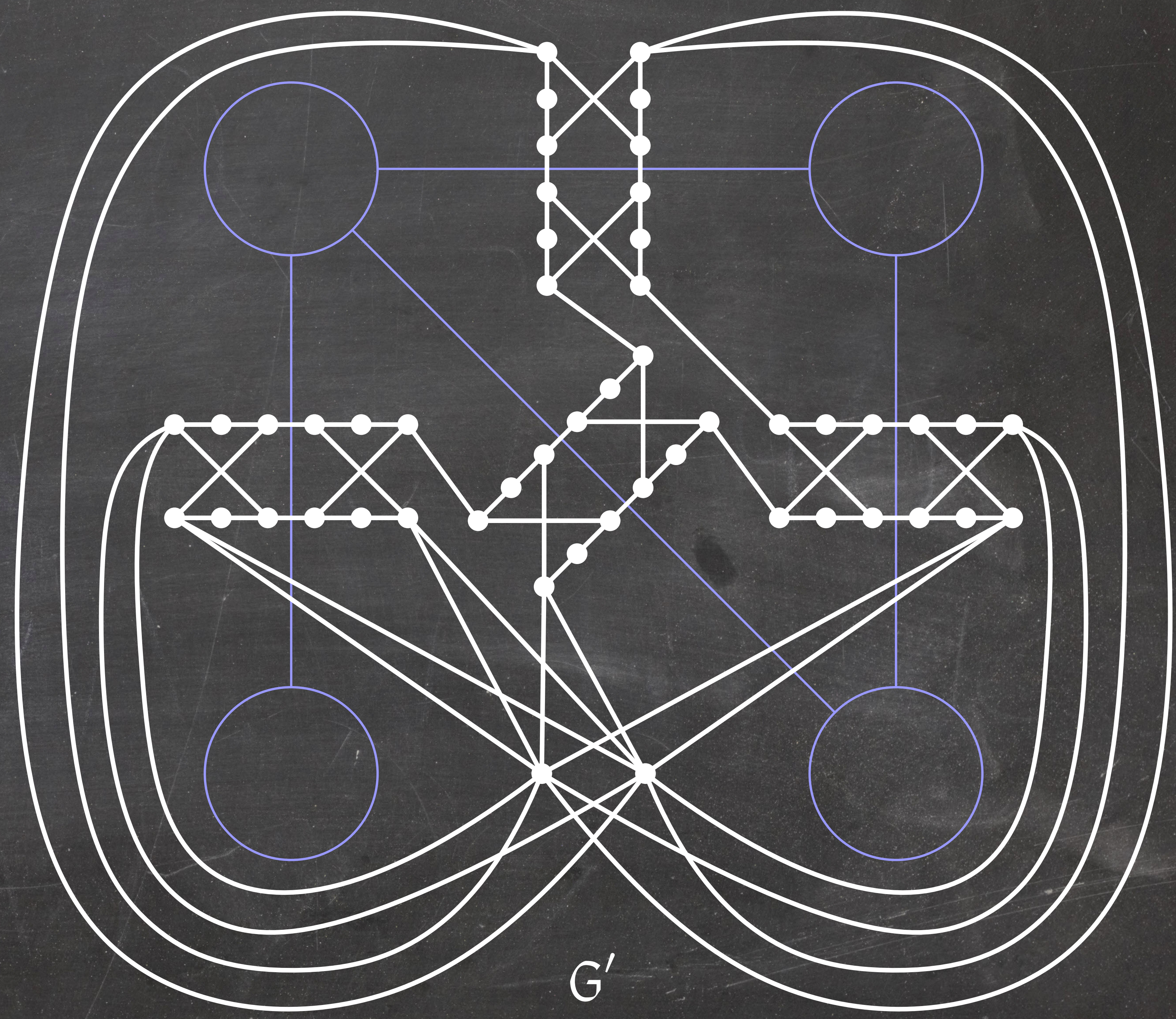

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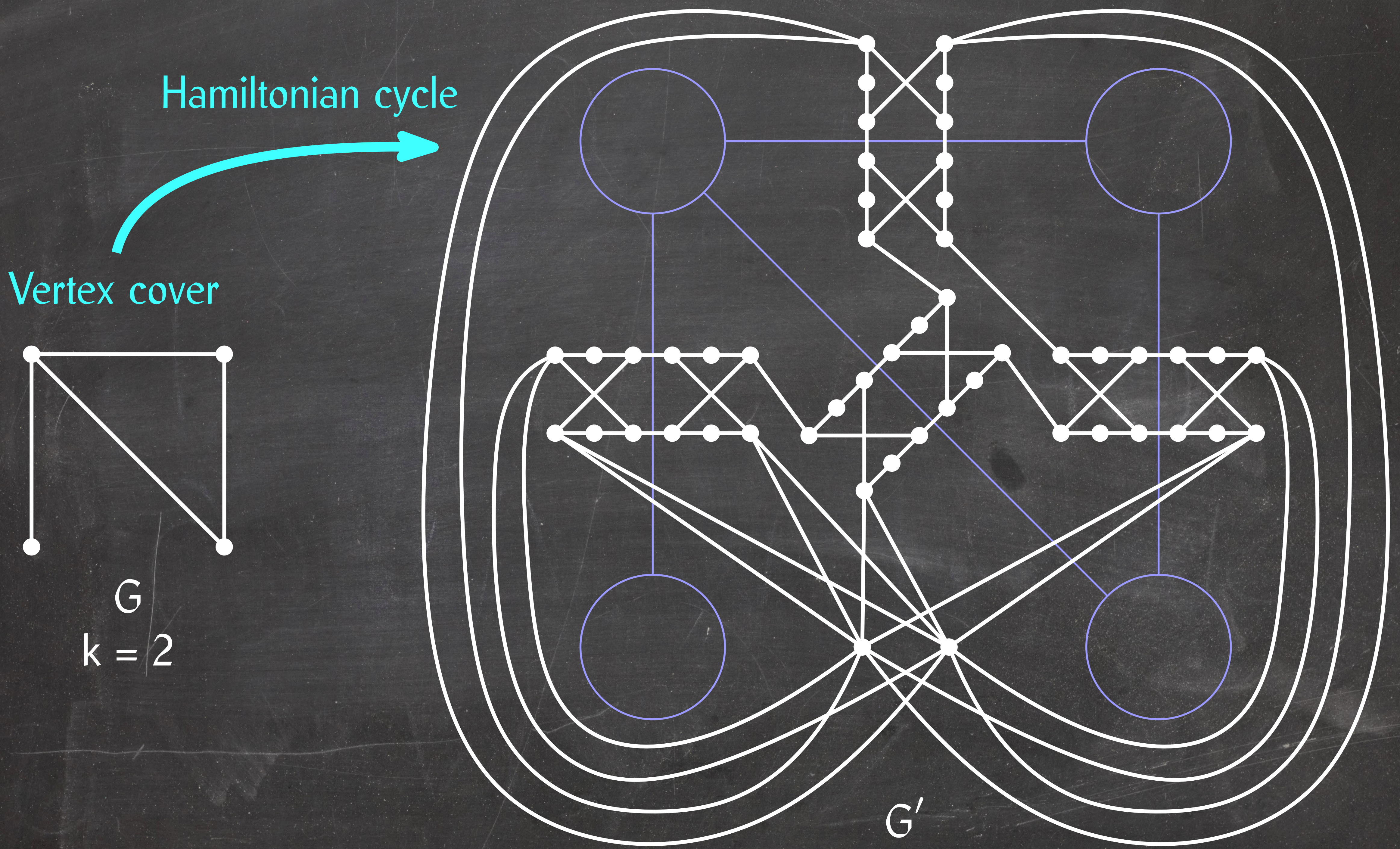

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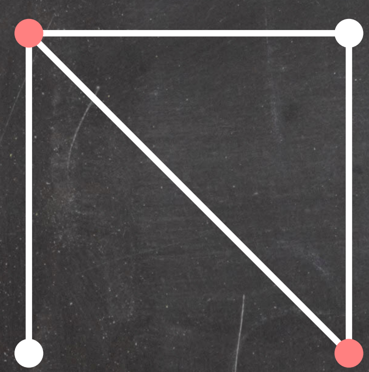
Hamiltonian cycle

Vertex cover



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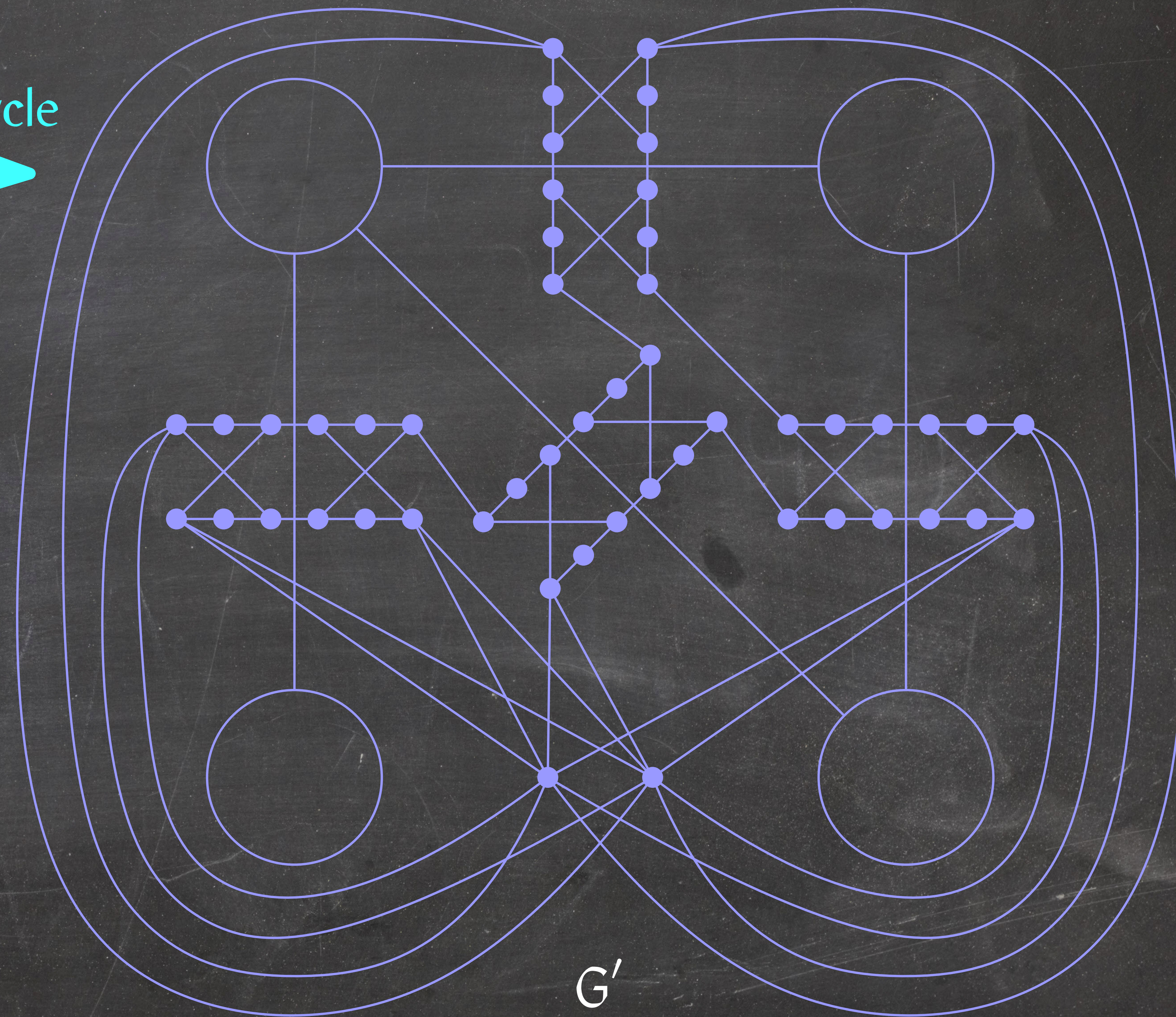

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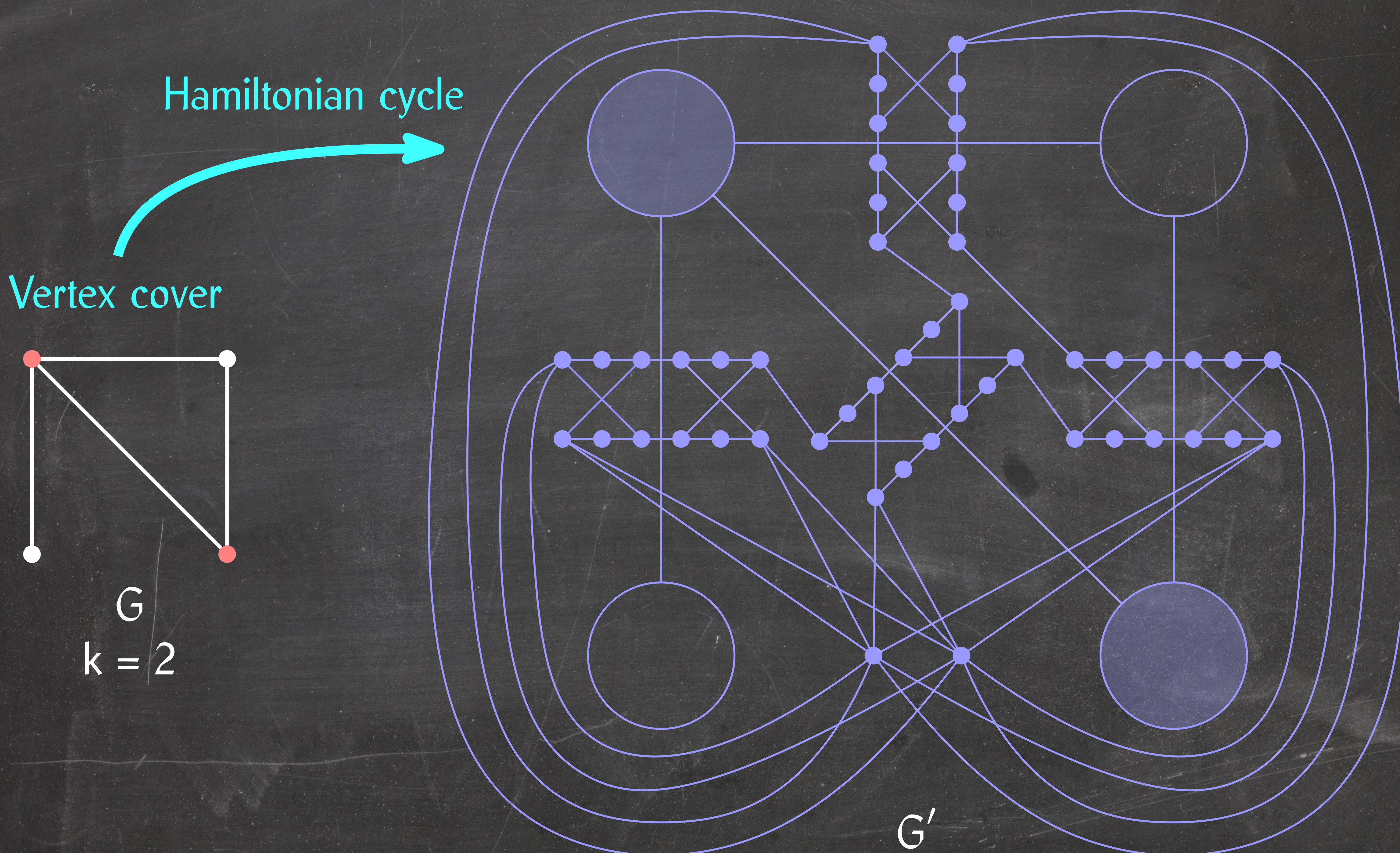

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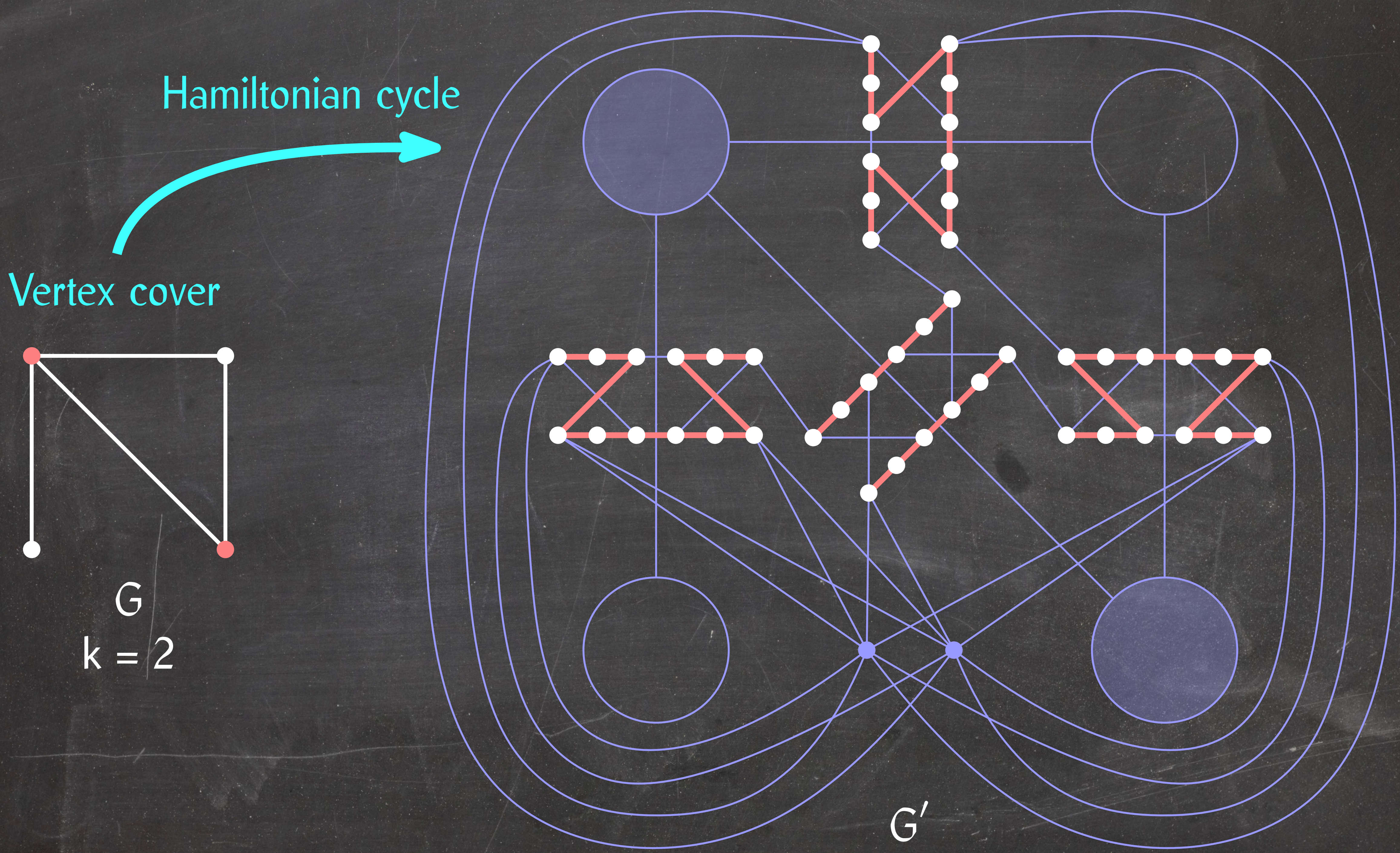

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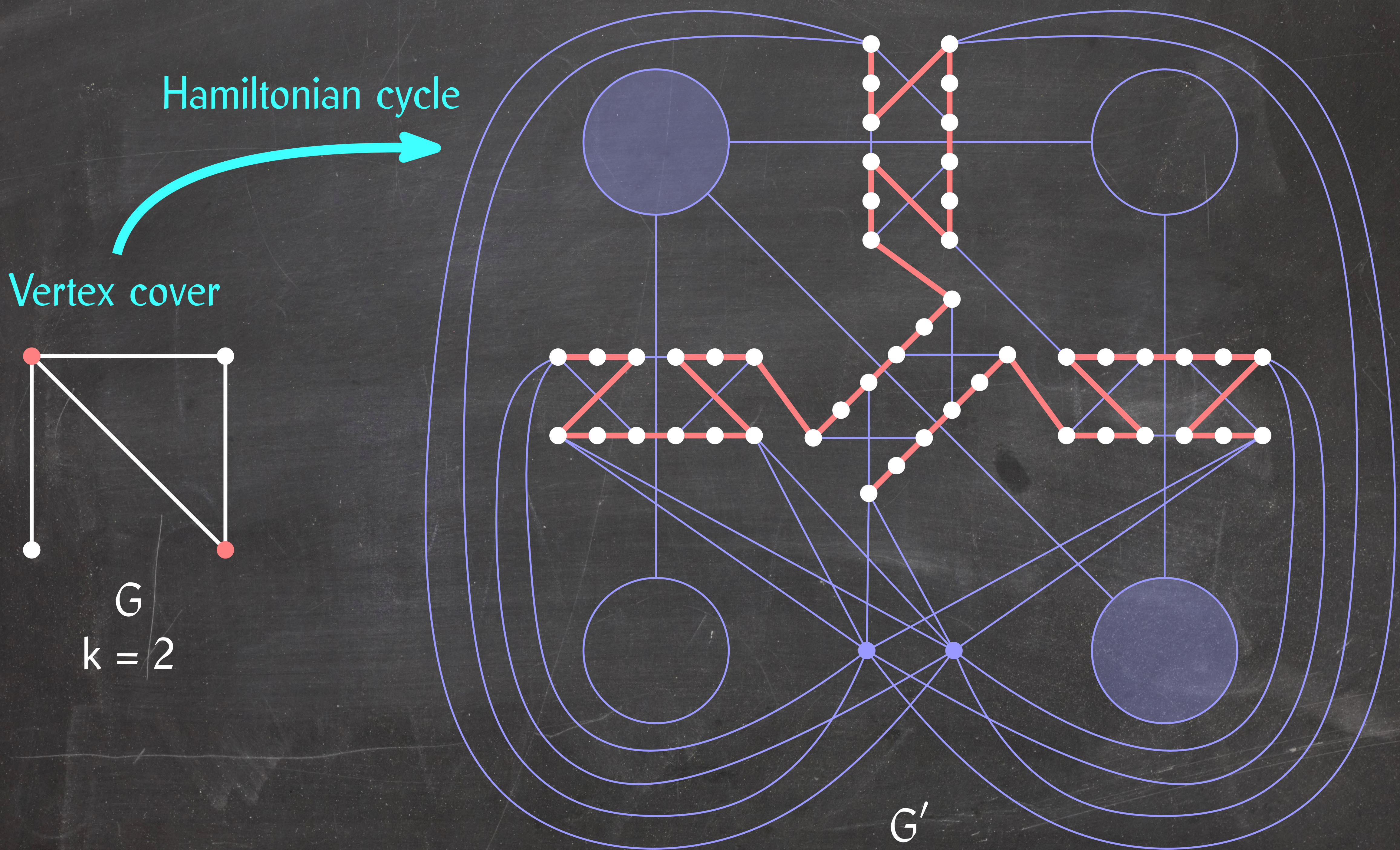

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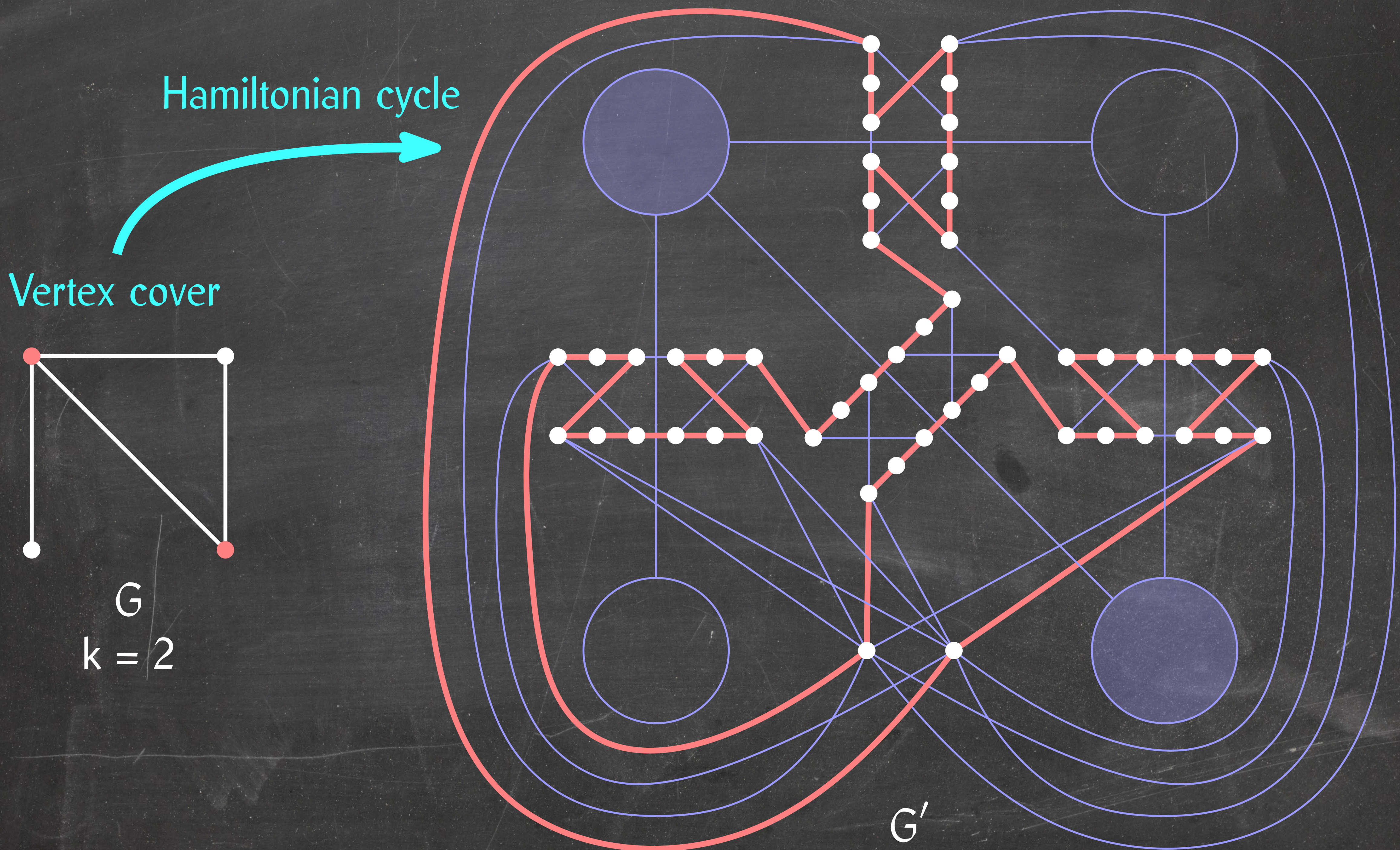

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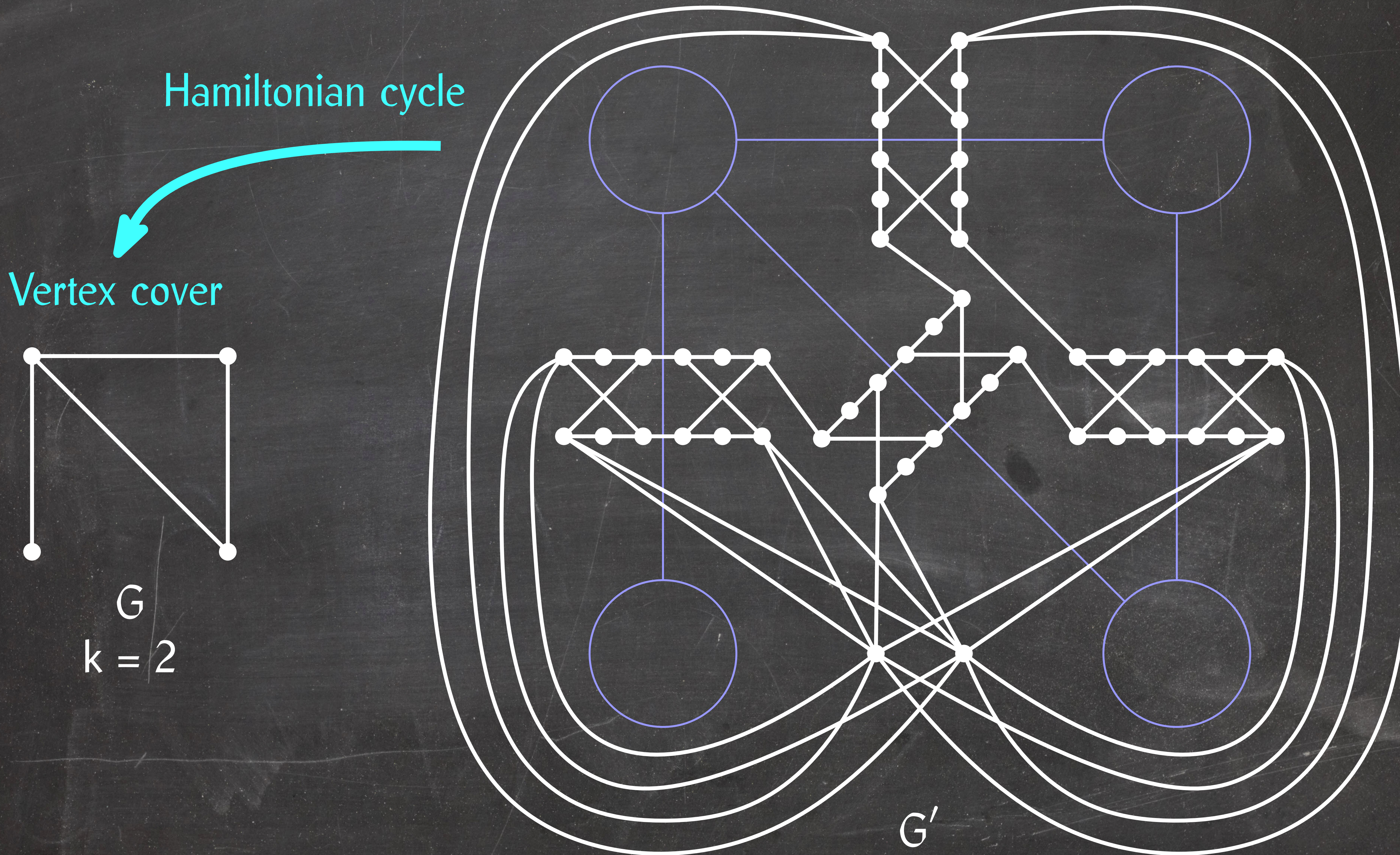

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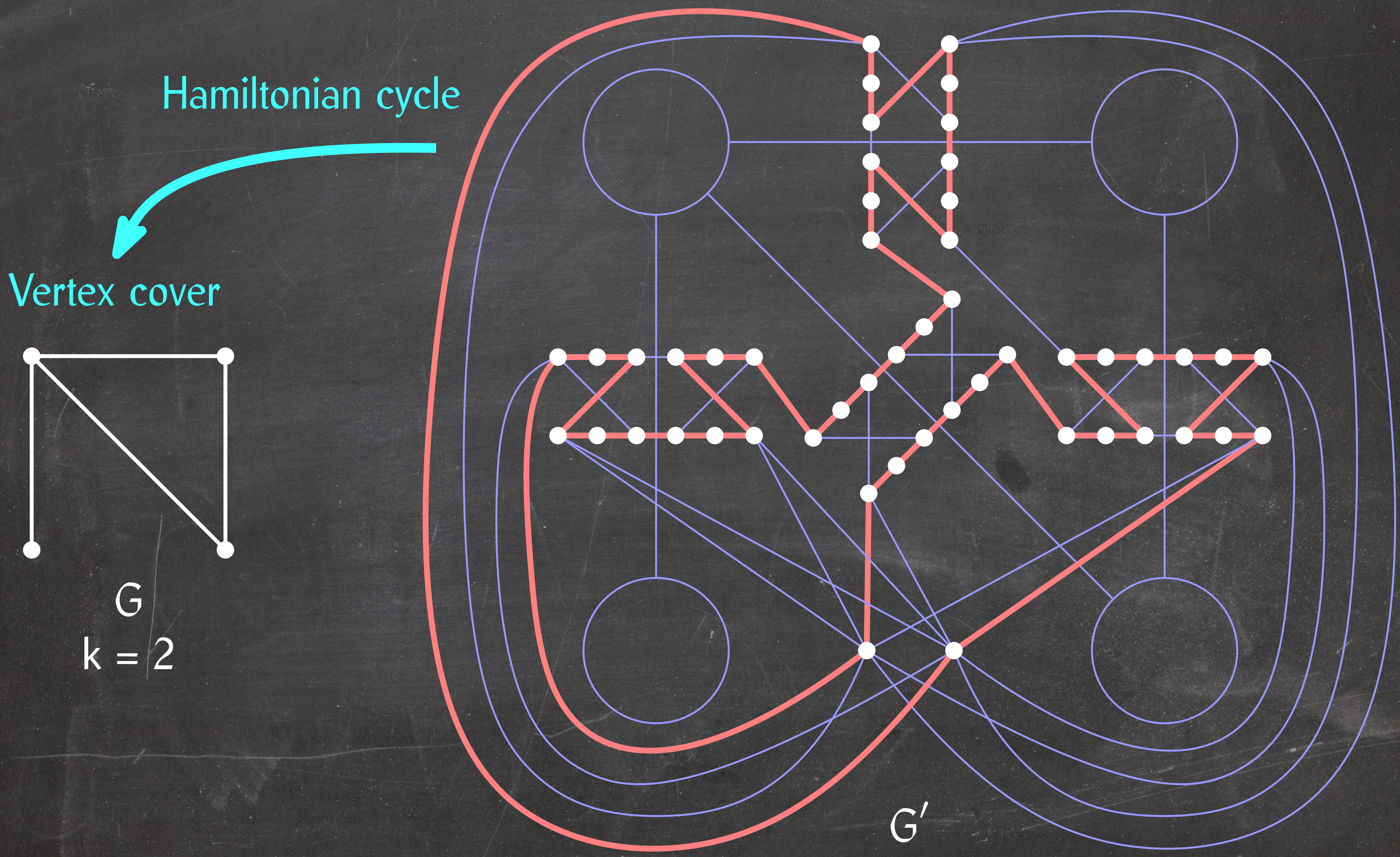

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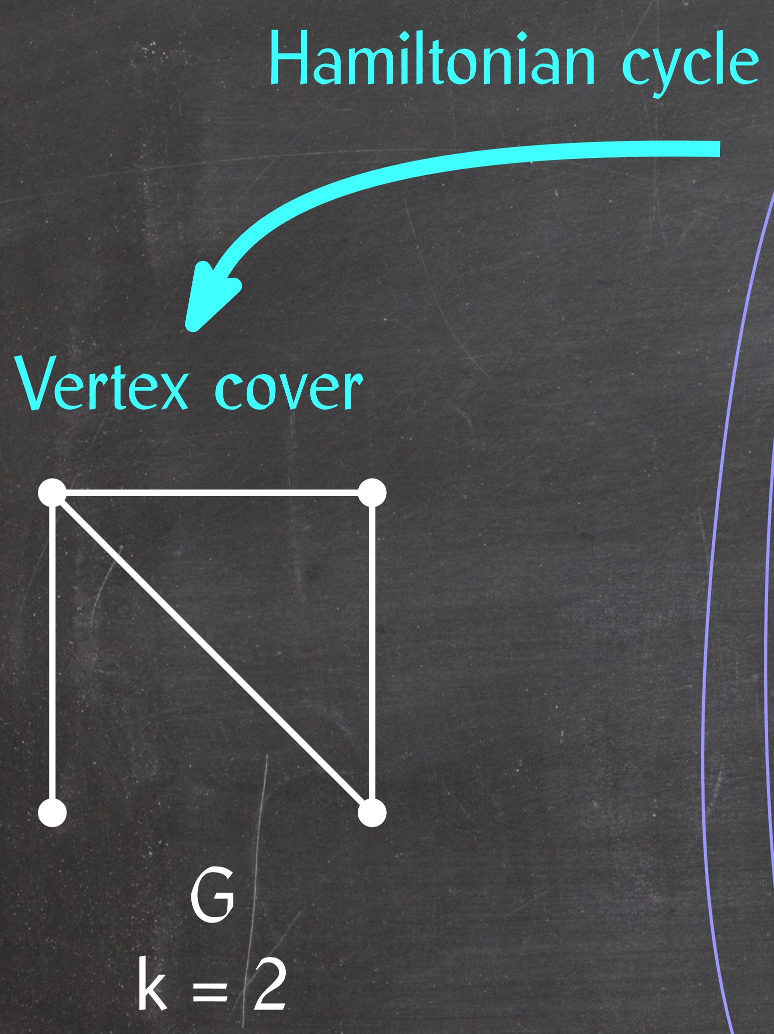

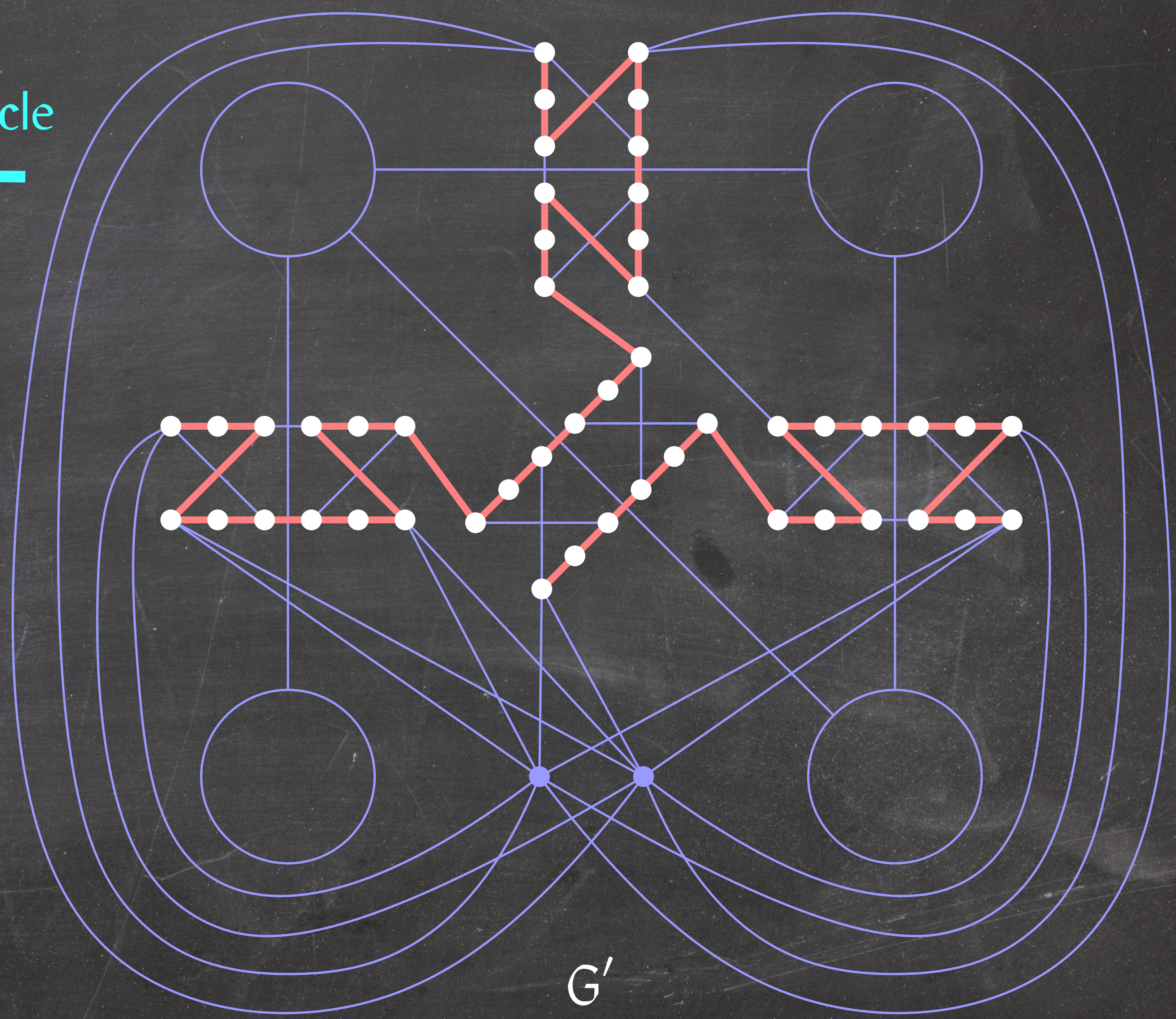

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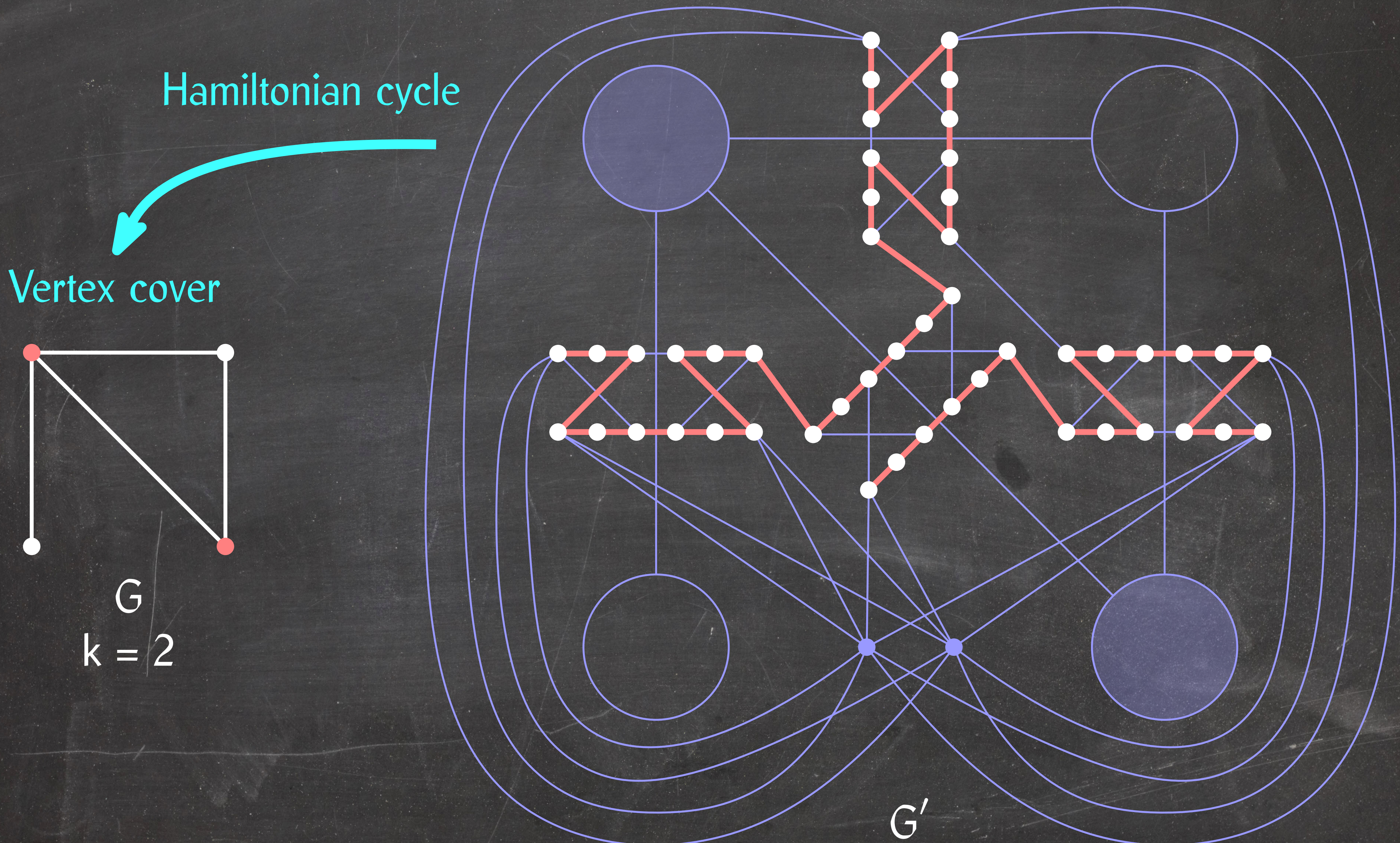

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Subset Sum

Given:

- A set $S = \{x_1, x_2, \ldots, x_n\}$ of distinct numbers
- A parameter t

Question:

Is there a subset $S' \subseteq S$ such that $\sum_{x \in S'} x = t$?

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Question:

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Example:

$$S = \{1, 2, 8, 13\}$$

S has a subset S' whose elements sum to 22, namely $S' = \{1, 8, 13\}$, but there is no subset whose elements sum to 12.

Subset Sum is NP-Complete

Exercise: Verify that Subset Sum is in NP.

Subset Sum is NP-Complete

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To prove: Subset Sum is NP-hard.

Subset Sum is NP-Complete

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To prove: Subset Sum is NP-hard.

Reduction from 3-SAT:

Given a formula F in 3-CNF, we construct a set S_F of $2n + 2m$ numbers with $n + m$ digits in base-10 notation and a number $t = \sum_{i=0}^{n-1} 10^{i+m} + \sum_{i=0}^{m-1} 4 \cdot 10^i$:

$$\underbrace{11 \cdots 1}_{\text{n variable digits}} \quad \underbrace{44 \cdots 4}_{\text{m clause digits}}$$

Subset Sum is NP-Complete

Exercise: Verify that Subset Sum is in NP.

To prove: Subset Sum is NP-hard.

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$$\underbrace{11 \cdots 1}_{n \text{ variable digits}} \quad \underbrace{44 \cdots 4}_{m \text{ clause digits}}$$

There will be a subset $S' \subseteq S_F$ such that $\sum_{x \in S'} x = t$ if and only if F is satisfiable.

Subset Sum is NP-Complete

$$F = (x_1 \lor x_2 \lor \bar{x}_3) \land (\bar{x}_1 \lor x_3 \lor x_4) \land (\bar{x}_2 \lor x_3 \lor \bar{x}_4) \land (x_1 \lor \bar{x}_3 \lor x_4)$$

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$$\text{Literal numbers} \left\{ \begin{array}{l} x_1 \\ \bar{x}_1 \\ x_2 \\ \bar{x}_2 \\ x_3 \\ \bar{x}_3 \\ x_4 \\ \bar{x}_4 \end{array} \right.$$

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t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1							
	$\bar{x}_1$	1							
	x_2								
	$\bar{x}_2$								
	x_3								
	$\bar{x}_3$								
	x_4								
	$\bar{x}_4$								
Slack numbers	s_1								
	s_1'								
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
	s_4'								
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Literal numbers	x_1	1							
	$\bar{x}_1$	1							
	x_2		1						
	$\bar{x}_2$		1						
	x_3			1					
	$\bar{x}_3$			1					
	x_4				1				
	$\bar{x}_4$				1				
Slack numbers	s_1								
	s_1'								
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
	s_4'								
	t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1							
	x_2		1						
	$\bar{x}_2$		1						
	x_3			1					
	$\bar{x}_3$			1					
	x_4				1				
	$\bar{x}_4$				1				
Slack numbers	s_1								
	s_1'								
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
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	t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1						
	$\bar{x}_2$		1						
	x_3			1					
	$\bar{x}_3$			1					
	x_4				1				
	$\bar{x}_4$				1				
Slack numbers	s_1								
	s_1'								
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
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	t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers	s_1								
	s_1'								
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
	s_4'								
	t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers	s_1					1			
	s_1'					2			
	s_2								
	s_2'								
	s_3								
	s_3'								
	s_4								
	s_4'								
	t	1	1	1	1	4	4	4	4

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Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers	s_1					1			
	s_1'					2			
	s_2						1		
	s_2'						2		
	s_3							1	
	s_3'							2	
	s_4								1
	s_4'								2
	t	1	1	1	1	4	4	4	4

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$$\begin{array}{lll}
\text{Literal numbers} & \left\{ \begin{array}{l} x_1 \\ \bar{x}_1 \\ x_2 \\ \bar{x}_2 \\ x_3 \\ \bar{x}_3 \\ x_4 \\ \bar{x}_4 \end{array} \right. & \begin{array}{cccc cccc} 1&0&0&0&1&0&0&1 \\ 1&0&0&0&0&1&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&0&0&1&0 \\ 0&0&1&0&0&1&1&0 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&1&0 \end{array} \\
\text{Slack numbers} & \left\{ \begin{array}{l} s_1 \\ s'_1 \\ s_2 \\ s'_2 \\ s_3 \\ s'_3 \\ s_4 \\ s'_4 \end{array} \right. & \begin{array}{cccc cccc} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&2&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&2 \end{array} \\
\hline
 & t & \begin{array}{cccc cccc} 1&1&1&1&4&4&4&4 \end{array}
\end{array}$$

Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Literal numbers	x_1	1	0	0	0	1	0	0	1
	$\bar{x}_1$	1	0	0	0	0	1	0	0
	x_2	0	1	0	0	1	0	0	0
	$\bar{x}_2$	0	1	0	0	0	0	1	0
	x_3	0	0	1	0	0	1	1	0
	$\bar{x}_3$	0	0	1	0	1	0	0	1
	x_4	0	0	0	1	0	1	0	1
	$\bar{x}_4$	0	0	0	1	0	0	1	0
Slack numbers	s_1	0	0	0	0	1	0	0	0
	s_1'	0	0	0	0	2	0	0	0
	s_2	0	0	0	0	0	1	0	0
	s_2'	0	0	0	0	0	2	0	0
	s_3	0	0	0	0	0	0	1	0
	s_3'	0	0	0	0	0	0	2	0
	s_4	0	0	0	0	0	0	0	1
	s_4'	0	0	0	0	0	0	0	2
	t	1	1	1	1	4	4	4	4

Lemma: F is satisfiable if and only if there is a subset $S' \subseteq S_F$ such that $\sum_{x \in S'} x = t$.

Subset Sum is NP-Complete

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Truth assignment

$\downarrow$

Subset S'

Literal numbers	x_1	1 0 0 0	1 0 0 1
	$\bar{x}_1$	1 0 0 0	0 1 0 0
	x_2	0 1 0 0	1 0 0 0
	$\bar{x}_2$	0 1 0 0	0 0 1 0
	x_3	0 0 1 0	0 1 1 0
	$\bar{x}_3$	0 0 1 0	1 0 0 1
	x_4	0 0 0 1	0 1 0 1
	$\bar{x}_4$	0 0 0 1	0 0 1 0
Slack numbers	s_1	0 0 0 0	1 0 0 0
	s_1'	0 0 0 0	2 0 0 0
	s_2	0 0 0 0	0 1 0 0
	s_2'	0 0 0 0	0 2 0 0
	s_3	0 0 0 0	0 0 1 0
	s_3'	0 0 0 0	0 0 2 0
	s_4	0 0 0 0	0 0 0 1
	s_4'	0 0 0 0	0 0 0 2
	t	1 1 1 1	4 4 4 4

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$x_1 = x_2 = x_3 = x_4 = \text{true}$

Truth assignment

Subset S'

Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers									
	t	1	1	1	1	4	4	4	4

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Truth assignment

$\downarrow$

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers									
	t	1	1	1	1	4	4	4	4

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Truth assignment

↓

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers									
	t	1	1	1	1	4	4	4	4

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$x_1 = x_2 = x_3 = x_4 =$ true

Truth assignment

↓

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers									
	t	1	1	1	1	2	2	1	2

Lemma: F is satisfiable if and only if there is a subset $S' \subseteq S_F$ such that $\sum_{x \in S'} x = t$.

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$x_1 = x_2 = x_3 = x_4 = \text{true}$

Truth assignment

$\downarrow$

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers	s'_1					2			
	t	1	1	1	1	4	2	1	2

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$x_1 = x_2 = x_3 = x_4 = \text{true}$

Truth assignment

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers	s_1'					2			
	s_2'						2		
	s_3							1	
	s_3'							2	
	s_4'								2
	t	1	1	1	1	4	4	4	4

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Truth assignment

$\uparrow$

Subset S'

Literal numbers	x_1	1				1			1
	$\bar{x}_1$	1					1		
	x_2		1			1			
	$\bar{x}_2$		1					1	
	x_3			1			1	1	
	$\bar{x}_3$			1		1			1
	x_4				1		1		1
	$\bar{x}_4$				1			1	
Slack numbers	s_1					1			
	s_1'					2			
	s_2						1		
	s_2'						2		
	s_3							1	
	s_3'							2	
	s_4								1
	s_4'								2
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Lemma: F is satisfiable if and only if there is a subset $S' \subseteq S_F$ such that $\sum_{x \in S'} x = t$.

Subset Sum is NP-Complete

$$F = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_1 \vee x_3 \vee x_4) \wedge (\bar{x}_2 \vee x_3 \vee \bar{x}_4) \wedge (x_1 \vee \bar{x}_3 \vee x_4)$$

Truth assignment

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers	s_1'					2			
	s_2'						2		
	s_3							1	
	s_3'							2	
	s_4'								2
	t	1	1	1	1	4	4	4	4

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$x_1 = x_2 = x_3 = x_4 = \text{true}$

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Slack numbers	s_1'					2			
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	s_3							1	
	s_3'							2	
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Truth assignment

↑

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
	x_4				1		1		1
Slack numbers									
	t	1	1	1	1	2	2	1	2

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Truth assignment

Subset S'

Literal numbers	x_1	1				1			1
	x_2		1			1			
	x_3			1			1	1	
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Slack numbers									
	t	1	1	1	1	2	2	1	2

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Summary

Many important problems are NP-hard or NP-complete.

Examples:

- Satisfiability
- Vertex cover
- Subset sum
- Hamiltonian cycle
- Clique
- Independent set
- …

These problems are unlikely to be solvable in polynomial time.

Techniques to cope with NP-hardness:

- Parameterized algorithms
- Approximation algorithms
- Heuristics