REGULAR LANGUAGES, FINITE AUTOMATA, AND LEXICAL ANALYSIS

PRINCIPLES OF PROGRAMMING LANGUAGES

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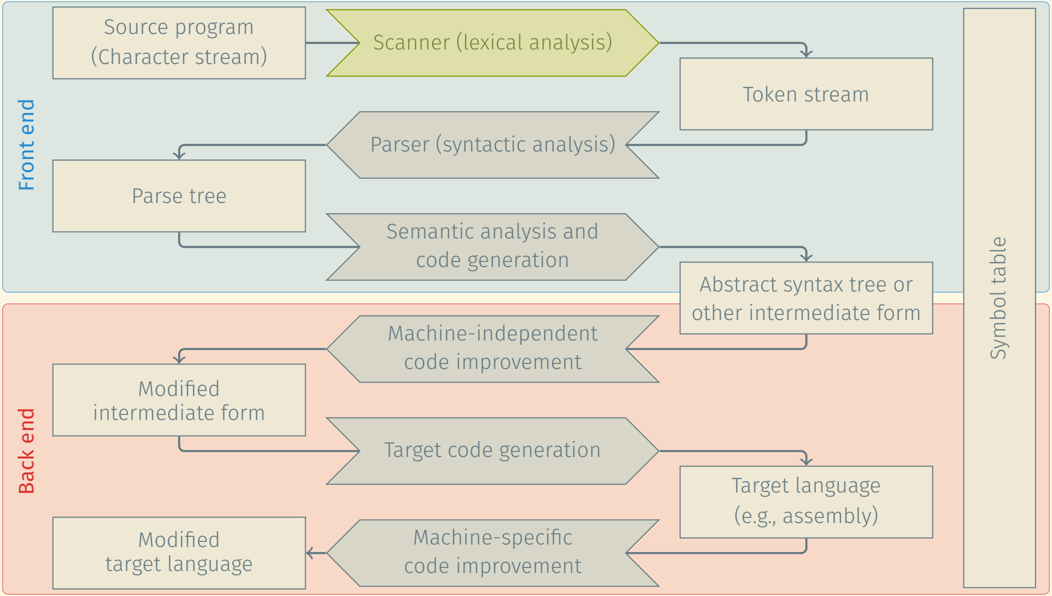
Dalhousie University

PROGRAM TRANSLATION FLOW CHART



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Transform input text into much more compact token stream (keywords, parentheses, operators, identifiers, ...).

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    [int] [key] [;] // The key
    [float] [value] [;] // The associated value
[}] [;]
```

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class DictEntry {
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    float value ; // The associated value
} ;
```

```
kwClass identifier '{'
    identifier identifier ';'
    identifier identifier ';'
'}' ';'
```

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Tools

- Regular expressions
- **Finite automata:** Very simple and efficient machines just powerful enough to carry out lexical analysis

ROAD MAP

- Regular languages
- Regular expressions
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA
- Building a scanner
  - Regular expression $\longrightarrow$ NFA $\longrightarrow$ DFA
  - Minimizing the DFA
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  - $A^0 = \{\epsilon\}$
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This is the only way to produce infinite regular languages!

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Interpretation:

- Precedence: Kleene star (*), Concatenation, Union (|)
- Parentheses indicate grouping

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REGULAR EXPRESSIONS MODEL REGULAR LANGUAGES

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Some capabilities beyond regular languages:

- Allow, for example, recognition of languages such as $\alpha\beta\alpha$, for $\alpha, \beta \in \Sigma^*$.

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- **Identifier in C:** $(a|b|\cdots|z|A|B|\cdots|Z|\_)(a|b|\cdots|z|A|B|\cdots|Z|0|1|\cdots|9|\_)*$
  $\rightarrow$ [a–zA–Z_][a–zA–Z0–9_]*
- **Anything but a lowercase letter:** [^a–z]
- **Any letter:** .
- **Digit, non-digit:** \d, \D
- **Whitespace, non-whitespace:** \s, \S
- **Word character, non-word character:** \w, \W

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ROAD MAP

- Regular languages
- Regular expressions
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA
- Building a scanner
  - Regular expression $\rightarrow$ NFA $\rightarrow$ DFA
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DETERMINISTIC FINITE AUTOMATON (DFA)

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Definition:
Deterministic finite automaton (DFA)

A tuple $D = (S, \Sigma, \delta, s_0, F)$:

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S	s_1	
	$*s_2$	

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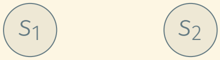

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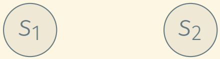
Σ

	0	1
s_1		
$*s_2$		

Graph representation:



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δ	0	1
s_1	s_1	s_2
$*s_2$	s_2	s_1

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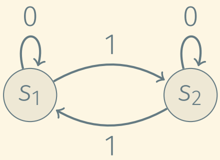

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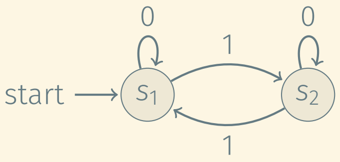

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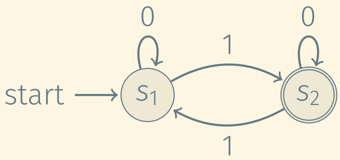

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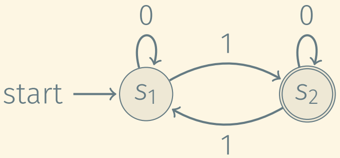

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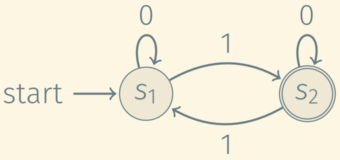

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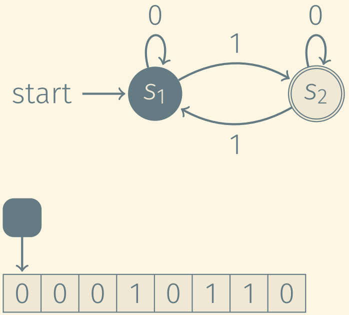

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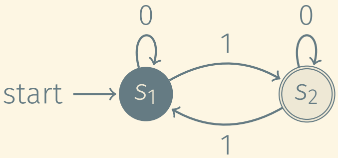

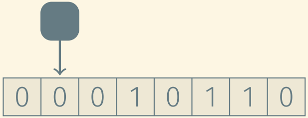

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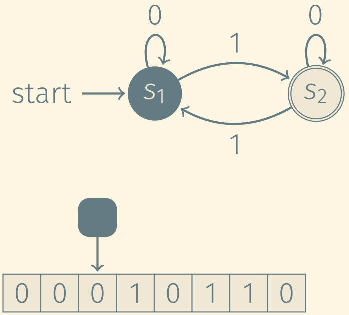

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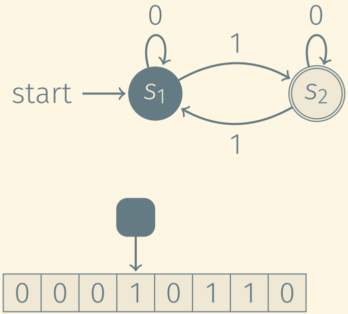

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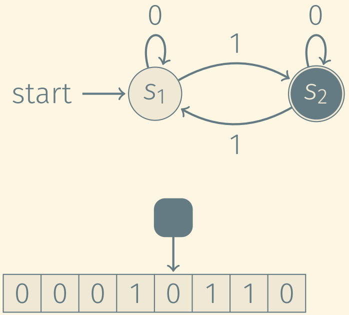

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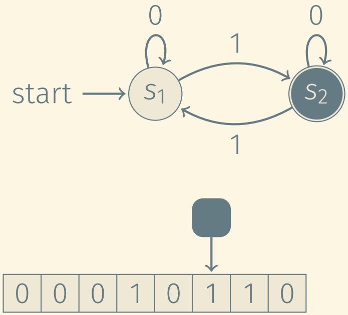

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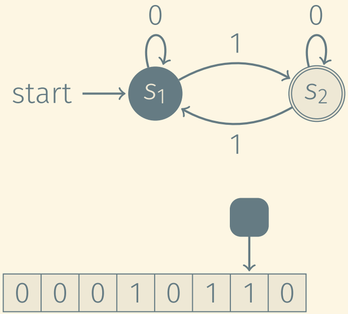

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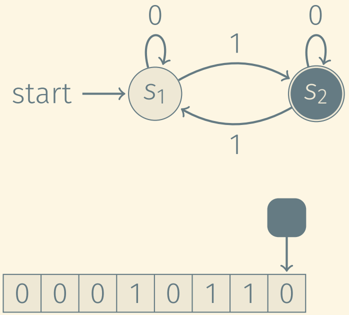

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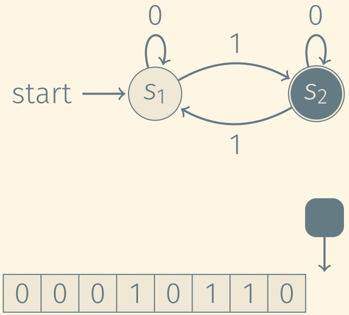

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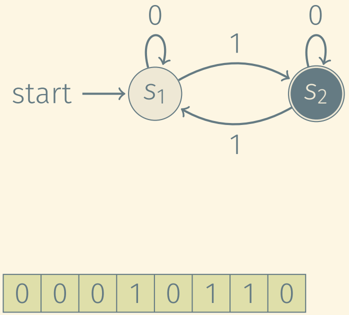

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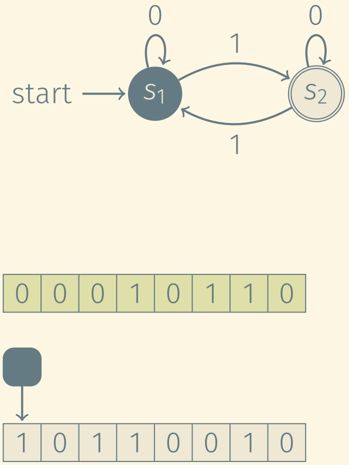

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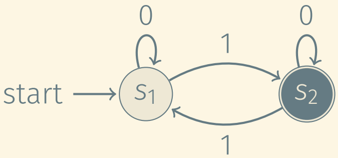

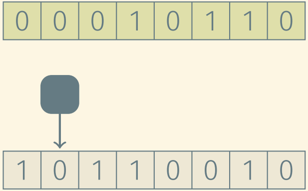

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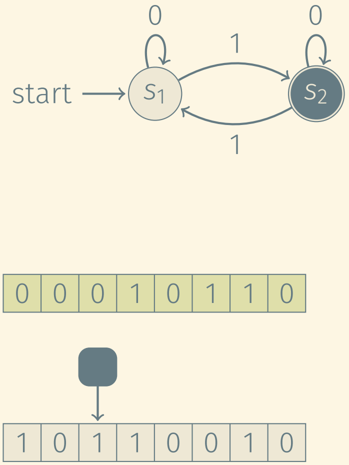

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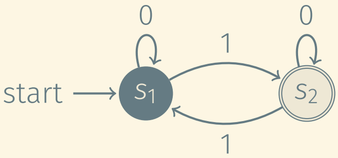

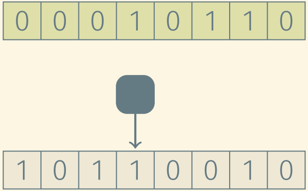

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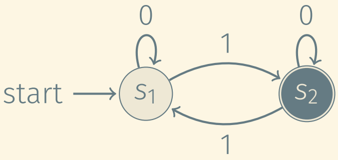

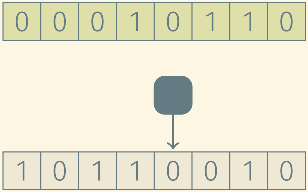

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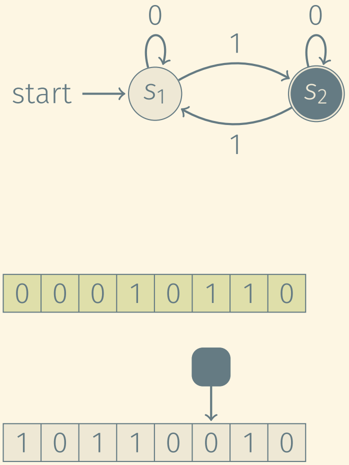

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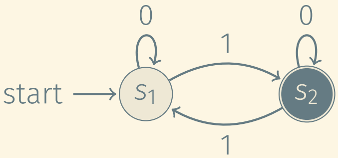

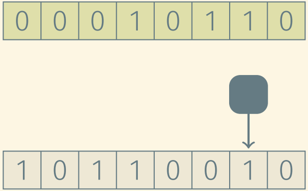

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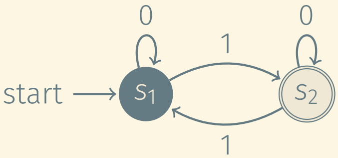

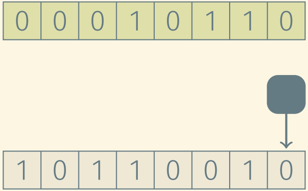

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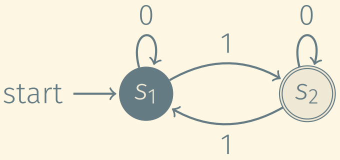

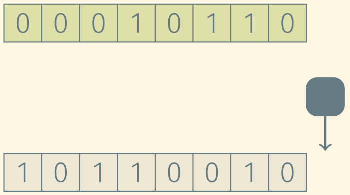

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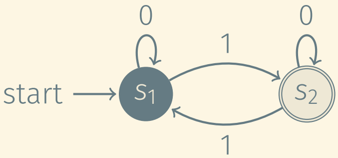

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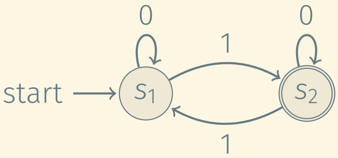

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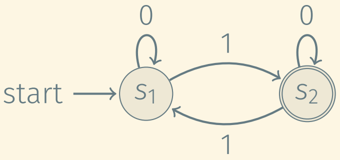

0	0	0	1	0	1	1	0

1	0	1	1	0	0	1	0

This DFA decides the language of all binary strings with ...

THE LANGUAGE DECIDED BY A DFA (1)

Acceptance/rejection of a string: Intuition

A DFA $D = (S, \Sigma, \delta, s_0, F)$ accepts a string $\sigma \in \Sigma^*$ if, after starting in state s_0 and reading σ, it finishes in an accepting state. Otherwise, it rejects σ.

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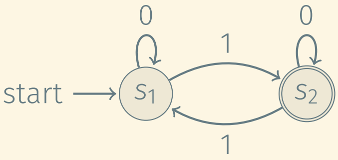

0	0	0	1	0	1	1	0

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This DFA decides the language of all binary strings with an odd number of 1s.

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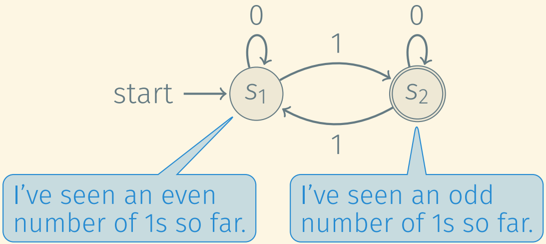

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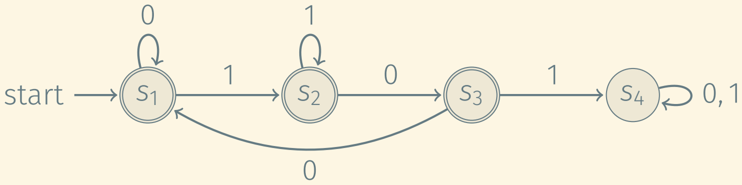
Language decided by a DFA

For a DFA $D = (S, \Sigma, \delta, s_0, F)$,

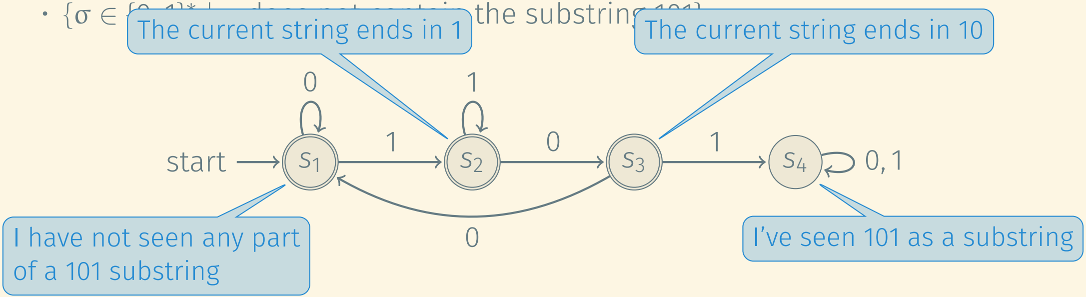
$$\mathcal{L}(D) = \{\sigma \in \Sigma^* \mid \delta^*(s_0, \sigma) \in F\}.$$

EXAMPLES OF DFA

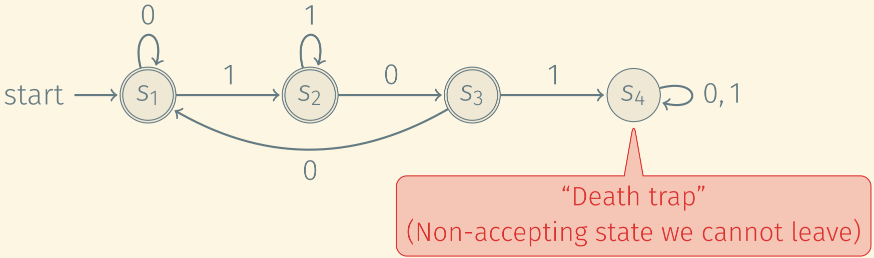
- $\{\sigma \in \{0,1\}^* \mid \sigma \text{ does not contain the substring } 101\}$



- $\{\sigma \in \{0,1\}^* \mid \sigma$ does not contain the substring $101\}$



- $\{\sigma \in \{0,1\}^* \mid \sigma$ does not contain the substring 101$\}$



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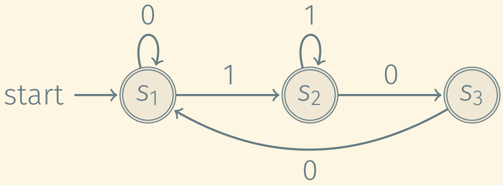

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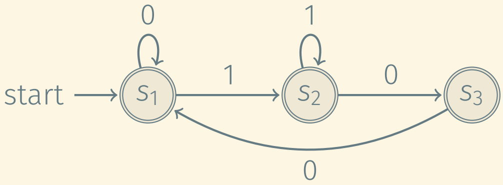
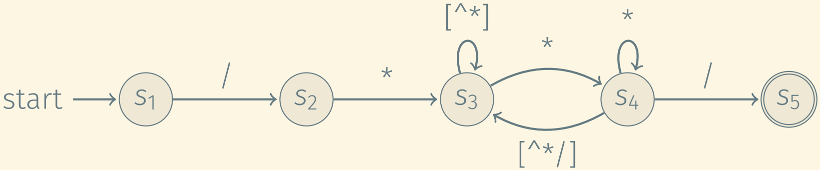


- Valid C comments (/*...*/)



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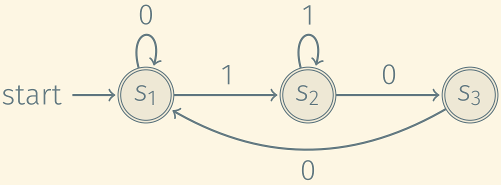

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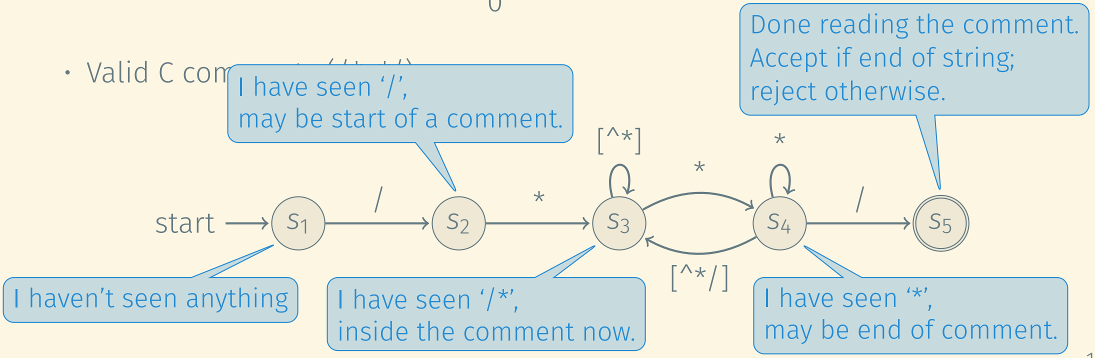

ROAD MAP

- Regular languages
- Regular expressions
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA
- Building a scanner
  - Regular expression $\longrightarrow$ NFA $\longrightarrow$ DFA
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NON-DETERMINISTIC FINITE AUTOMATON (NFA) (1)

For a DFA $D = (S, \Sigma, \delta, s_0, F)$, reading a string σ puts the DFA into a unique state $\delta^*(s_0, \sigma)$. That's why it's called "deterministic".

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An NFA $N = (S, \Sigma, \delta, s_0, F)$ accepts σ if $\delta^*(s_0, \sigma) \cap F \neq \emptyset$.
(N has the ability to reach an accepting state while reading σ, assuming it makes the right choices.)

NON-DETERMINISTIC FINITE AUTOMATON (NFA) (2)

Definition:
Non-deterministic finite automaton (NFA)

A tuple $D = (S, \Sigma, \delta, s_0, F)$:

- Set of states S
- Finite alphabet Σ
- Transition function $\delta : S \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^S$
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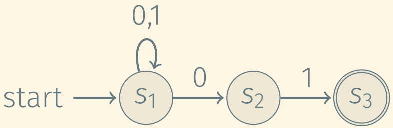
δ	0	1	ϵ
$\to s_1$	$\{s_1, s_2\}$	$\{s_1\}$	$\emptyset$
s_2	$\emptyset$	$\{s_3\}$	$\emptyset$
$*s_3$	$\emptyset$	$\emptyset$	$\emptyset$



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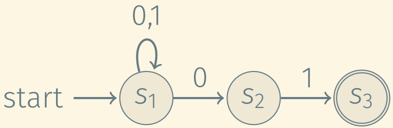
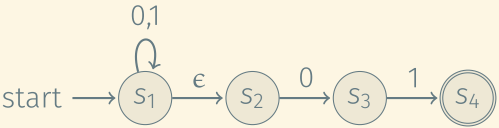
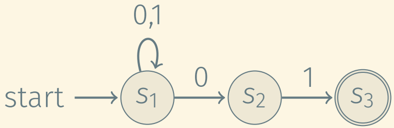
s_3	$\emptyset$	$\{s_4\}$	$\emptyset$
$*s_4$	$\emptyset$	$\emptyset$	$\emptyset$





THE LANGUAGE DECIDED BY AN NFA (1)



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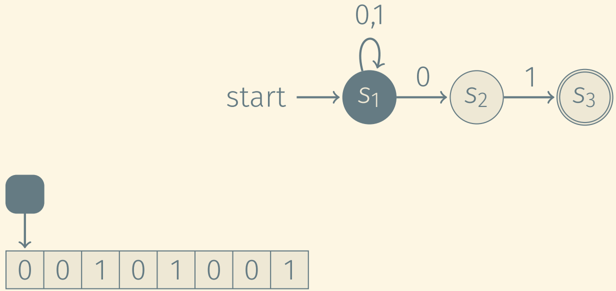

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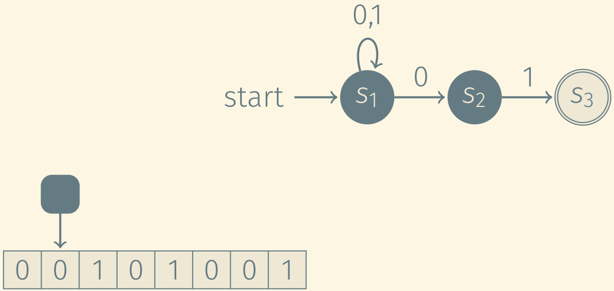
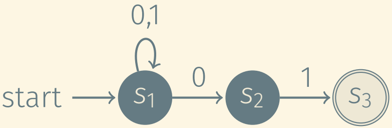
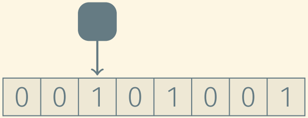



0,1
start
s_1
0
s_2
1
s_3


0 0 1 0 1 0 0 1

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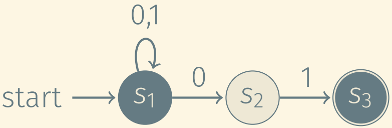

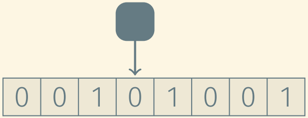

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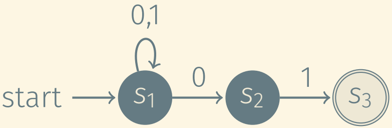

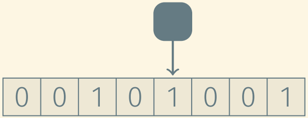

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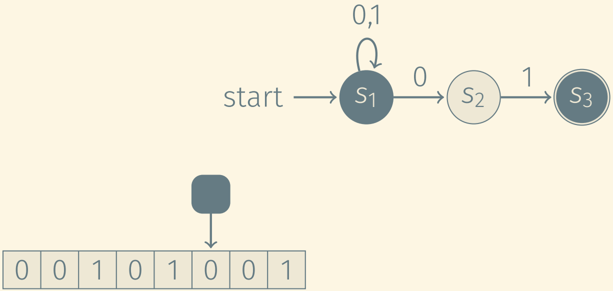

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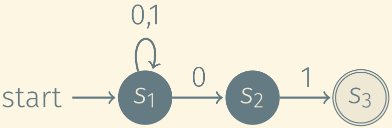

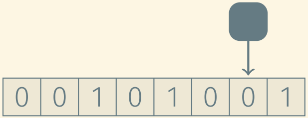

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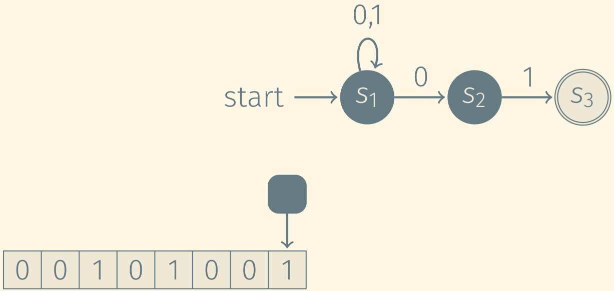

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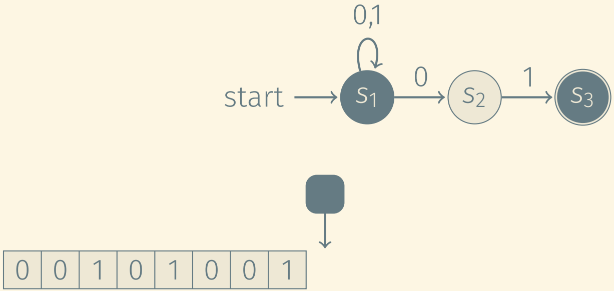

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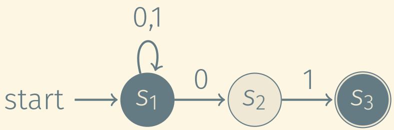

0	0	1	0	1	0	0	1

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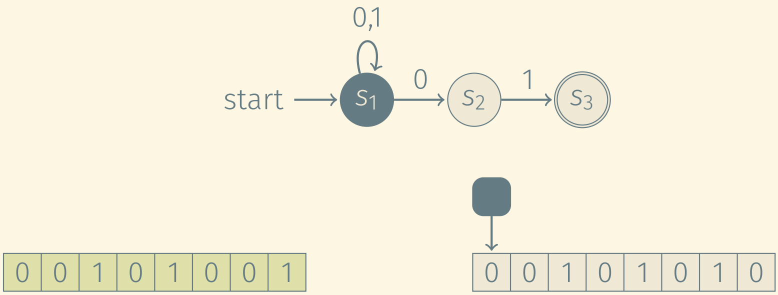

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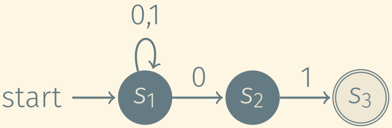

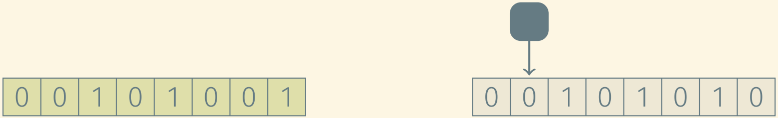

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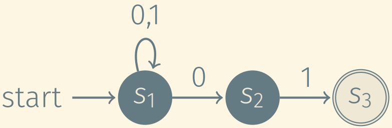

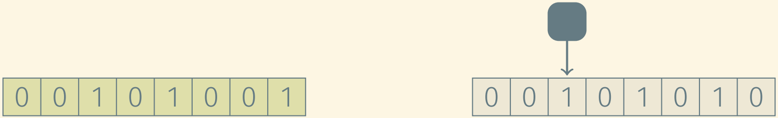

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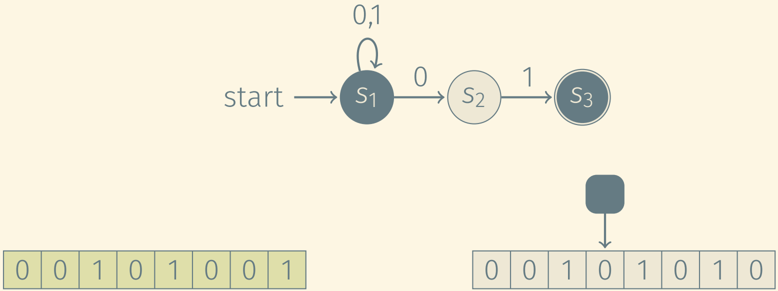

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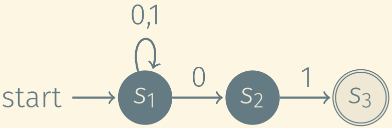

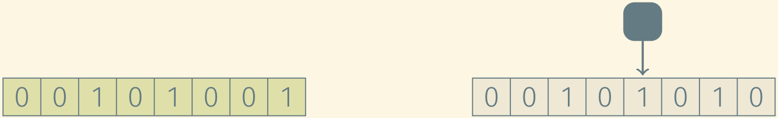

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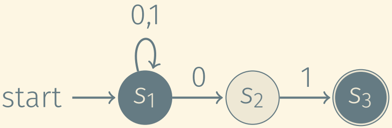

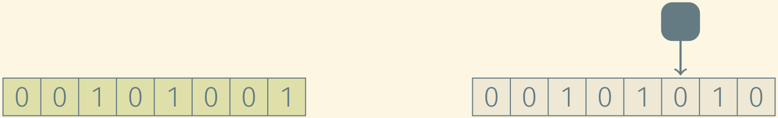

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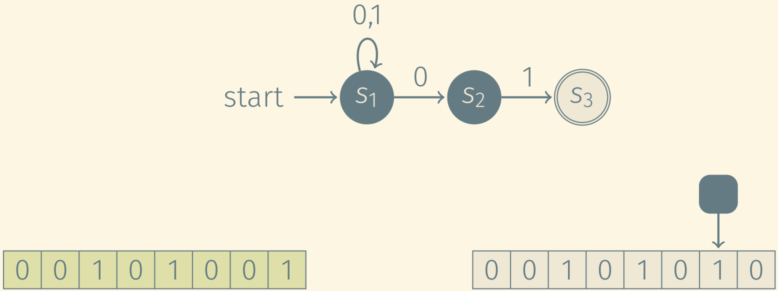

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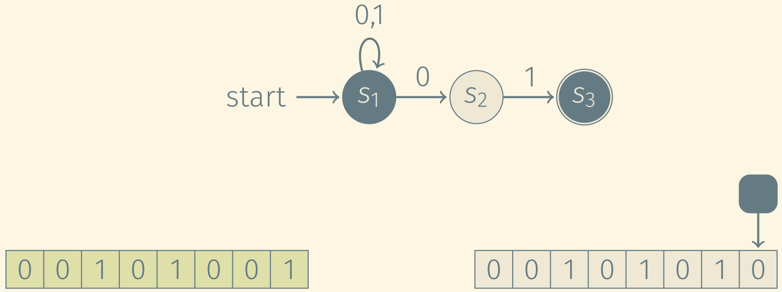

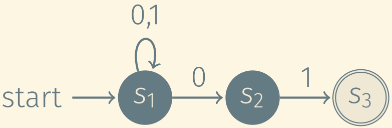
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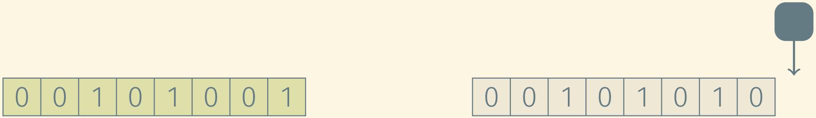
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0 0 1 0 1 0 1 0

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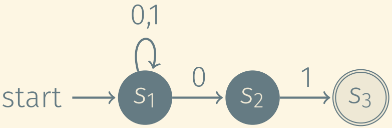

0	0	1	0	1	0	0	1

0	0	1	0	1	0	1	0

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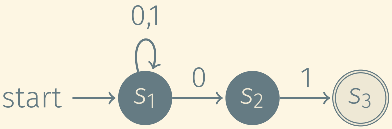
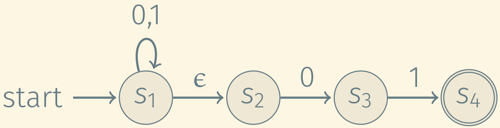


0	0	1	0	1	0	0	1

0	0	1	0	1	0	1	0

$$\mathcal{L}(N) = \mathcal{L}((0|1)*01)$$

THE LANGUAGE DECIDED BY AN NFA (2)



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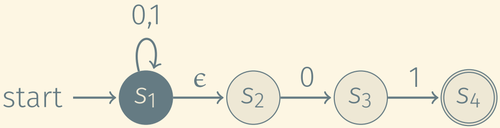

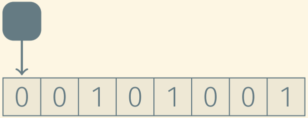

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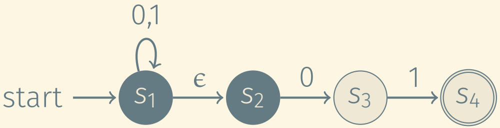

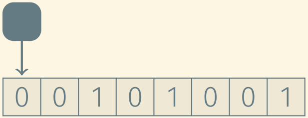

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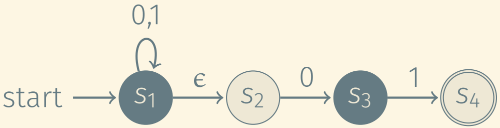

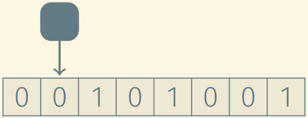

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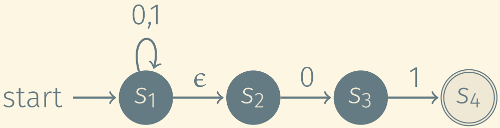

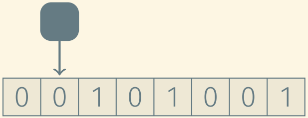

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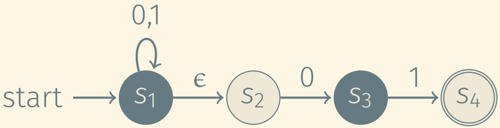

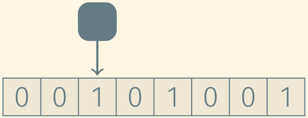

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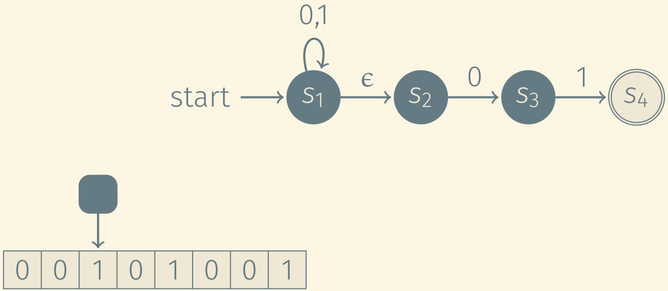

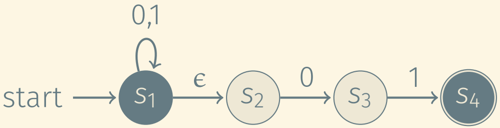
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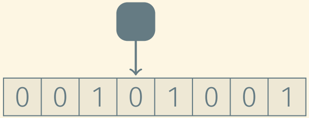
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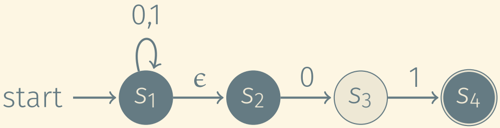

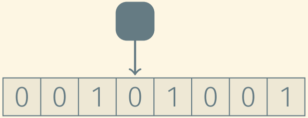

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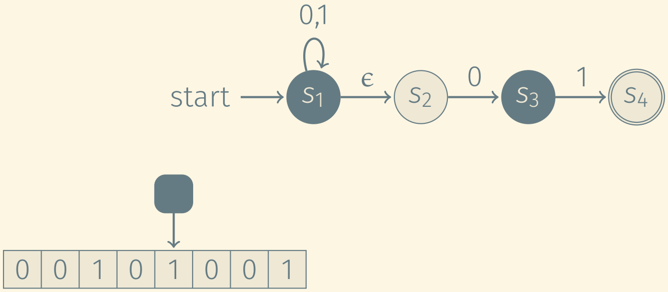

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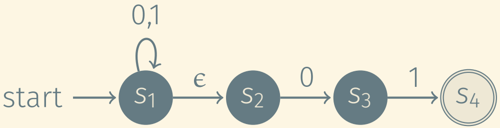

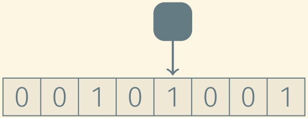

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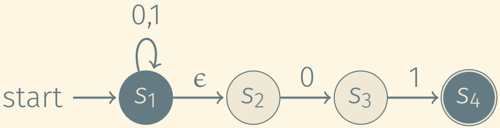

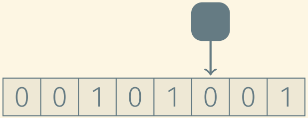

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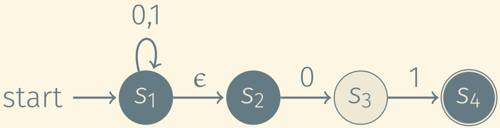

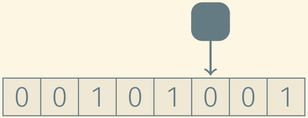

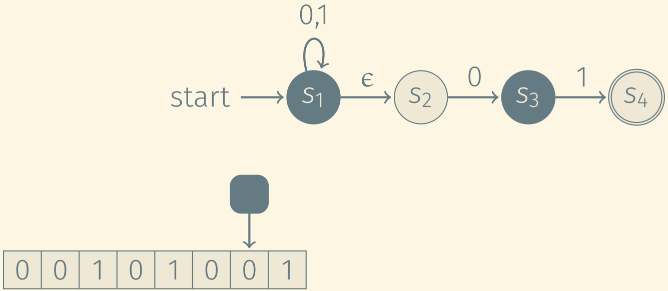
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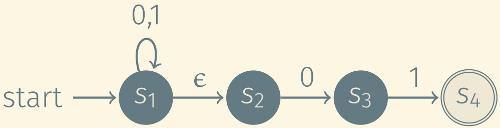

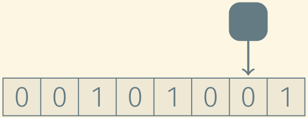

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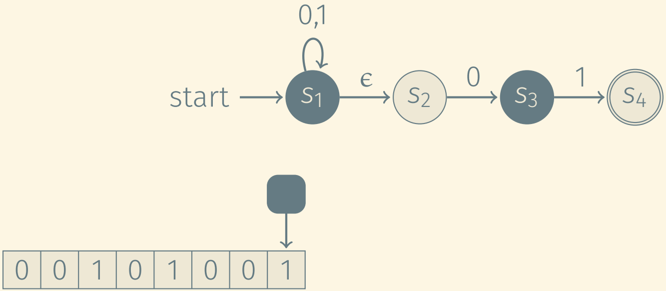

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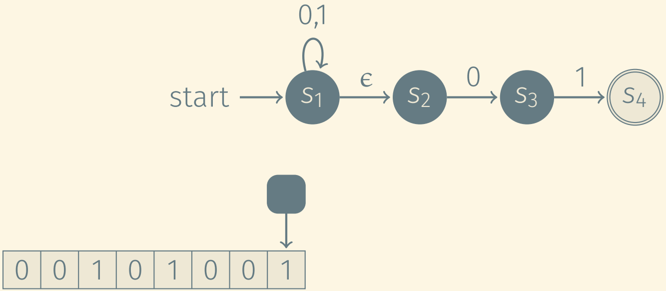

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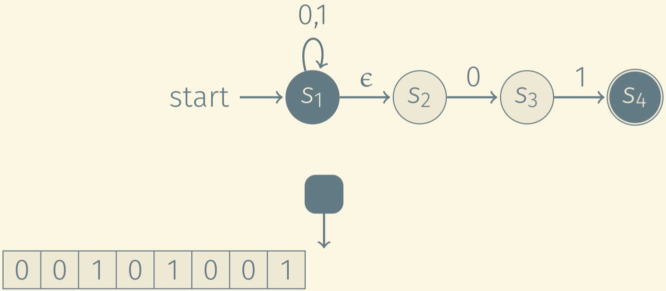

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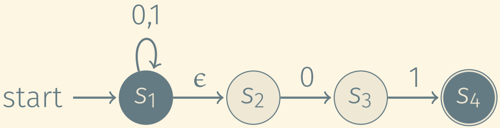

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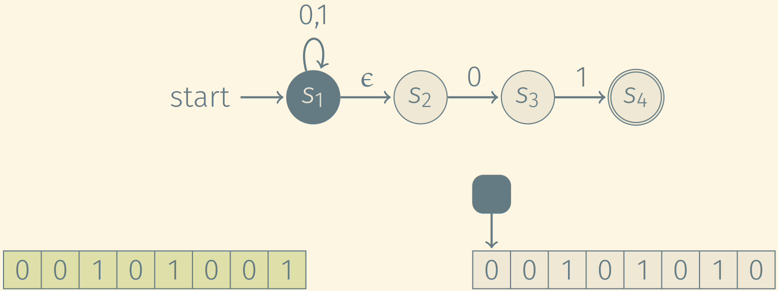

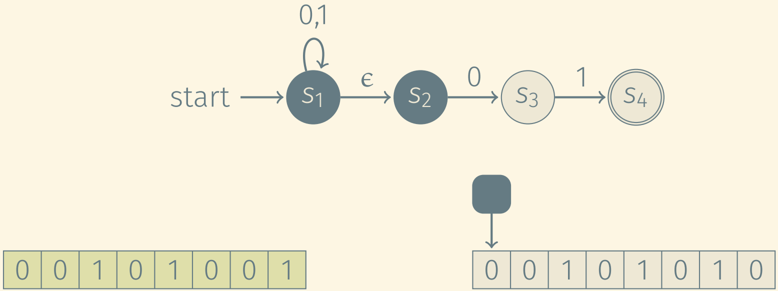

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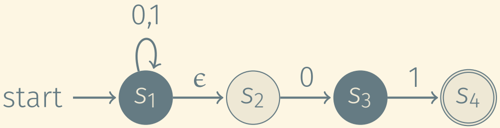

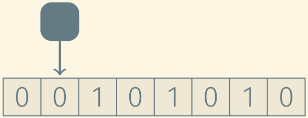

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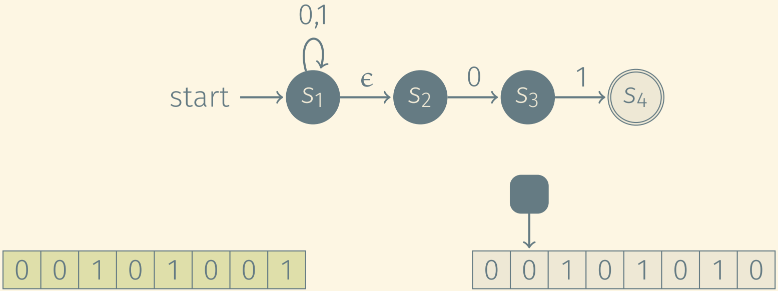

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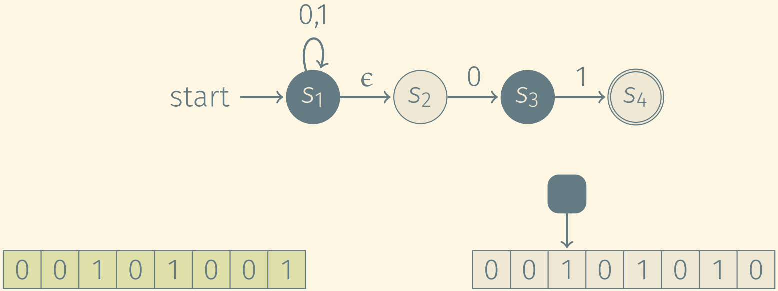

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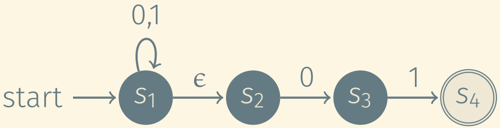

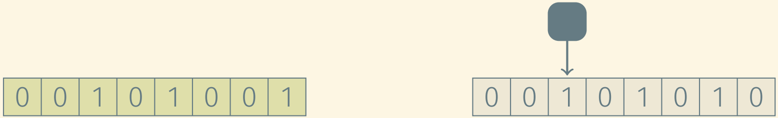

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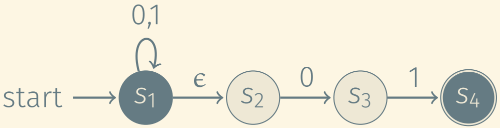

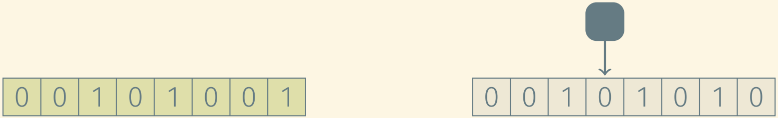

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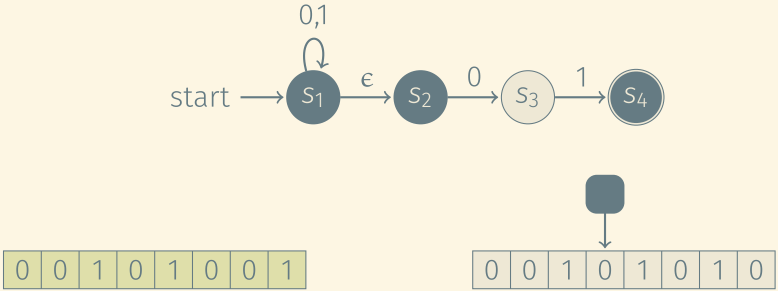

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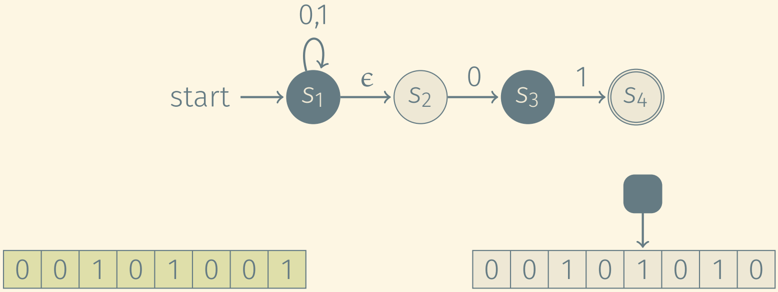

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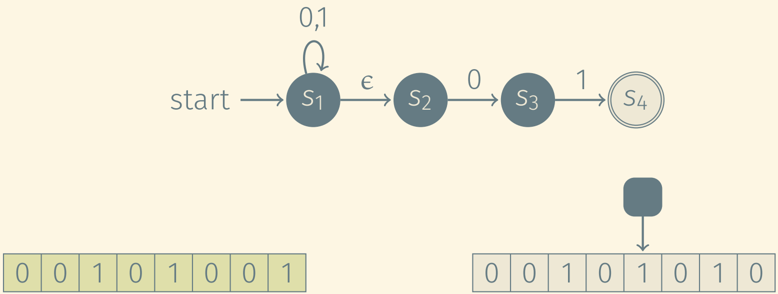

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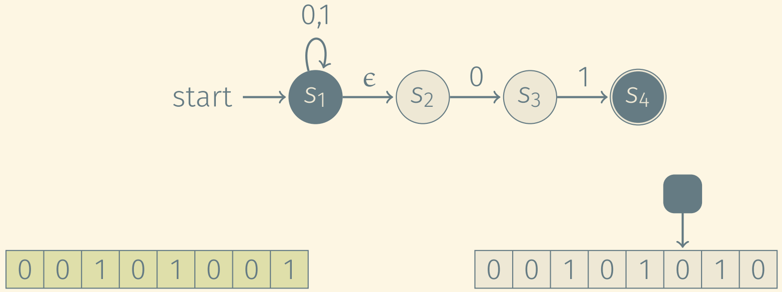

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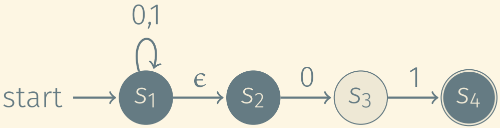

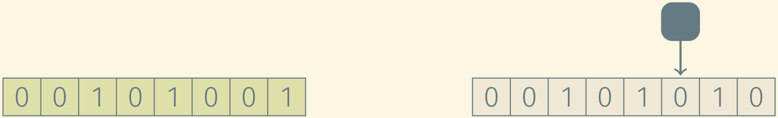

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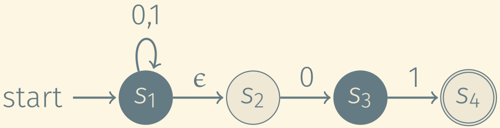

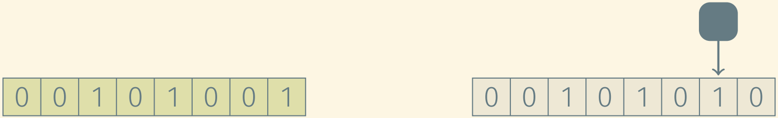

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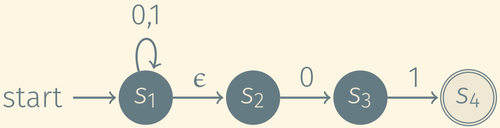

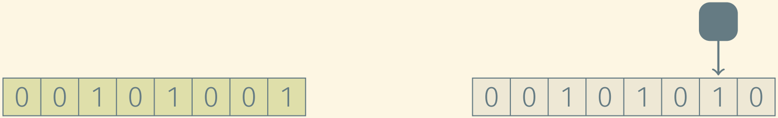

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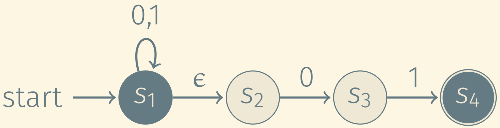

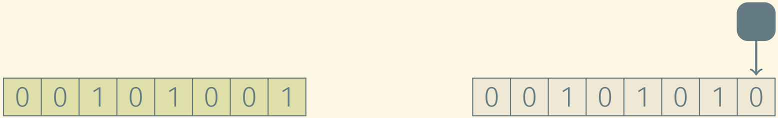

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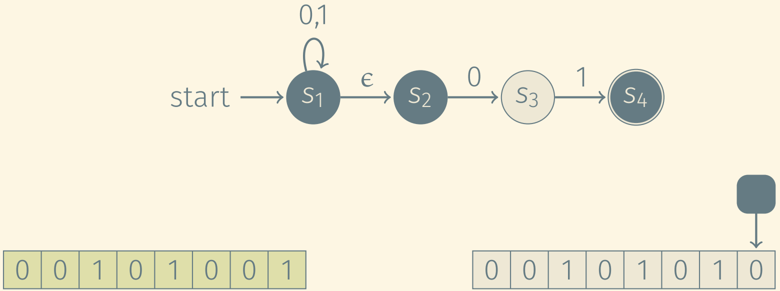

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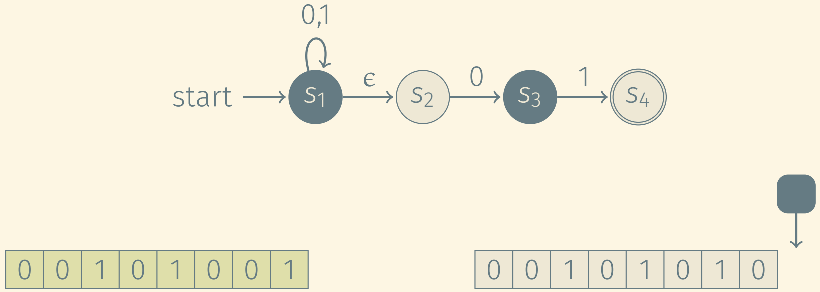

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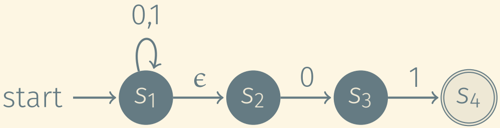

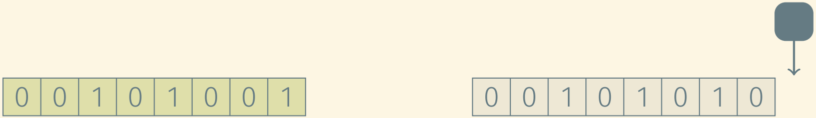

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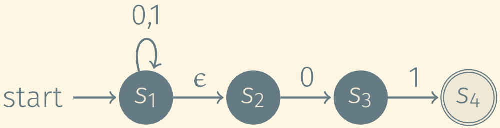

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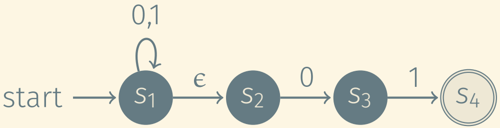

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$$\mathcal{L}(N) = \mathcal{L}((0|1)*01)$$

THE LANGUAGE DECIDED BY AN NFA (3)

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Language decided by a DFA

For a DFA $D = (S, \Sigma, \delta, s_0, F)$,

$$\mathcal{L}(D) = \{\sigma \in \Sigma^* \mid \delta^*(s_0, \sigma) \in F\}.$$

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What should this definition look like for an NFA?

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- What *is* $\delta^*(s_0, \sigma)$?

- How should it be related to F for N to accept σ?

THE LANGUAGE DECIDED BY AN NFA (3)

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What should this definition look like for an NFA?

- What *is* $\delta^*(s_0, \sigma)$?

  The set of states reachable from s_0 by reading σ.

- How should it be related to F for N to accept σ?

Language decided by an NFA

For an NFA $N = (S, \Sigma, \delta, s_0, F)$,

$$\mathcal{L}(N) = \{\sigma \in \Sigma^* \mid \delta^*(s_0, \sigma) \cap F \neq \emptyset\}.$$

What should this definition look like for an NFA?

- What *is* $\delta^*(s_0, \sigma)$?

  The set of states reachable from s_0 by reading σ.

- How should it be related to F for N to accept σ?

ϵ-CLOSURE

Definition: ϵ-Closure

For some subset $S' \subseteq S$ of states, $\text{ECLOSE}(S')$ is the set of all states that can be reached from states in S' using only ϵ-transitions.

Formally, $\text{ECLOSE}(S')$ is the smallest superset $\text{ECLOSE}(S') \supseteq S'$ such that $\delta(s, \epsilon) \subseteq \text{ECLOSE}(S')$ for all $s \in \text{ECLOSE}(S')$.

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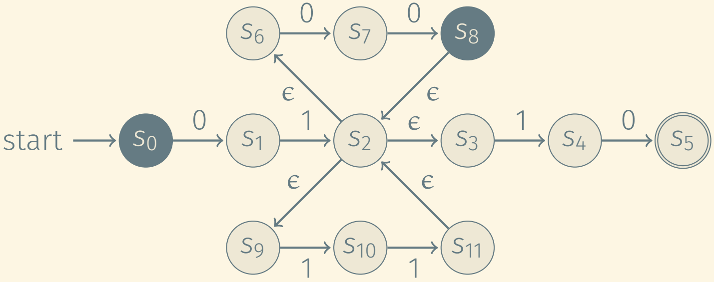

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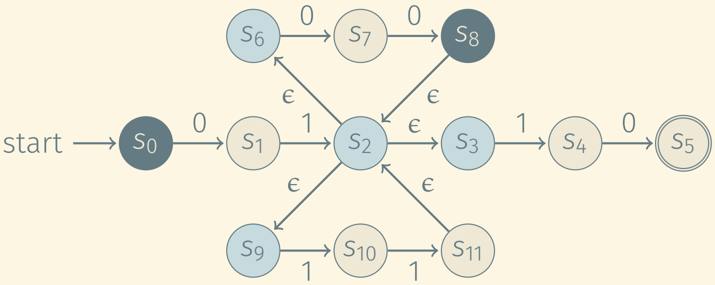

A transition function for strings

$\delta^*(s, \sigma) =$ the set of states reachable from s by reading σ.

THE LANGUAGE DECIDED BY AN NFA (4)

A transition function for strings

$\delta^*(s, \sigma)$ = the set of states reachable from s by reading σ.

- $\delta^*(s, \epsilon) =$

THE LANGUAGE DECIDED BY AN NFA (4)

A transition function for strings

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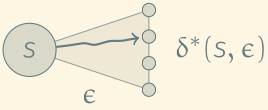

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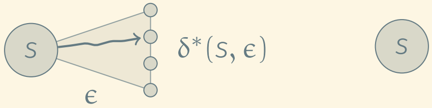

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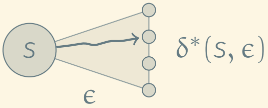

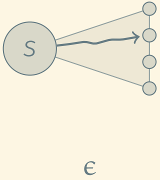

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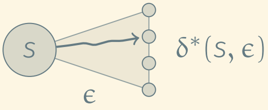

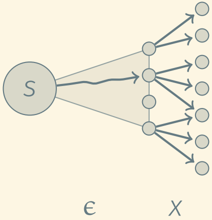

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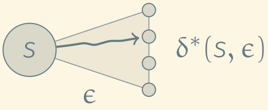

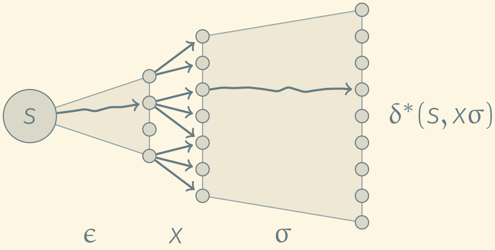

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- Deterministic finite automata (DFA)
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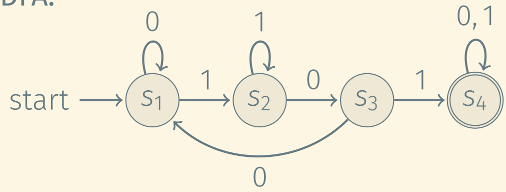

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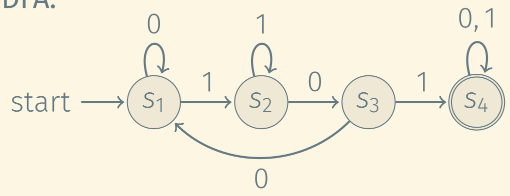

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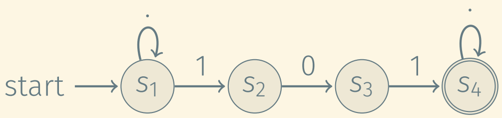

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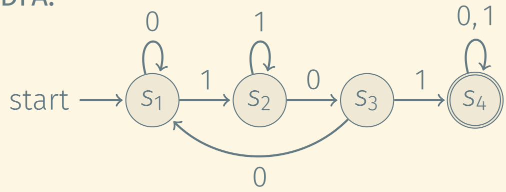

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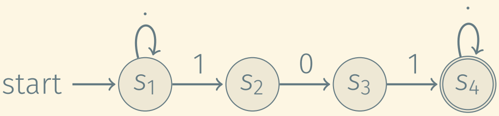

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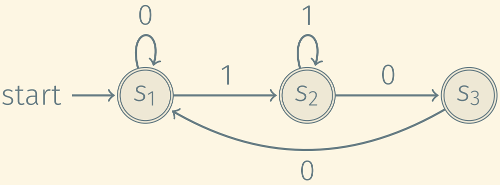

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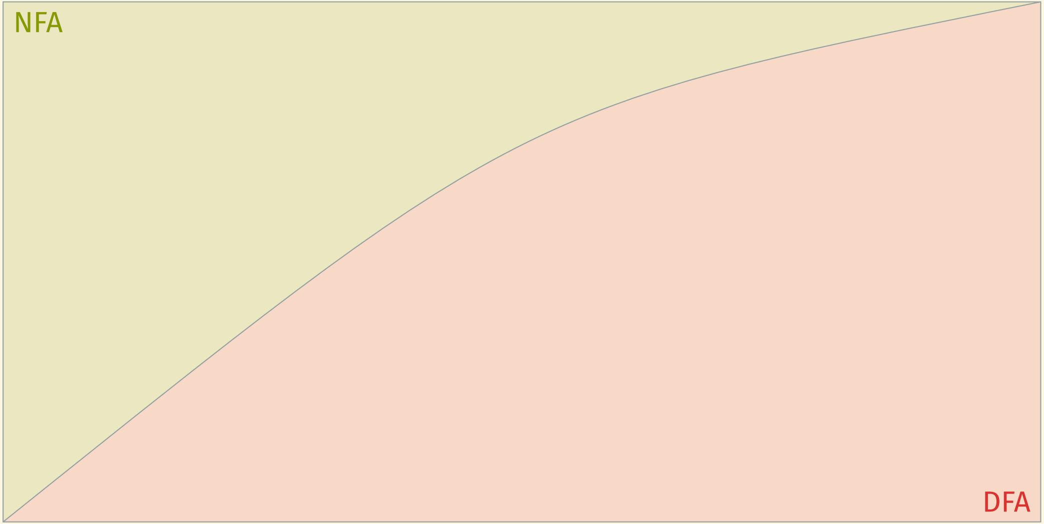

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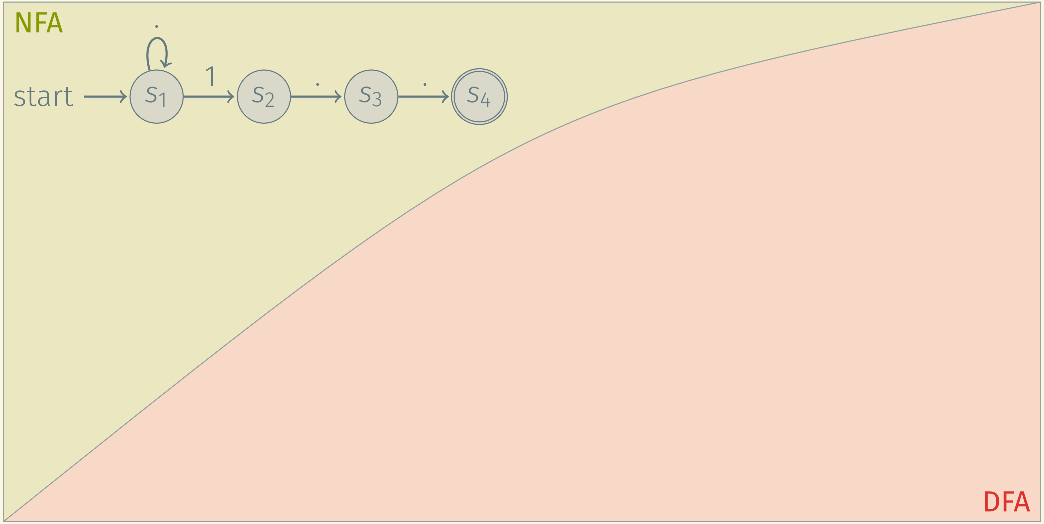

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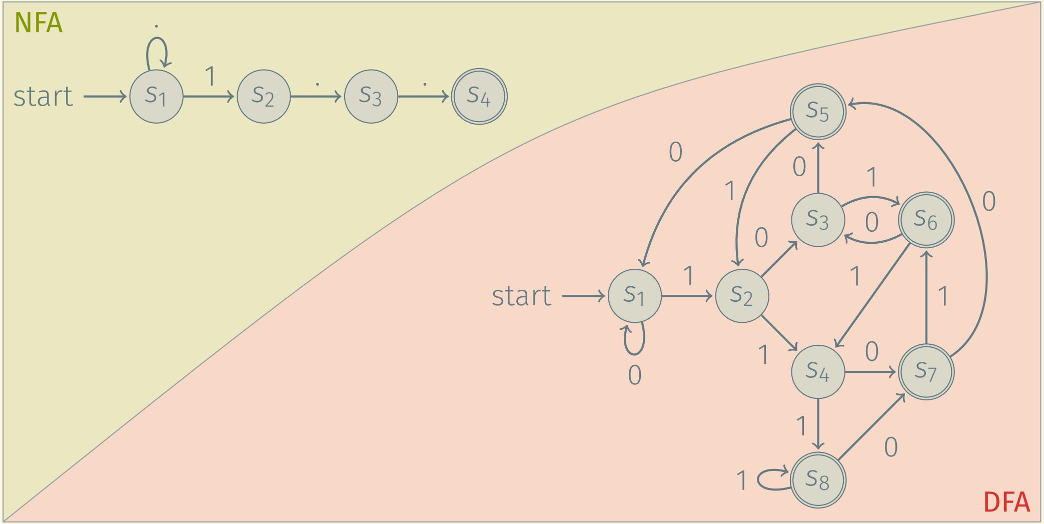

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$\longrightarrow$ We need to construct DFA.

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ARE NFA MORE POWERFUL THAN DFA?

No!

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Theorem

The following statements are equivalent:

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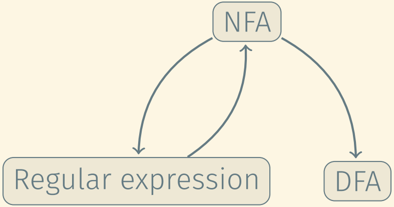

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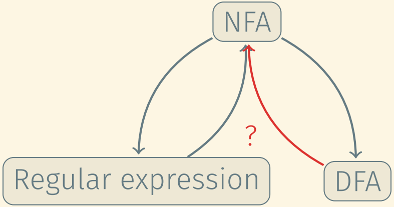

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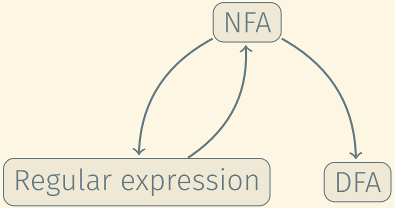

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We want $\gamma^*(t_0, \sigma) = \gamma^*(\mathrm{ECLOSE}(\{s_0\}), \sigma) = \delta^*(s_0, \sigma)$.

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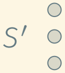

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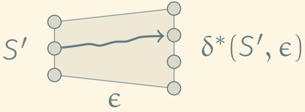

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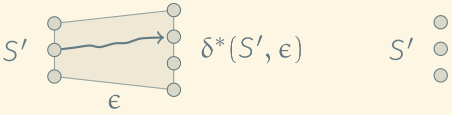

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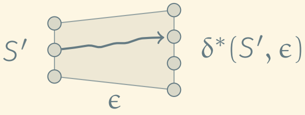

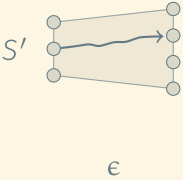

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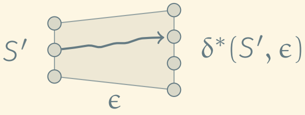

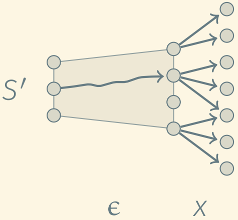

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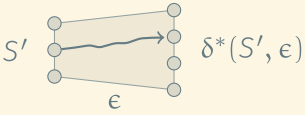

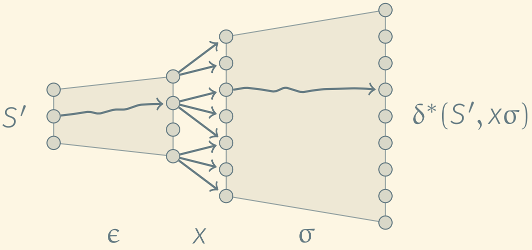

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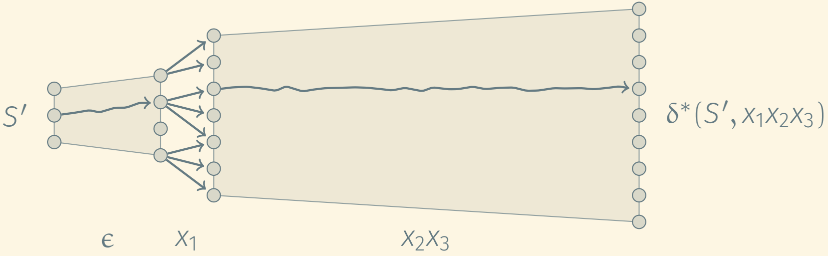

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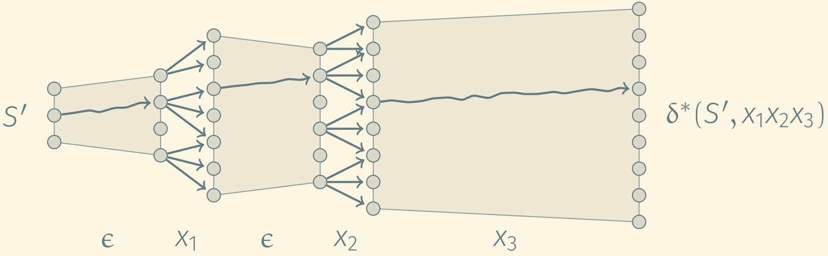

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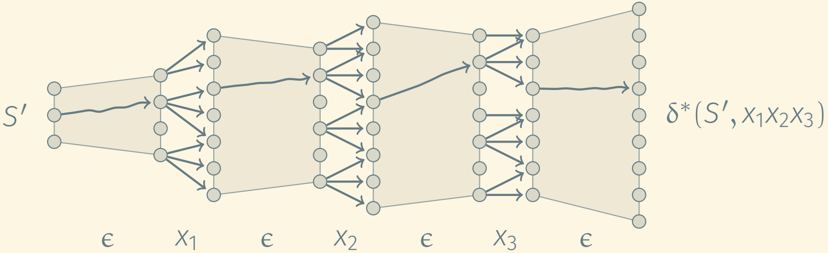

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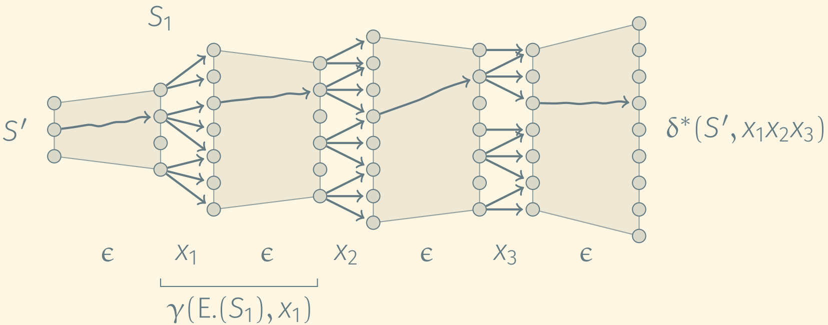

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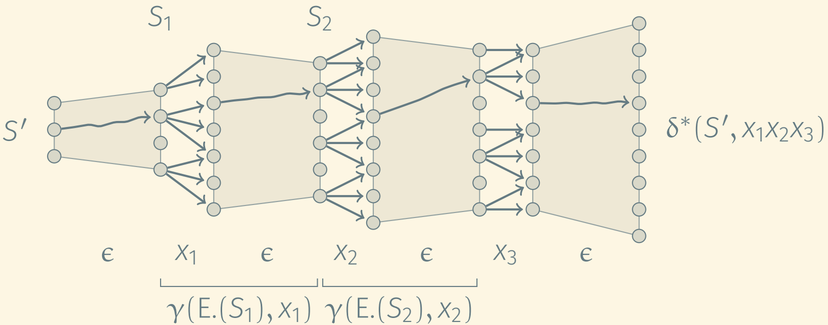
- $\gamma(S', x) = \bigcup_{s \in S'} \text{ECLOSE}(\delta(s, x))$,
- $\gamma^*(S', \epsilon) = S'$, and
- $\gamma^*(S', x\sigma) = \gamma^*(\gamma(S', x), \sigma)$.

Then

$$\delta^*(S', \sigma) = \gamma^*(\text{ECLOSE}(S'), \sigma).$$



TRANSITION FUNCTION OF THE DFA (3)

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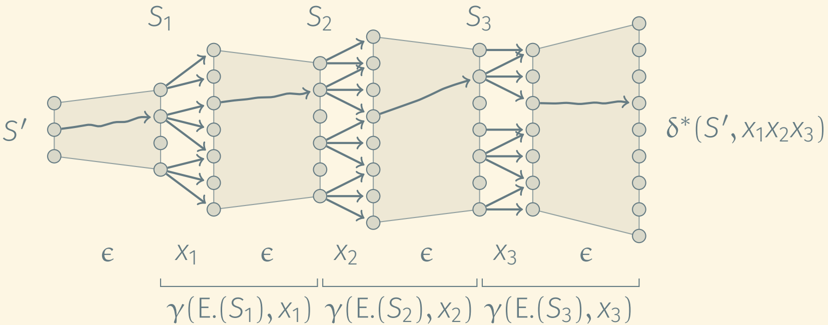

Transition function of the DFA

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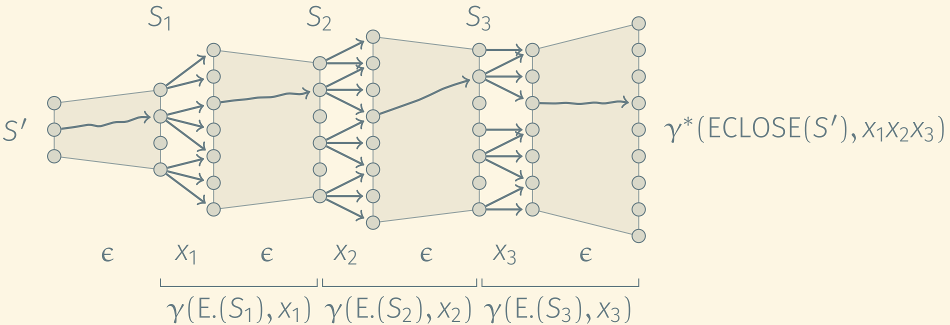

FROM NFA TO DFA: CONSTRUCTING ONLY THE STATES WE NEED

$N = (S, \Sigma, \delta, s_0, F) \rightarrow D = (T, \Sigma, \gamma, t_0, G)$

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So far, $T = 2^S$, but only a subset of states may be reachable from t_0.
$\longrightarrow$ We would like to choose T to be this subset.

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FROM NFA TO DFA: CONSTRUCTING ONLY THE STATES WE NEED

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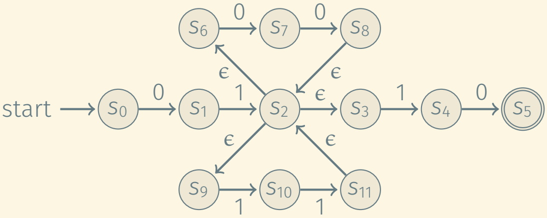
So far, $T = 2^S$, but only a subset of states may be reachable from t_0.
$\longrightarrow$ We would like to choose T to be this subset.

Obvious idea: Construct D with $T = 2^S$, then throw away the states not reachable from t_0.
Too costly!

Almost as obvious: Generate only the states we can reach from t_0:

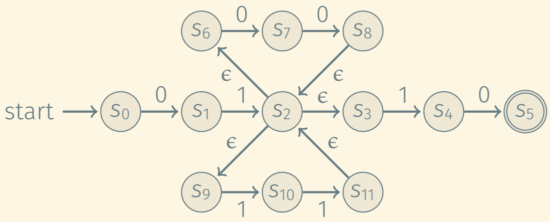
- Start with $T = \{t_0\}$ and a queue $Q = \{t_0\}$ of new states.
- While $Q \neq \emptyset$:
  - Remove some $t \in Q$ from Q.
  - For each $x \in \Sigma$, add $\gamma(t, x)$ to T, and to Q if $\gamma(t, x)$ was not in T before.

FROM NFA TO DFA: EXAMPLE

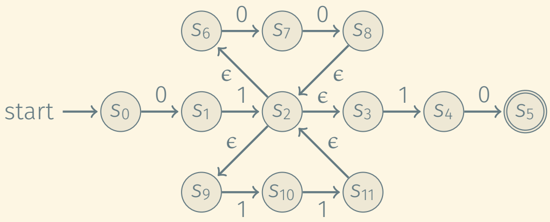


	δ	0	1
$\rightarrow$			

FROM NFA TO DFA: EXAMPLE

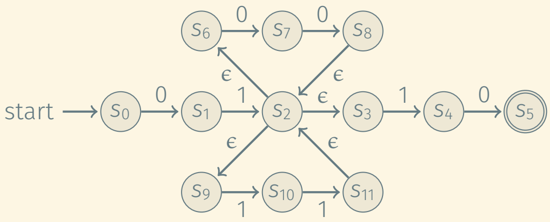


	δ	0	1
$\rightarrow$	$\{s_0\}$		



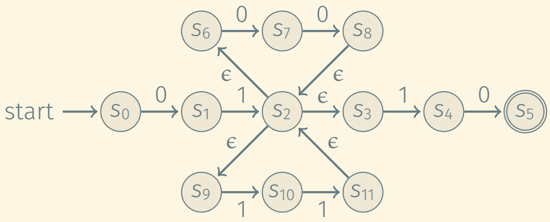
	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$		
	$\emptyset$		

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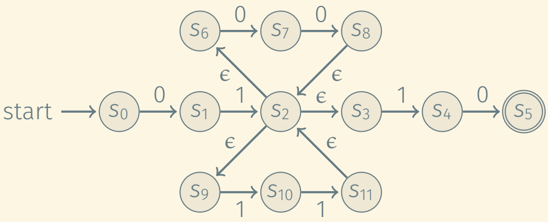
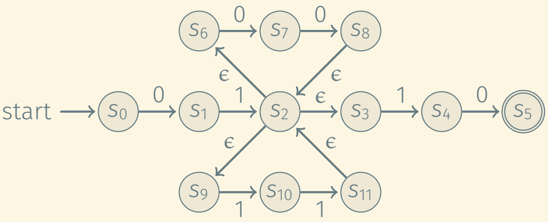


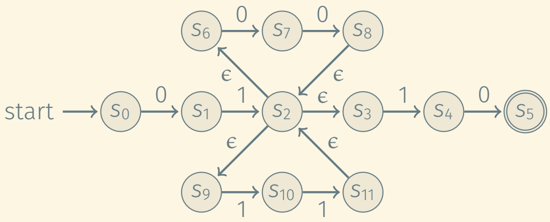
	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	
	$\emptyset$		

FROM NFA TO DFA: EXAMPLE



	δ	0	1
→	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$		

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$		
	$\{s_2, s_3, s_6, s_9\}$		

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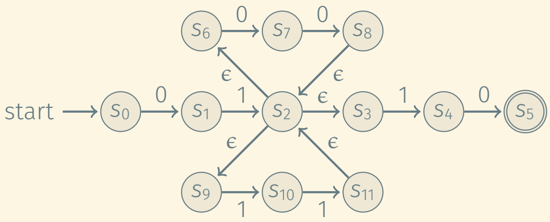
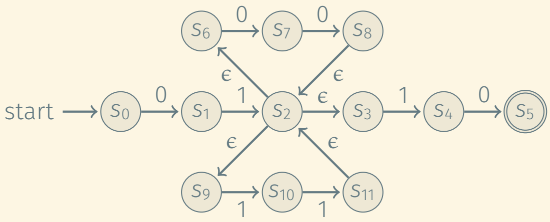
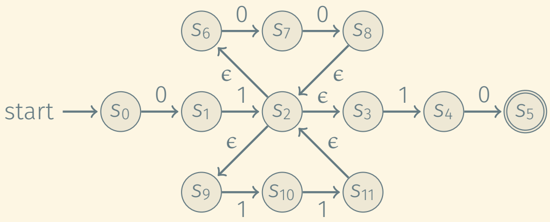


	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$		



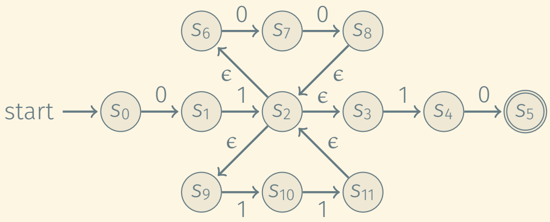
	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$		
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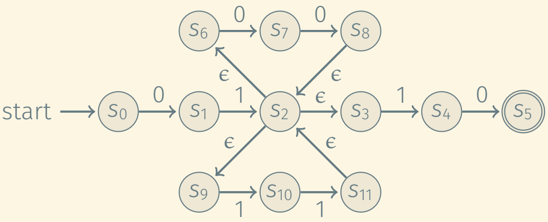

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$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
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	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	
	$\{s_4, s_{10}\}$		

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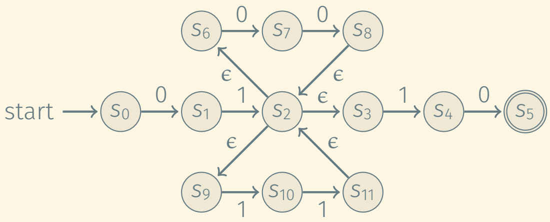

	δ	0	1
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	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$		

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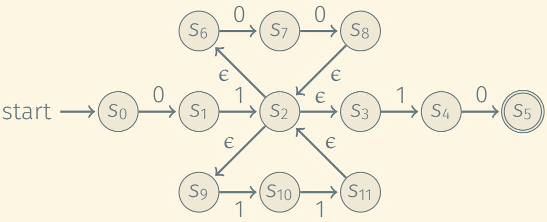

	δ	0	1
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	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$		
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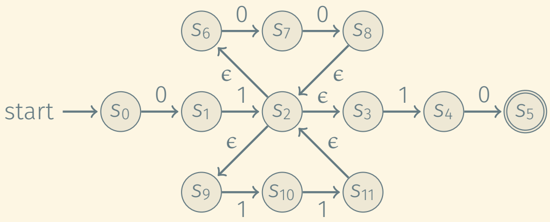

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	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	
	$\{s_2, s_3, s_6, s_8, s_9\}$		

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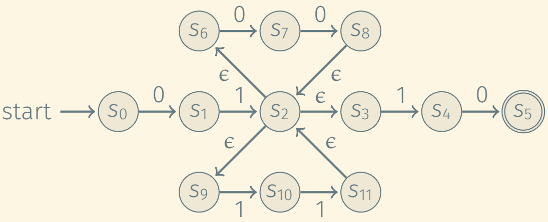

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	$\emptyset$	$\emptyset$	$\emptyset$
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	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$		

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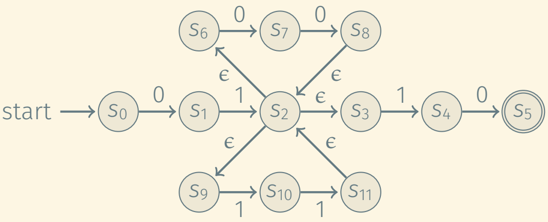

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	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$		
	$\{s_5\}$		
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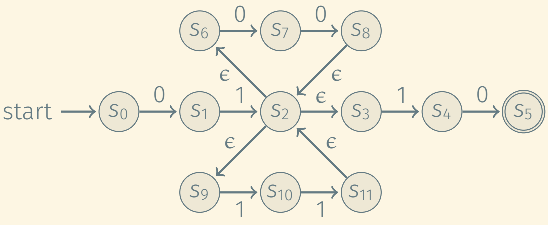

	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$	$\{s_7\}$	
	$\{s_5\}$		
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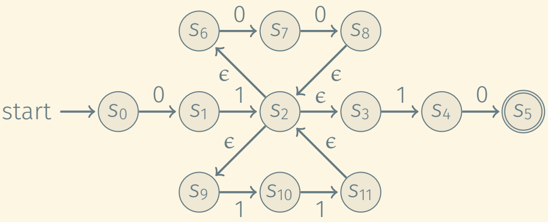

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$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
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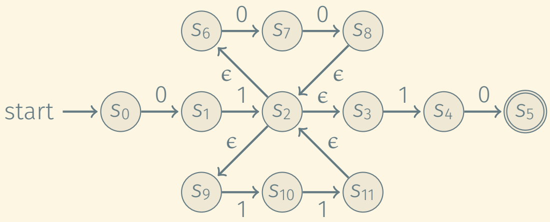

	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
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	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_5\}$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9, s_{11}\}$		

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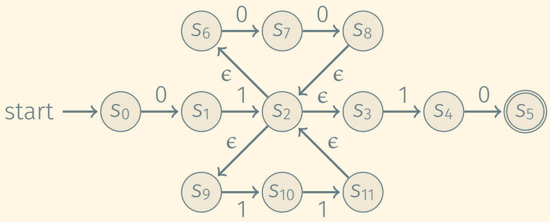
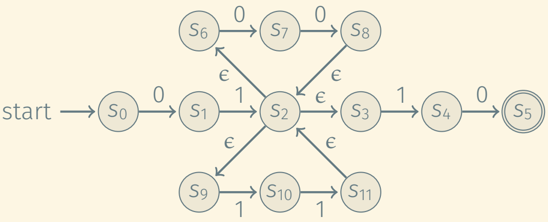


	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_5\}$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9, s_{11}\}$	$\{s_7\}$	

FROM NFA TO DFA: EXAMPLE



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
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	$\emptyset$	$\emptyset$	$\emptyset$
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	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_5\}$	$\emptyset$	$\emptyset$
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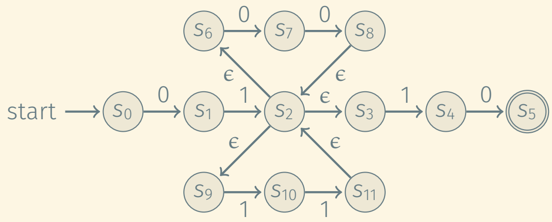

	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
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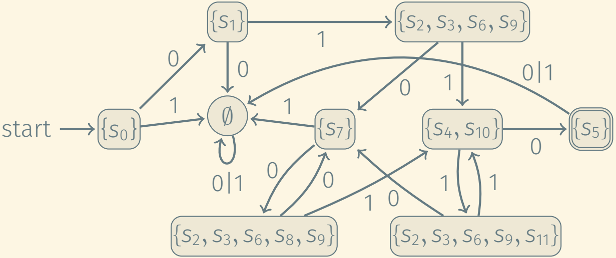
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
	$\{s_2, s_3, s_6, s_8, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
*	$\{s_5\}$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9, s_{11}\}$	$\{s_7\}$	$\{s_4, s_{10}\}$



	δ	0	1
$\rightarrow$	$\{s_0\}$	$\{s_1\}$	$\emptyset$
	$\{s_1\}$	$\emptyset$	$\{s_2, s_3, s_6, s_9\}$
	$\emptyset$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9\}$	$\{s_7\}$	$\{s_4, s_{10}\}$
	$\{s_7\}$	$\{s_2, s_3, s_6, s_8, s_9\}$	$\emptyset$
	$\{s_4, s_{10}\}$	$\{s_5\}$	$\{s_2, s_3, s_6, s_9, s_{11}\}$
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*	$\{s_5\}$	$\emptyset$	$\emptyset$
	$\{s_2, s_3, s_6, s_9, s_{11}\}$	$\{s_7\}$	$\{s_4, s_{10}\}$

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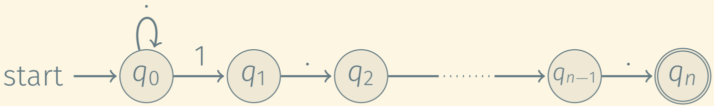

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- However, $\sigma_1 0^m \notin \mathcal{L}$ and $\sigma_2 0^m \in \mathcal{L}$.

$\Rightarrow$ D does not decide $\mathcal{L}$.

FROM REGULAR EXPRESSION TO NFA

Base cases:

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$\emptyset$

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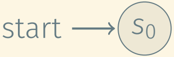

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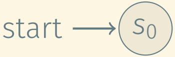

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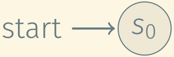

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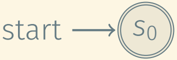

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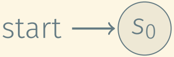

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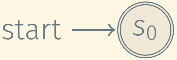

$x \quad (x \in \Sigma)$

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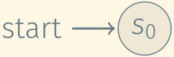

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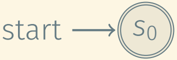

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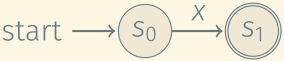

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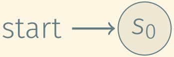

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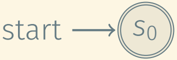

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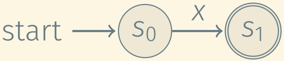

Inductive steps:

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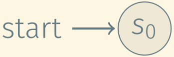

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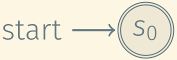

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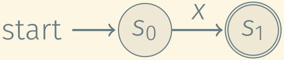

Inductive steps:

$A|B$

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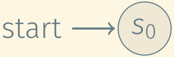

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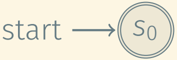

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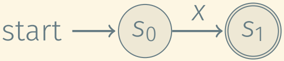

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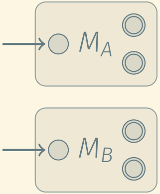

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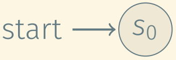

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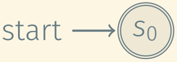

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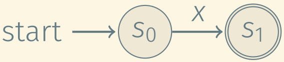

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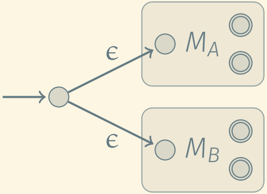

FROM REGULAR EXPRESSION TO NFA

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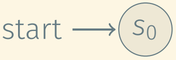

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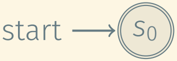

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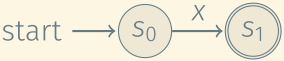

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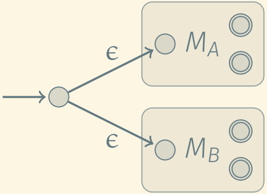

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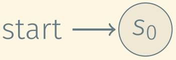

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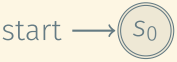

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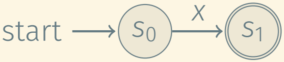

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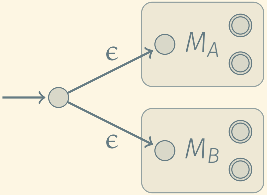
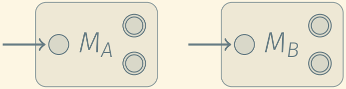
$A|B$



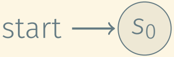
AB



FROM REGULAR EXPRESSION TO NFA

Base cases:

$\emptyset$



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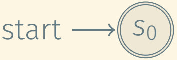

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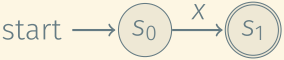

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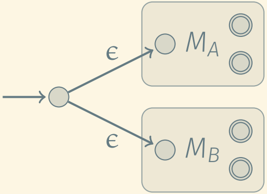

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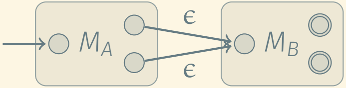

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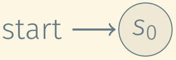

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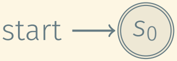

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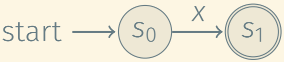

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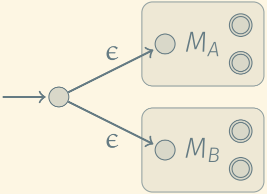

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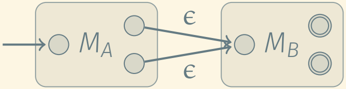
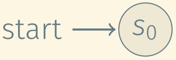


$A*$

Base cases:

$\emptyset$



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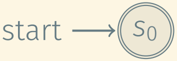

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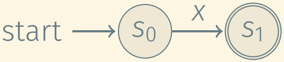
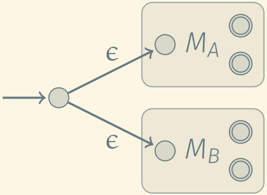


Inductive steps:

$A|B$



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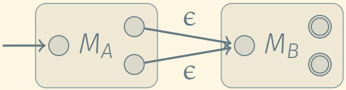
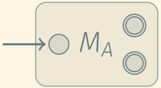


$A*$



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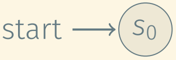
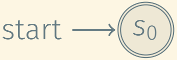
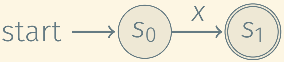


ϵ

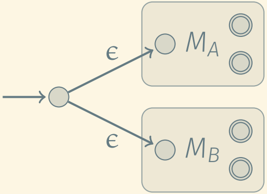


$x \quad (x \in \Sigma)$

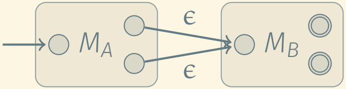


Inductive steps:

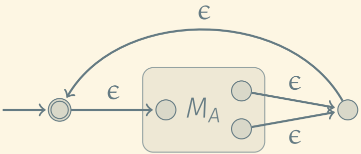
$A|B$



AB



$A*$



FROM REGULAR EXPRESSION TO NFA: EXAMPLE

$$(0|\epsilon)(1|000*)*(0|\epsilon)$$

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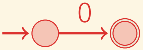

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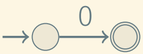

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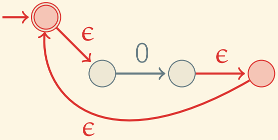

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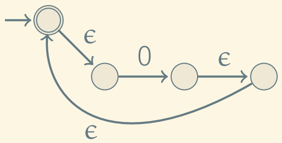

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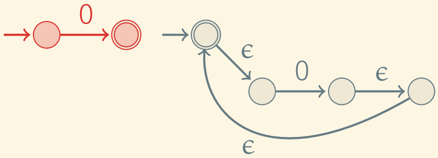

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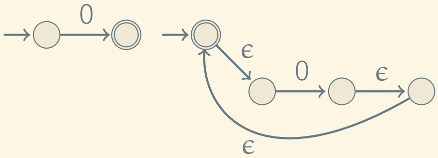

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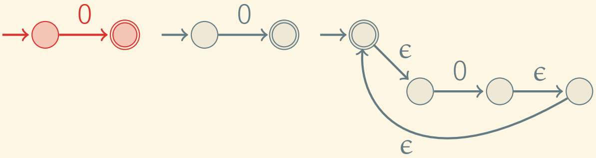

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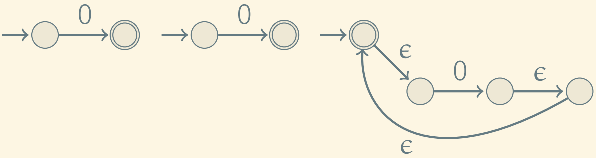

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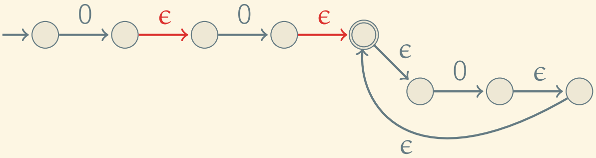

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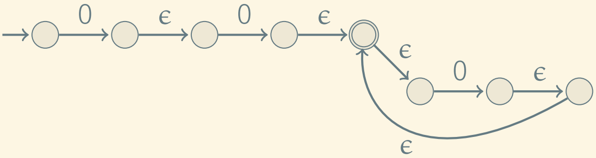

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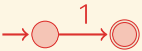

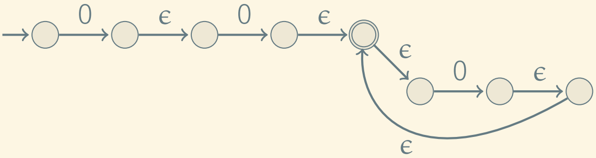

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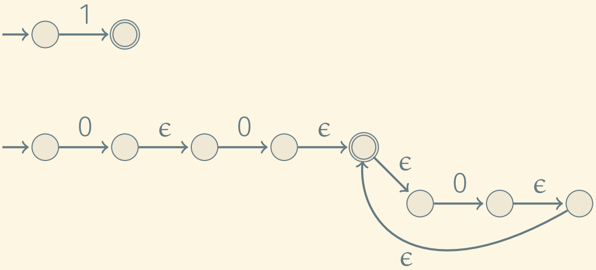

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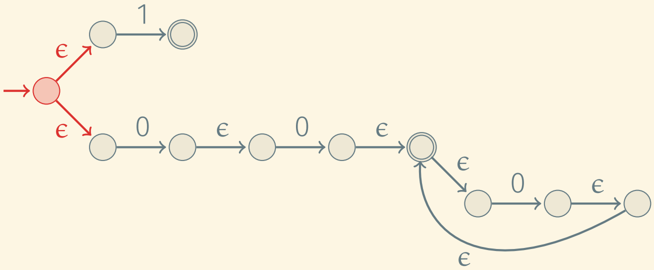

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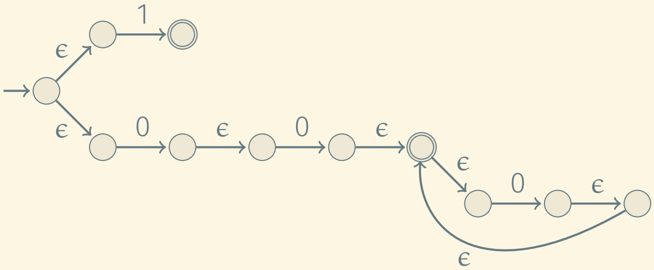

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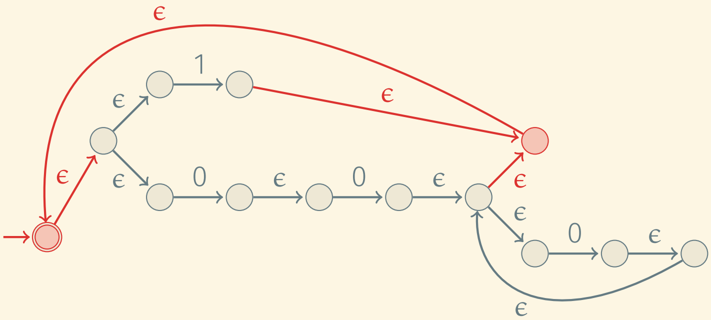

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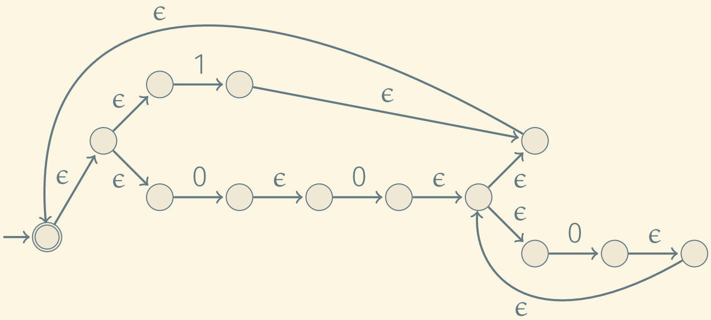

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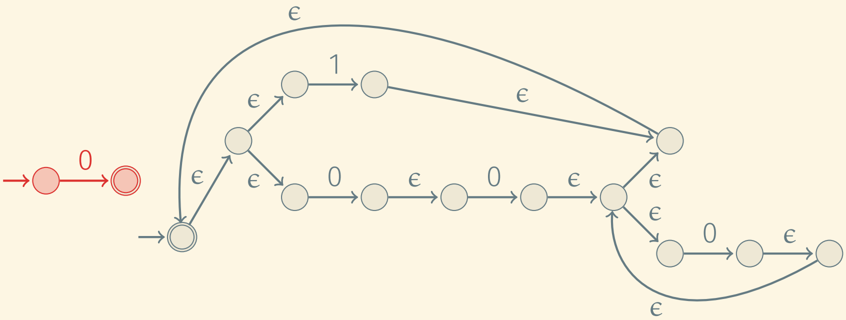

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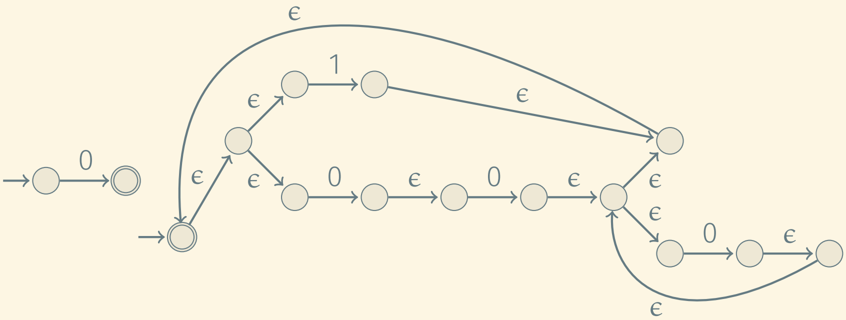

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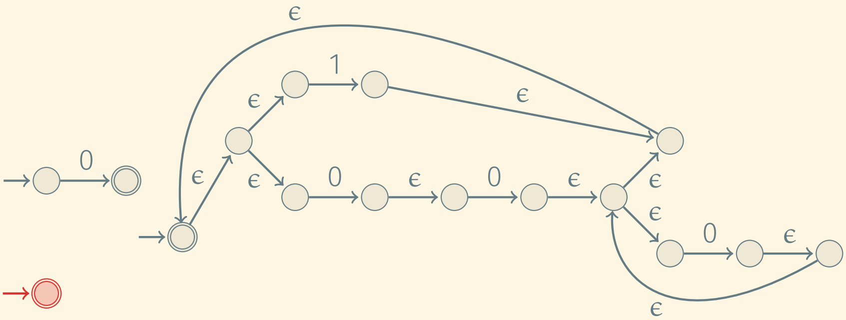

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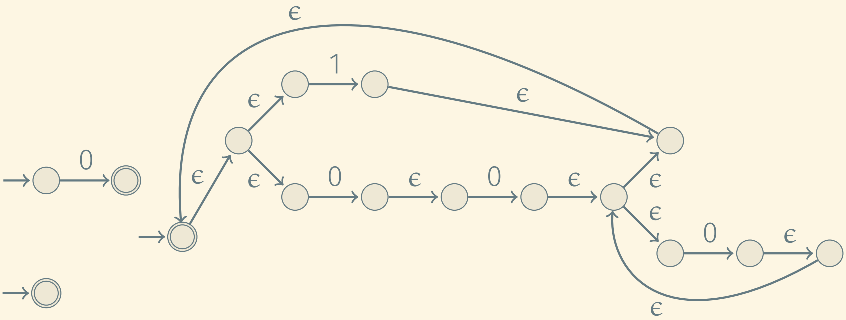

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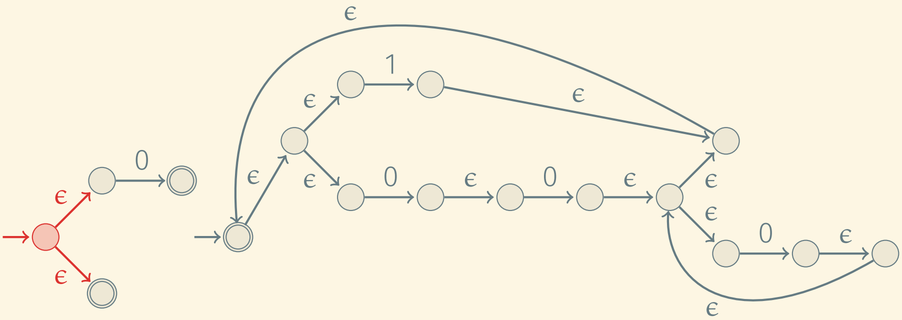
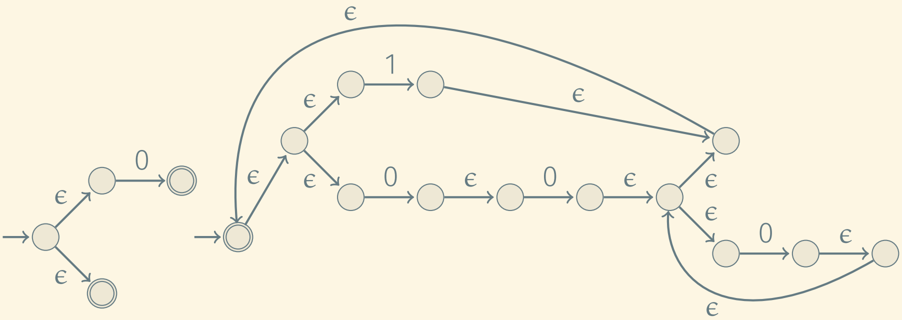


FROM REGULAR EXPRESSION TO NFA: EXAMPLE

$(0|\epsilon)(1|000*)*\color{red}{(0|\epsilon)}$



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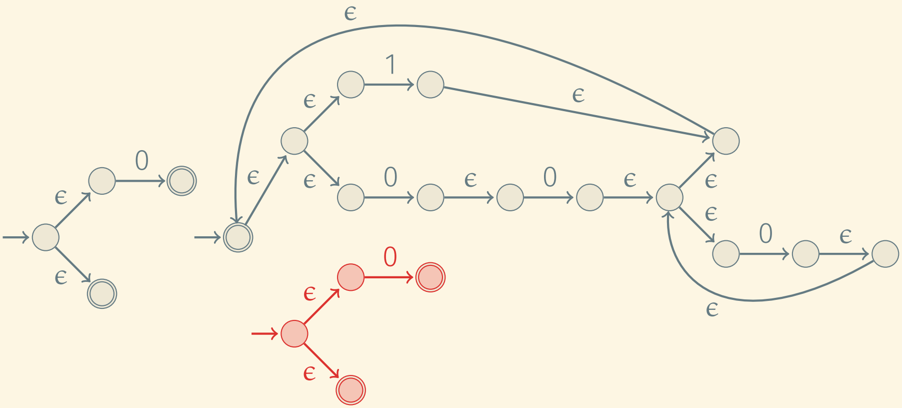

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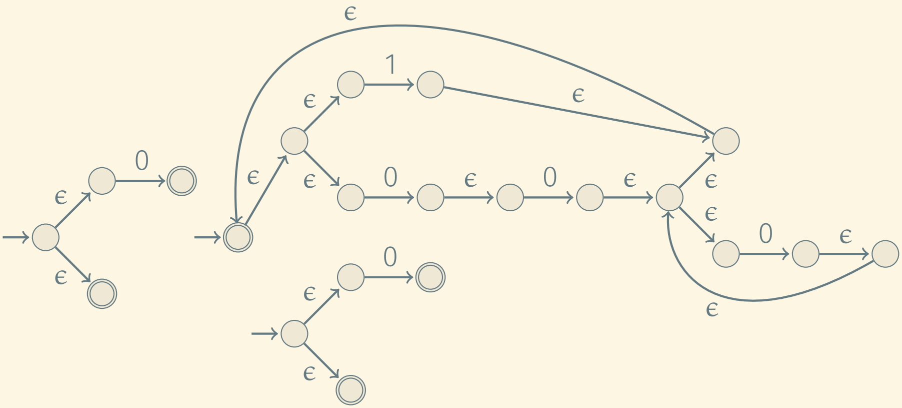

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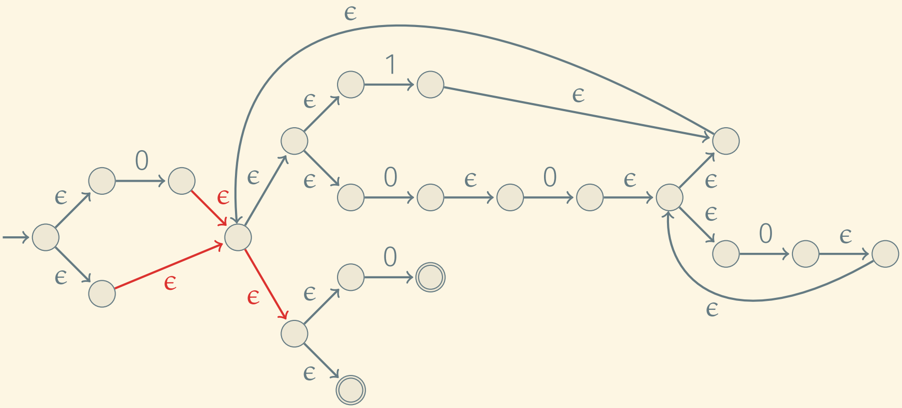

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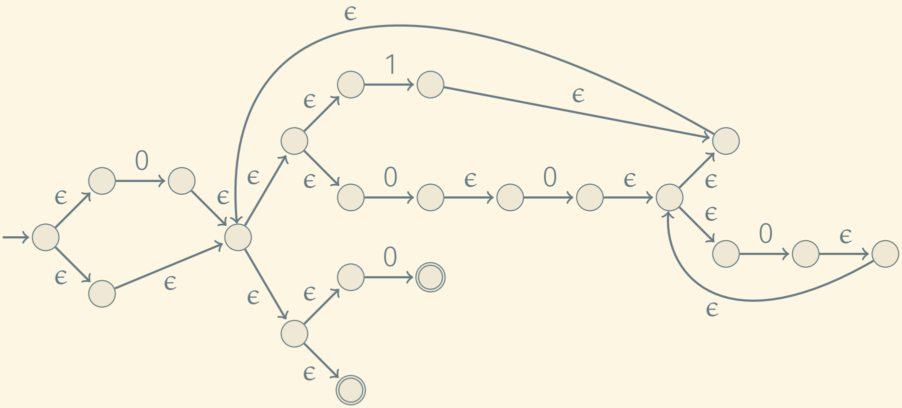
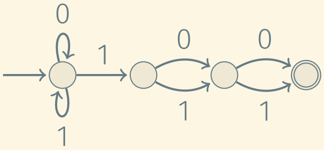


FROM NFA TO REGULAR EXPRESSION (1)

Acceptance of a string by (D/N)FA: Intuition

Concatenating the labels of the edges of a path in a (D/N)FA $N = (S, \Sigma, \delta, s_0, F)$ produces a string, called the label of the path.

N accepts a string σ if there exists a path from s_0 to an accepting state whose label is σ.



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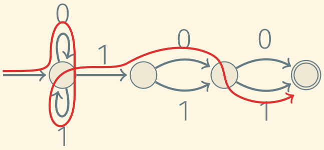

Accepts 01101

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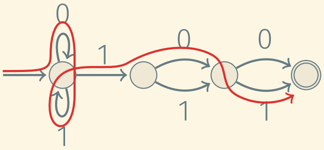
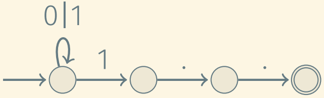


Accepts 01101

Definition: Regular Expression NFA (RNFA)

An RNFA N is a finite automaton whose edges are labelled with regular expressions.

N accepts a string σ if there exists a path from s_0 to an accepting state whose label is R and $\sigma \in \mathcal{L}(R)$.



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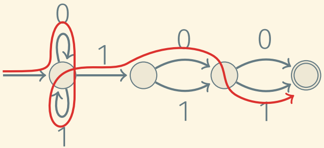

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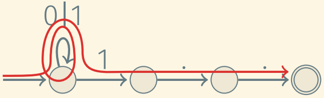

Accepts $01101 \in \mathcal{L}((0|1)(0|1)1..)$

FROM NFA TO REGULAR EXPRESSION (2)

Proof idea:

$$\text{NFA} \longrightarrow \text{RFA}_1 \longrightarrow \text{RFA}_2 \longrightarrow \cdots \longrightarrow \text{RFA}_n$$

$$\mathcal{L}(\text{NFA}) = \mathcal{L}(\text{RFA}_1) = \mathcal{L}(\text{RFA}_2) = \cdots = \mathcal{L}(\text{RFA}_n)$$

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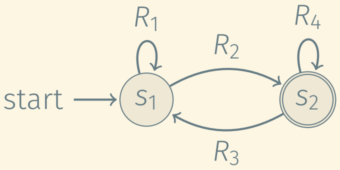
RFA_n has two states, an initial state and an accepting state:



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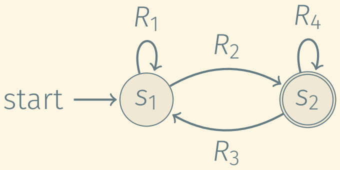

$$\mathcal{L}(\mathrm{NFA}) = \mathcal{L}(\mathrm{RFA}_n) = \mathcal{L}((R_1|R_2R_4{*}R_3){*}R_2R_4{*})$$

NFA $\longrightarrow$ RFA$_1$:

- $\mathcal{L}(\text{NFA}) = \mathcal{L}(\text{RFA}_1)$
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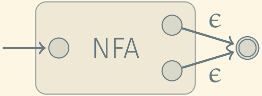

FROM NFA TO REGULAR EXPRESSION (4)

$\text{RFA}_k \longrightarrow \text{RFA}_{k+1}$:

- $\mathcal{L}(\text{RFA}_k) = \mathcal{L}(\text{RFA}_{k+1})$
- RFA_{k+1} has one state less than RFA_k.

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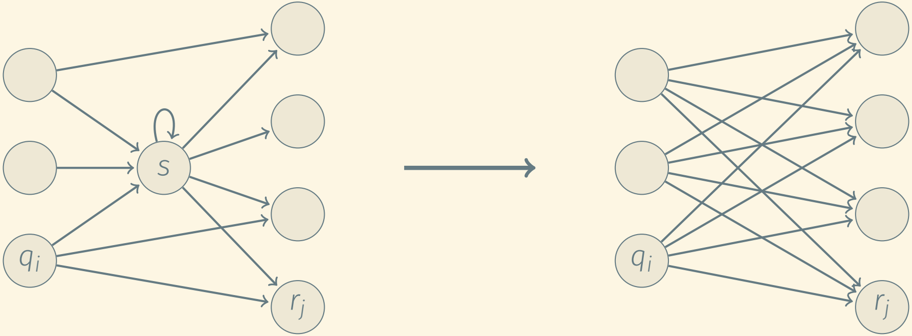

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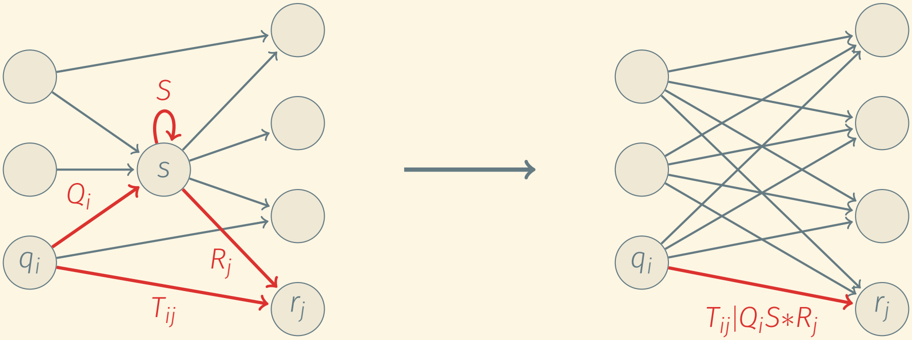

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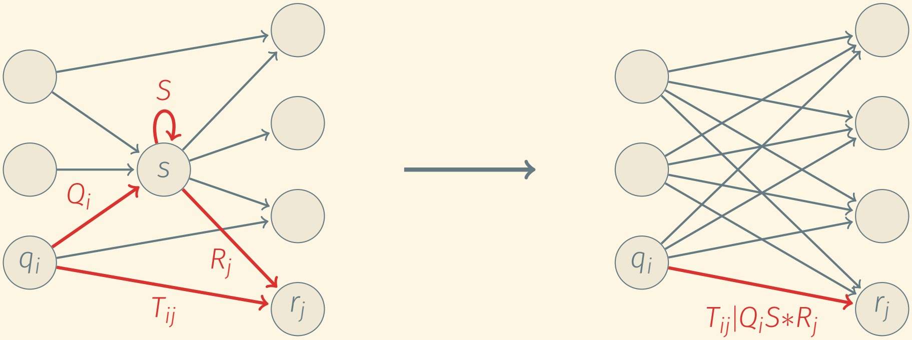
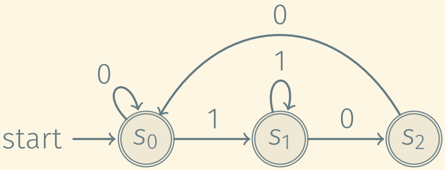


Note: This may create loops because some states may simultaneously be in- and out-neighbours of s.

FROM NFA TO REGULAR EXPRESSION: EXAMPLE

$\mathcal{L}((0|\epsilon)(1|000*)*(0|\epsilon))$: All strings that do not contain 101 as a substring



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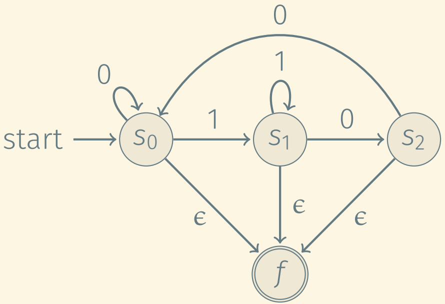

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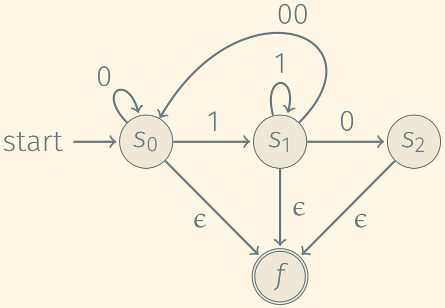

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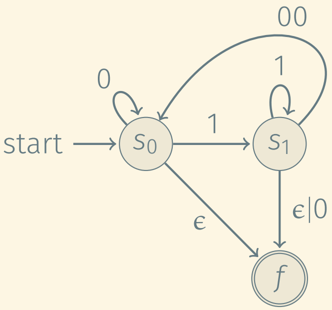

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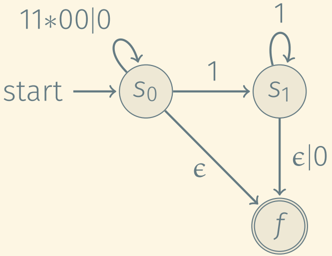

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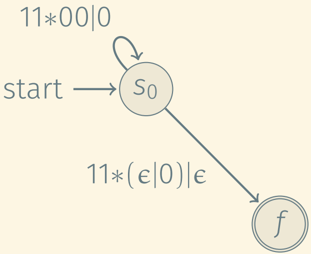

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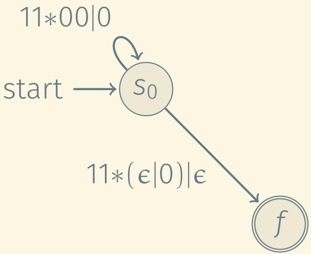

Regular expression: $(11*00|0)*(11*(\epsilon|0)|\epsilon)$

- Regular languages
- Regular expressions
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA

- Building a scanner
  - Regular expression $\rightarrow$ NFA $\rightarrow$ DFA
  - Minimizing the DFA

- Limitations of regular languages

ROAD MAP

- Regular languages
- Regular expressions
- Deterministic finite automata (DFA)
- Non-deterministic finite automata (NFA)
- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA

- Building a scanner
  - Regular expression $\longrightarrow$ NFA $\longrightarrow$ DFA
  - Minimizing the DFA

- Limitations of regular languages

Scanner

A scanner produces a token (token type, value) stream from a character stream.

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Modes of operation

- Complete pass produces token stream, which is then passed to the parser.
- Parser calls scanner to request next token

In either case, the scanner greedily recognizes the longest possible token.

Scanner implementation

- Hand-written, ad-hoc: Usually when speed is a concern.
- From regular expression using scanner generator: More convenient. Result:
  - Case statements representing transitions of the DFA.
  - Table representing the DFA's transition function plus driver code to implement the DFA.

Workflow

Regular expression $\longrightarrow$ NFA $\longrightarrow$ DFA $\longrightarrow$ minimized DFA

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- Keywords are not identifiers:
  - Look up identifier in keyword table (e.g., hash table) to see whether it is in fact a keyword
- “Look ahead” to distinguish tokens with common prefix (e.g., 100 vs 100.5):
  - Try to find the longest possible match by continuing to scan from an accepting state.
  - Backtrack to last accepting state when “stuck”.

EXTENDED EXAMPLE: AN INCOMPLETE SCANNER FOR PASCAL (1)

Regular expressions for the different tokens:

lparen:	`\(`
rparen:	`\)`
lbrac:	`\[`
rbrac:	`\]`
comma:	`,`
dot:	`\.`
dotdot:	`\.\.`
lt:	`<`
le:	`<=`
ident:	`[A-Za-z][A-Za-z0-9_]*`
int:	`[+-]?[0-9]+`
real:	`[+-]?[0-9]+(\.[0-9]+)?([Ee][+-]?[0-9]+)?`

…

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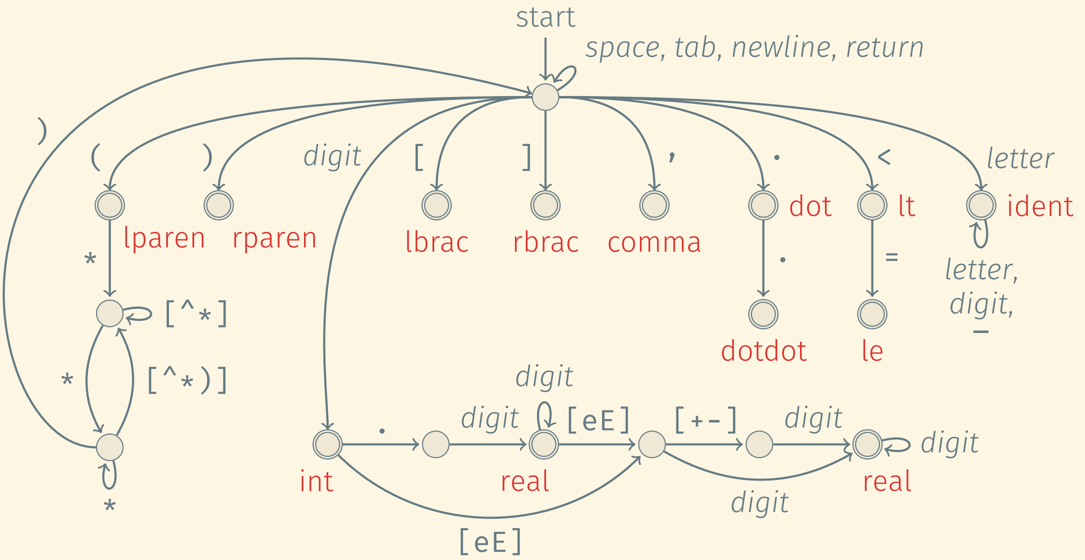
- Minimize the DFA.


start
space, tab, newline, return
)
(
)
digit
[
]
,
.
<
letter
lparen
rparen
lbrac
rbrac
comma
dot
lt
ident
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=
letter,
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*
[^*)]
digit
digit
[eE]
[+-]
digit
.
digit
int
real
real
*
digit
[eE]

EXTENDED EXAMPLE: AN INCOMPLETE SCANNER FOR PASCAL (4)

Driver code:

- Whenever the scan reaches an accepting state of the spaces/comments NFA, set a start marker.
- Whenever the scan reaches an accepting state of any other NFA, set an end marker and remember the token.
- Whenever the scan reaches state $\emptyset$,
  - Go back to the end marker.
  - Report the remembered token.
  - Turn the text between start and end marker into a representation of the scanned token (integer, identifier string, ...).
  - Set the start marker to be equal to the end marker.

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Given a DFA D, produce a DFA D' with the minimum number of states and such that $\mathcal{L}(D) = \mathcal{L}(D')$.

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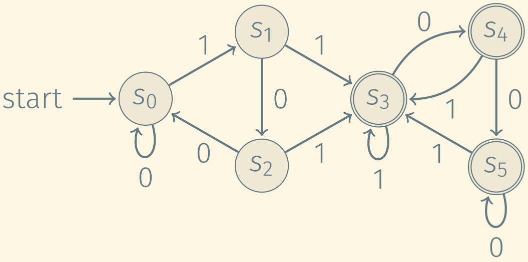
- The final set of equivalence classes is the set of states of the minimized DFA.

MINIMIZING THE DFA: EXAMPLE



$$S_0 \xleftarrow{0} S_0 \xrightarrow{1} S_1$$
$$S_2 \xleftarrow{0} S_1 \xrightarrow{1} S_3$$
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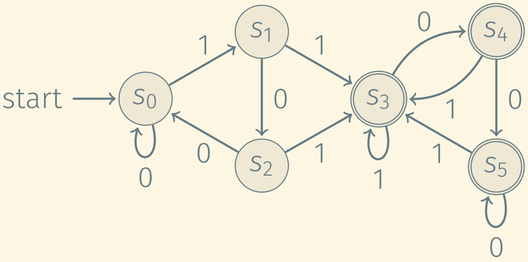
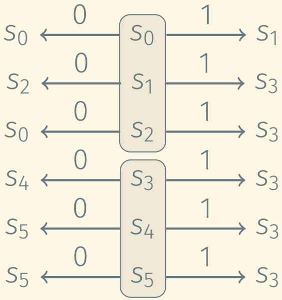
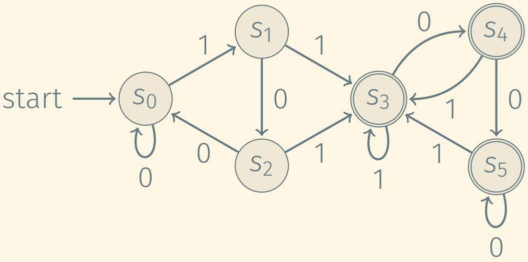
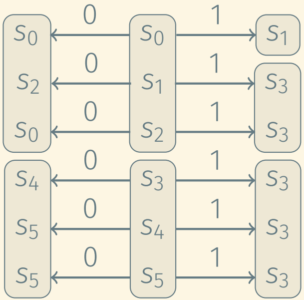





start
s_0
s_1
s_2
s_3
s_4
s_5
0
1


s_0
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1

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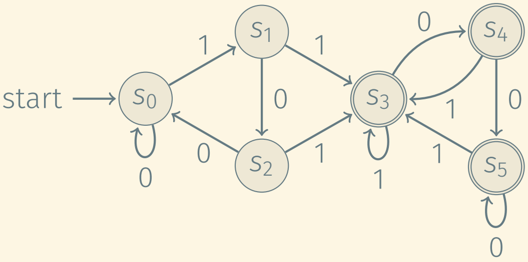

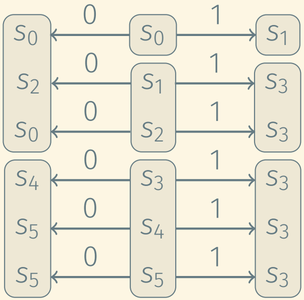

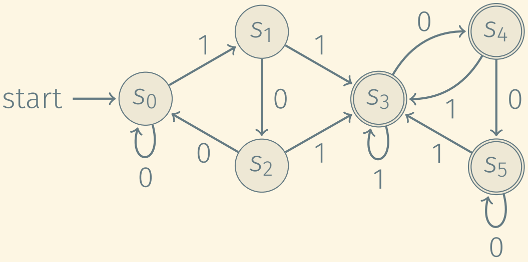
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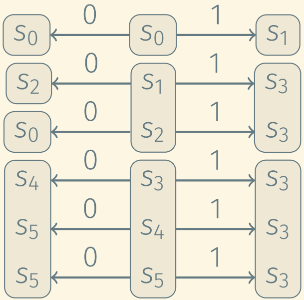
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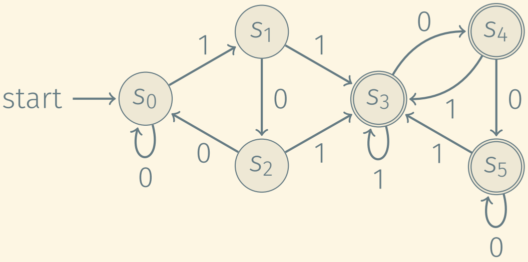
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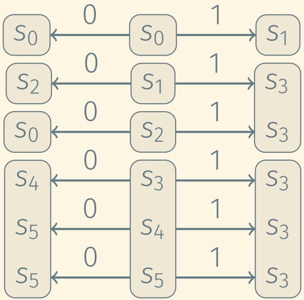
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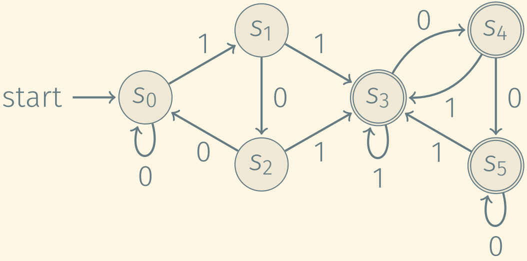

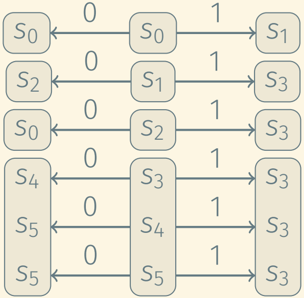

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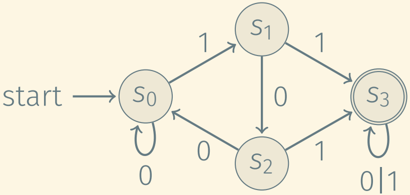

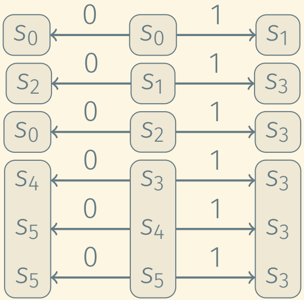

MINIMIZING THE DFA (2)

The described procedure ensures that $\mathcal{L}(D) = \mathcal{L}(D')$ but does not distinguish between different types of accepting states (corresponding to tokens).

To distinguish between different types of accepting states, start with one equivalence class per type of accepting state.

ROAD MAP

- Regular languages
- Regular expressions
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- Expressive power of DFA and NFA
- Equivalence of regular expressions, DFA, and NFA

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Pumping Lemma

For every regular language $\mathcal{L}$, there exists a constant $n_{\mathcal{L}}$ such that every $\sigma \in \mathcal{L}$ with $|\sigma| \geq n_{\mathcal{L}}$ can be written as $\sigma = \alpha\beta\gamma$ with

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- Since $|\alpha\beta| \leq n_{\mathcal{L}}$, we have $\alpha = 0^k$ and $\beta = 0^m$, where $m = |\beta| > 0$.
- Thus, $\alpha\beta\beta\gamma = 0^{m+n_{\mathcal{L}}}1^{n_{\mathcal{L}}} \notin \mathcal{L}$, a contradiction.

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PROOF OF THE PUMPING LEMMA

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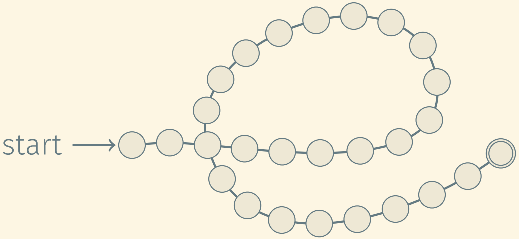

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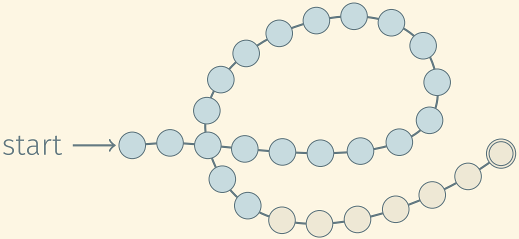

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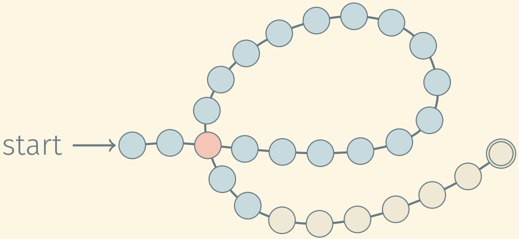

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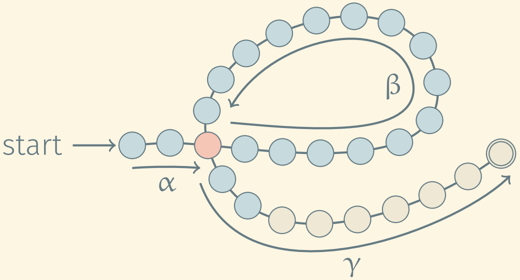

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PUMPING LEMMA: MORE APPLICATIONS

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  - $\alpha\beta^c\gamma \in \mathcal{L}$, where $c = |\alpha\gamma| = p - b \geq 2$.
  - However, $|\alpha\beta^c\gamma| = (b + 1)c$, which is not prime because $b + 1 \geq 2$ and $c \geq 2$. Contradiction.

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- Lexical analysis requires some simple extensions to DFA because we need to know which token wa accepted and we need to support greediness/backtracking.