# Greedy Algorithms

Textbook Reading

Chapters 16, 17, 21, 23 & 24

# Overview

## Design principle:

Make progress towards a globally optimal solution by making locally optimal choices, hence the name.

## Problems:

- Interval scheduling
- Minimum spanning tree
- Shortest paths
- Minimum-length codes

## Proof techniques:

- Induction
- The greedy algorithm "stays ahead"
- Exchange argument

## Data structures:

- Priority queue
- Union-find data structure

# Interval Scheduling

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A set of activities competing for time intervals on a certain resource
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**Goal:**

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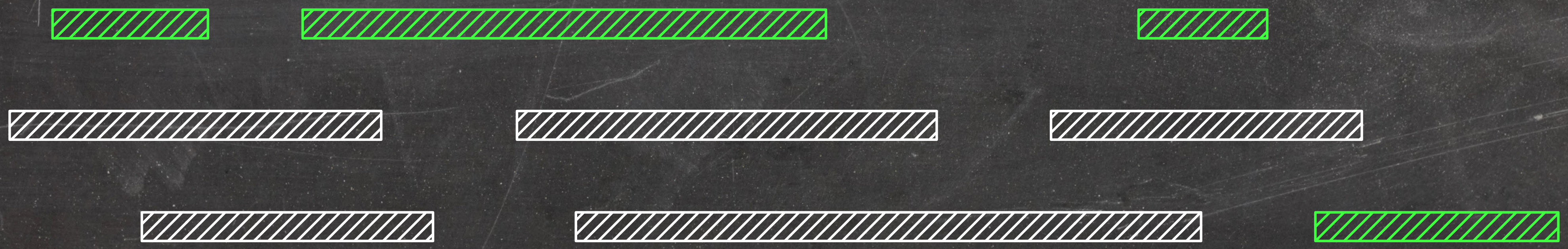

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# A Greedy Framework for Interval Scheduling

**FindSchedule(S)**

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**Main questions:**

- Can we choose an arbitrary interval I in each iteration?
- How do we choose interval I in each iteration?

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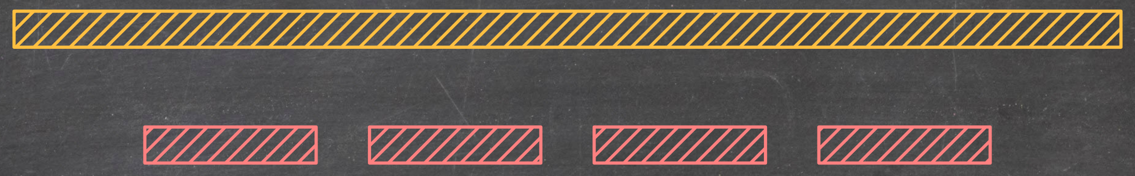

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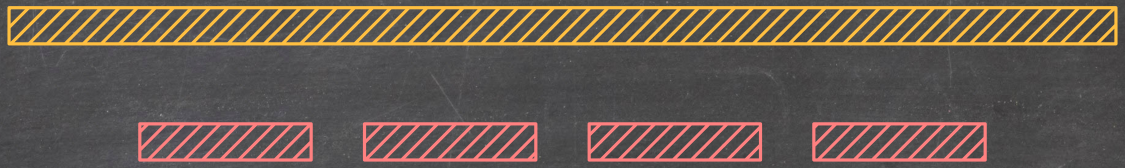

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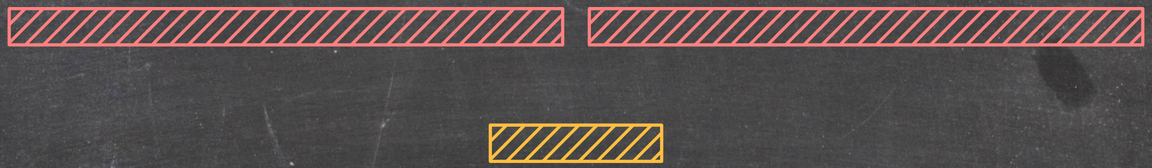

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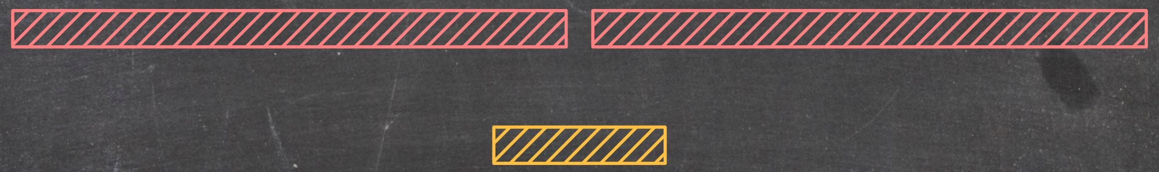

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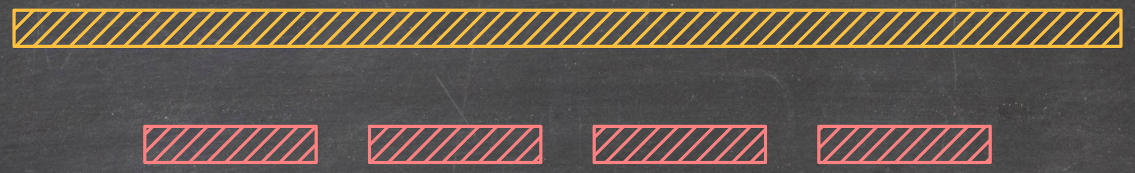

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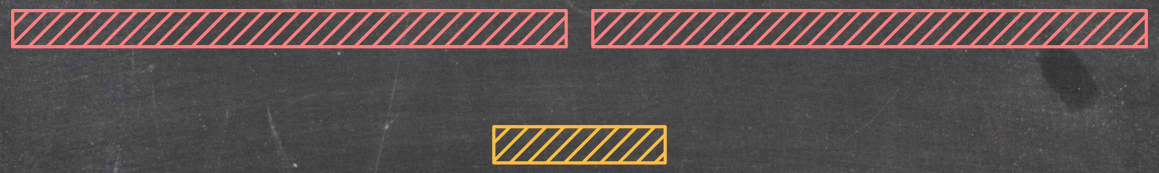

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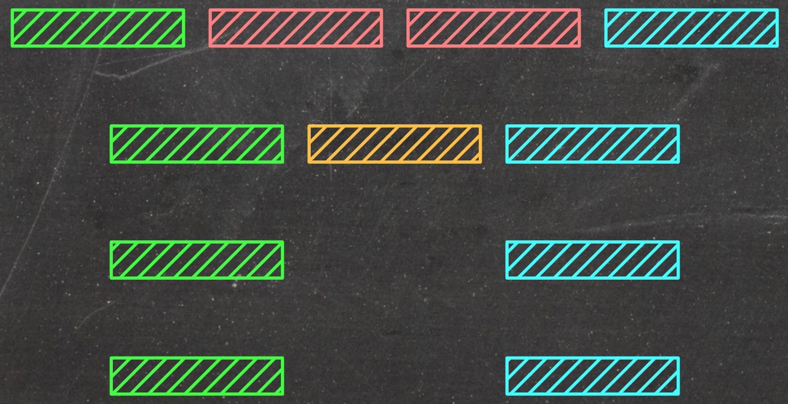

# The Strategy That Works

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Let $I_1 \prec I_2 \prec \cdots \prec I_k$ be the schedule we compute.

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$\Rightarrow$ Since $O_{j+1}$ starts after $O_j$ ends, it also starts after $I_j$ ends.

$\Rightarrow$ If $k < m$, FindSchedule inspects $O_{k+1}$ after $I_k$ and thus would have added it to its output, a contradiction.

# The Greedy Algorithm Stays Ahead

**Lemma:** FindSchedule finds a maximum-cardinality set of conflict-free intervals.

**Proof by induction:**

Base case(s): Verify that the claim holds for a set of initial instances.

Inductive step: Prove that, if the claim holds for the first k instances, it holds for the (k + 1)st instance.

# The Greedy Algorithm Stays Ahead

**Lemma:** FindSchedule finds a maximum-cardinality set of conflict-free intervals.

**Base case:** $I_1$ ends no later than $O_1$ because both $I_1$ and $O_1$ are chosen from $S$ and $I_1$ is the interval in $S$ that ends first.

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$\Rightarrow$ $I_{k+1}$ ends no later than $O_{k+1}$ because it is the interval that ends first among all intervals that do not conflict with $I_1, I_2, \ldots, I_k$.

# Implementing The Algorithm

**FindSchedule(S)**

```
S' = [ ]
sort the intervals in S by increasing finish times
S'.append(S[1])
f = S[1].f
for i = 2 to |S|
    do if S[i].s > f
          then S'.append(S[i])
               f = S[i].f
return S'
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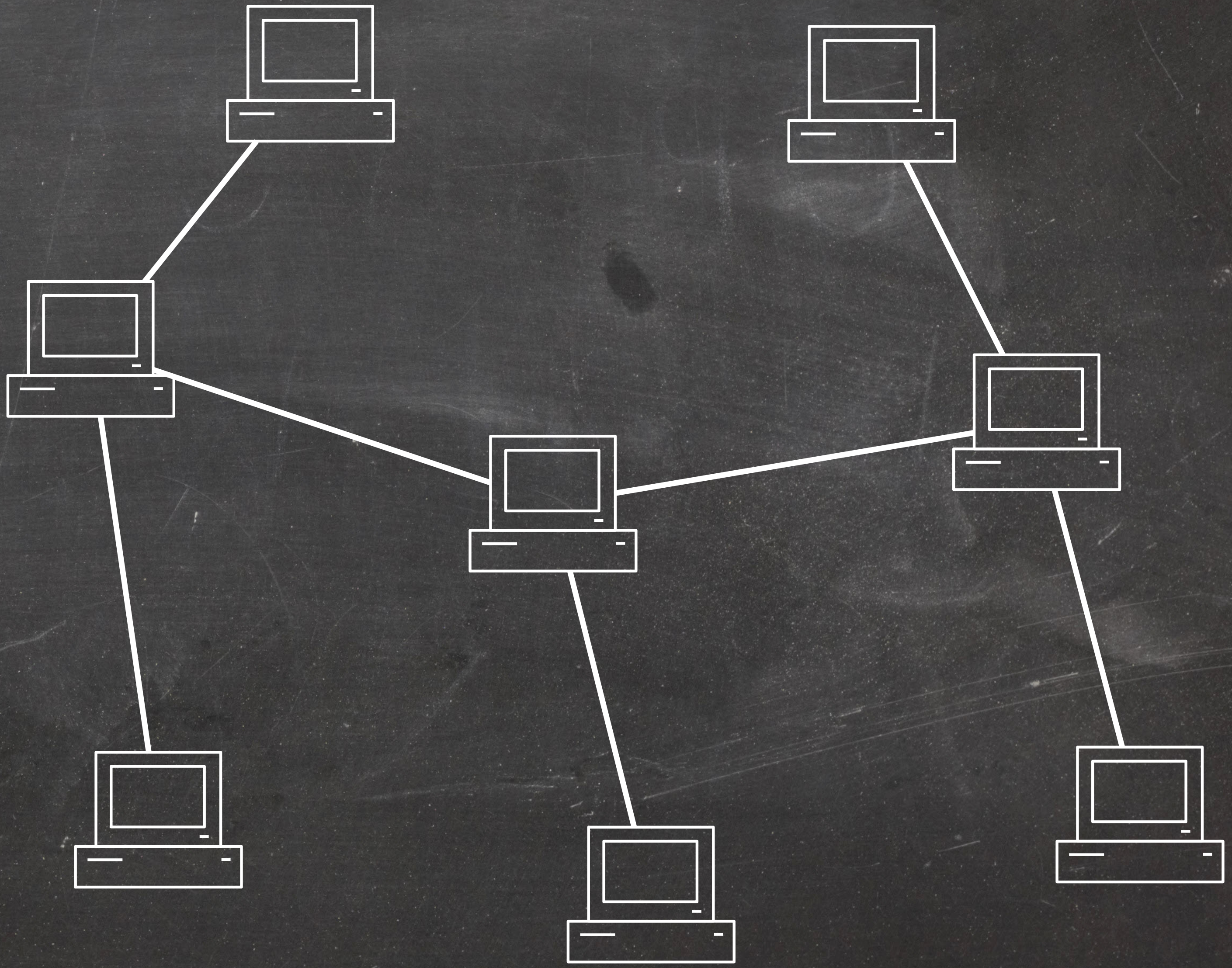
**Lemma:** A maximum-cardinality set of non-conflicting intervals can be found in $O(n \lg n)$ time.

# Minimum Spanning Tree

**Given:** n computers

**Goal:** Connect them so that every computer can communicate with every other computer.



We don't care whether the connection between any pair of computers is short.

We don't care about fault tolerance.

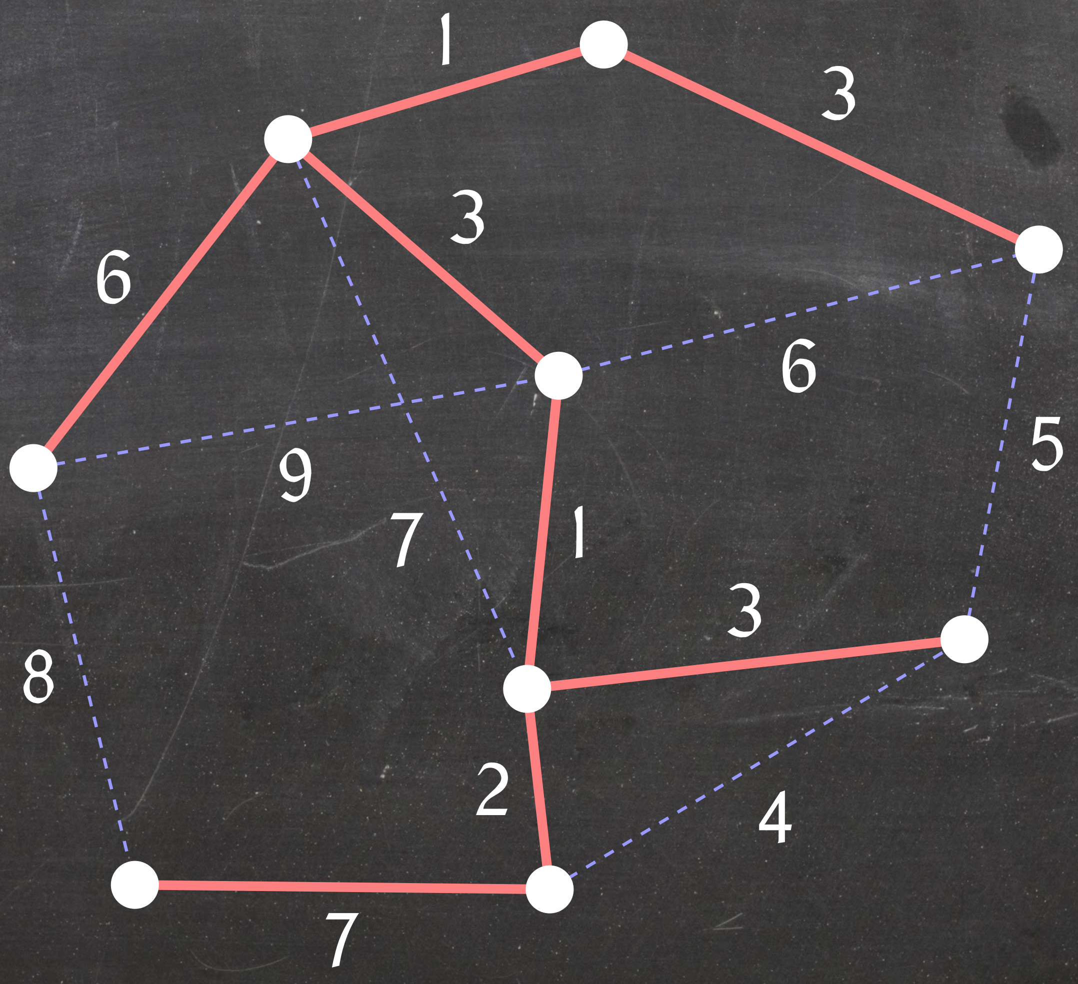
Every foot of cable costs us $1.

⇒ We want the cheapest possible network.

# Minimum Spanning Tree

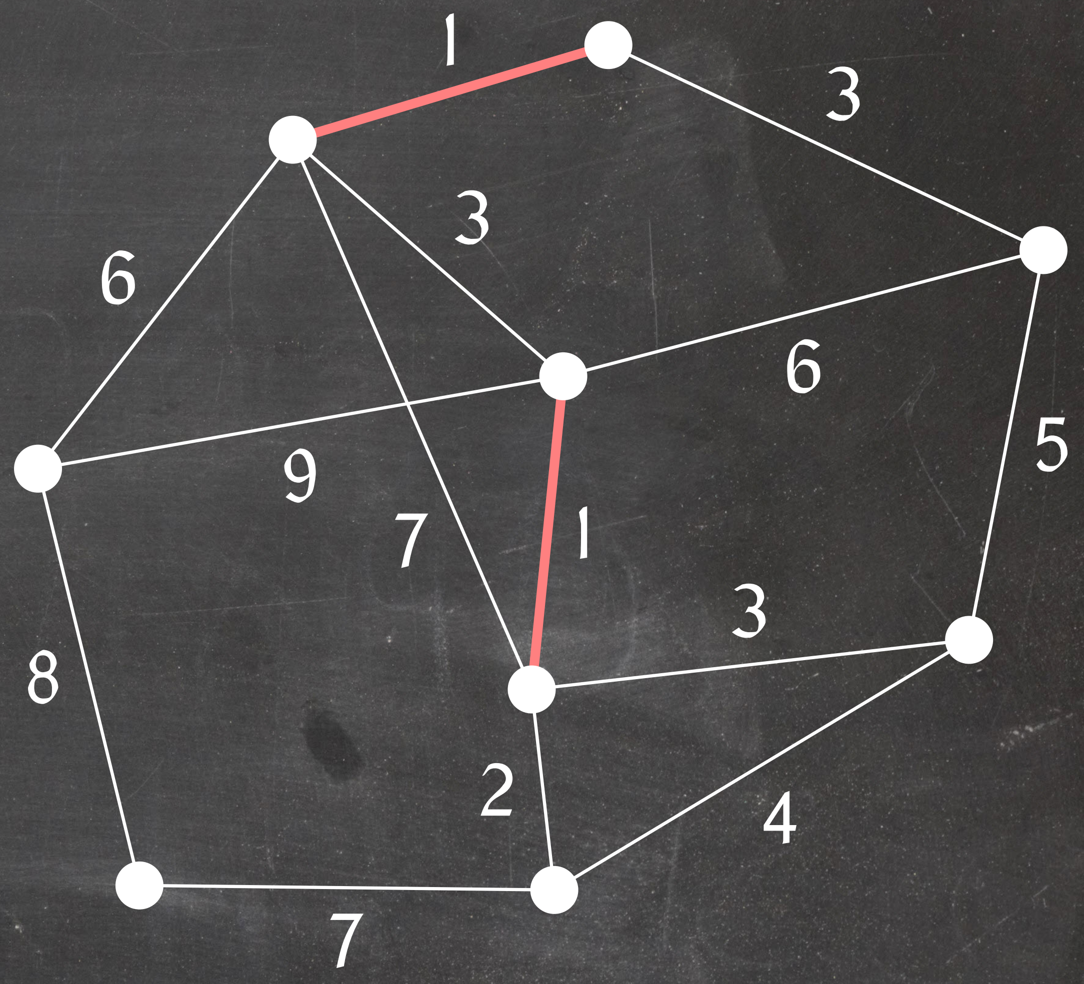
Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of $G$, a minimum spanning tree (MST) $T$ of $G$ is a spanning tree with minimum total weight

$$w(T) = \sum_{e \in T} w(e).$$



# Kruskal's Algorithm

**Greedy choice:** Pick the shortest edge



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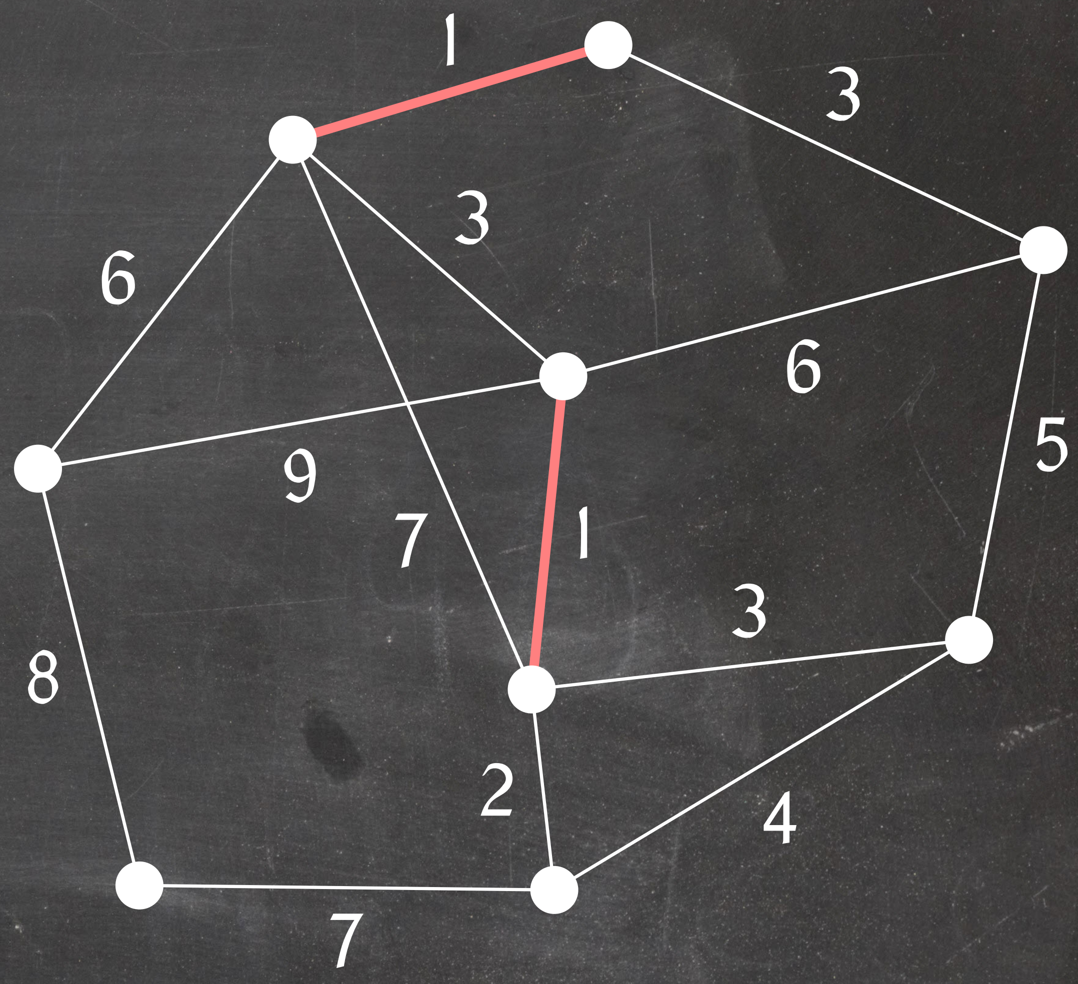
**Greedy choice:** Pick the shortest edge that connects two previously disconnected vertices.



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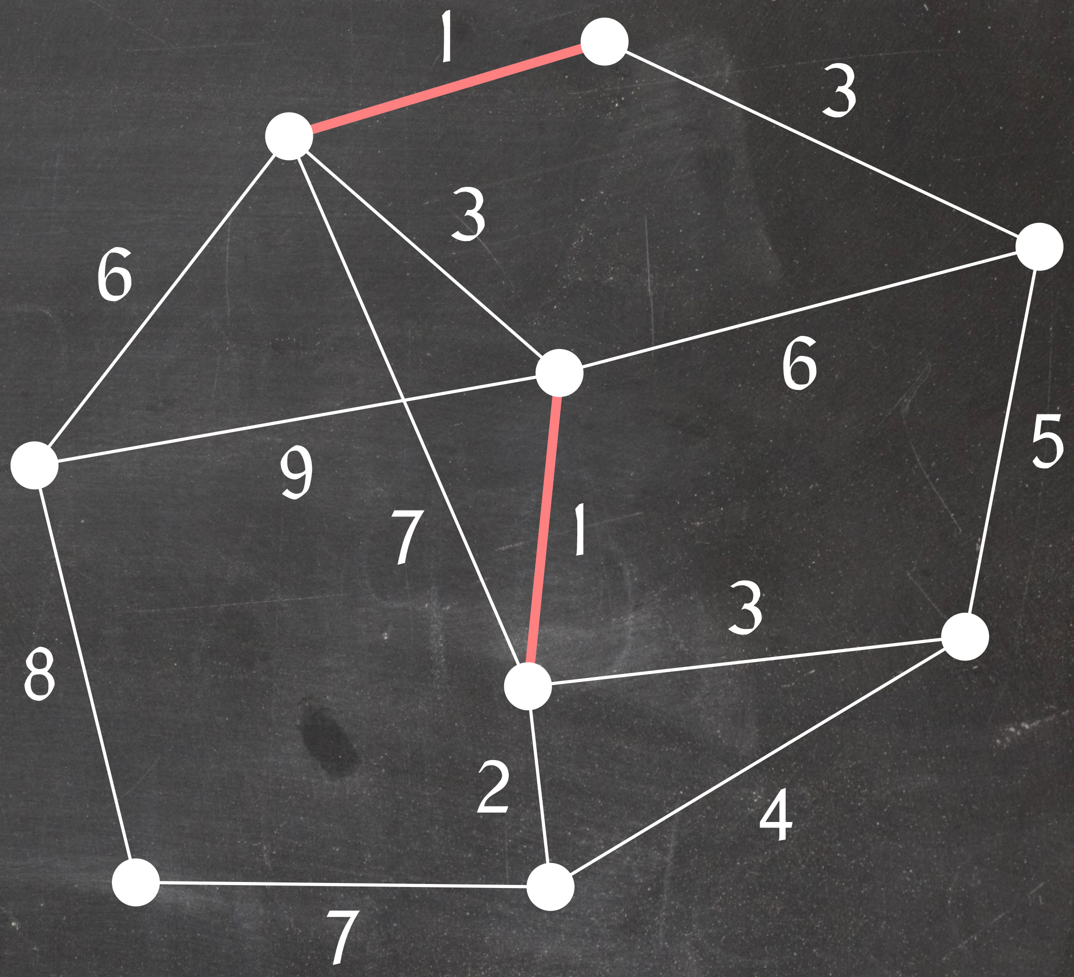

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T = (V, ∅)
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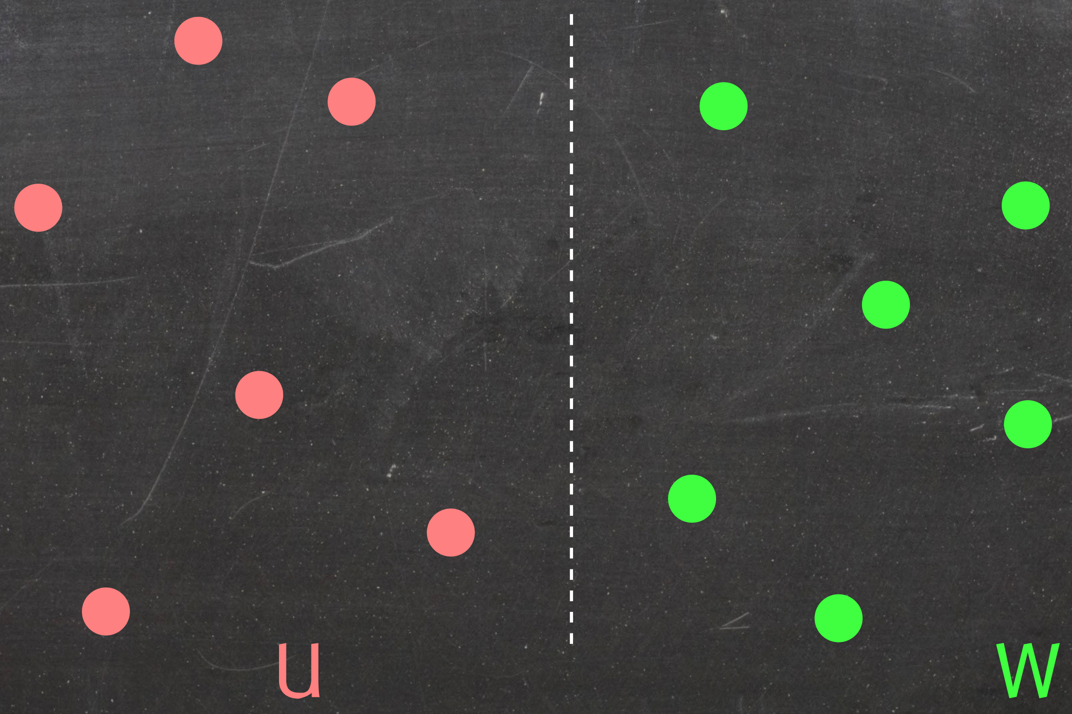
# A Cut Theorem

A cut is a partition $(U, W)$ of $V$ into two non-empty subsets: $\emptyset \subset U \subset V$ and $W = V \setminus U$.



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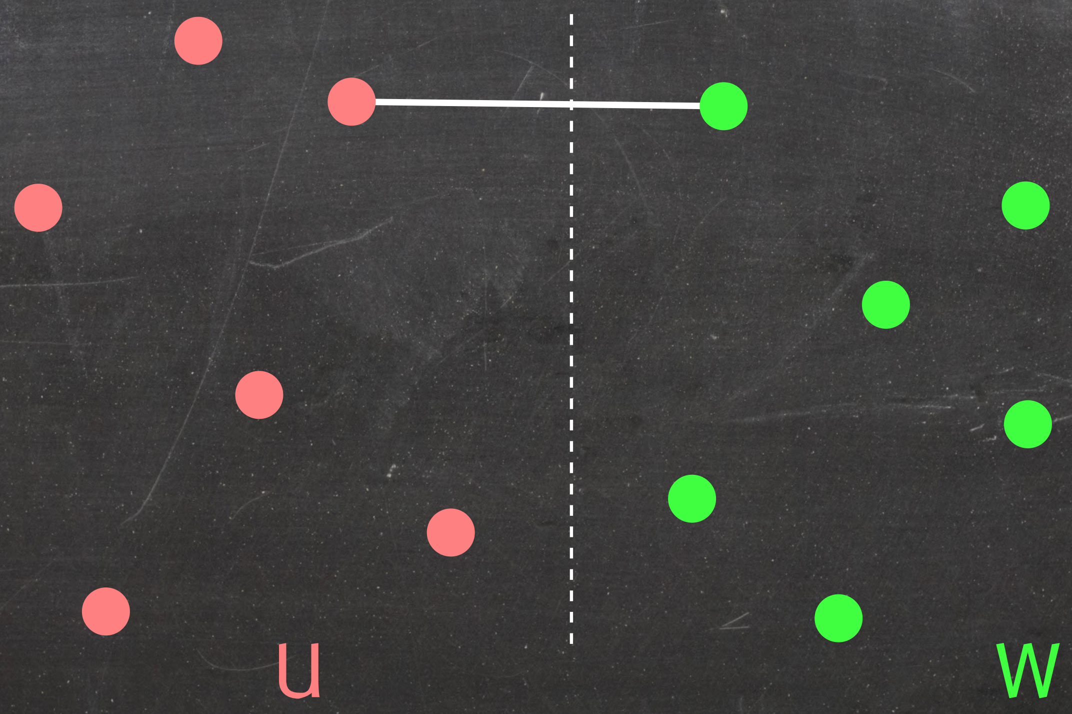

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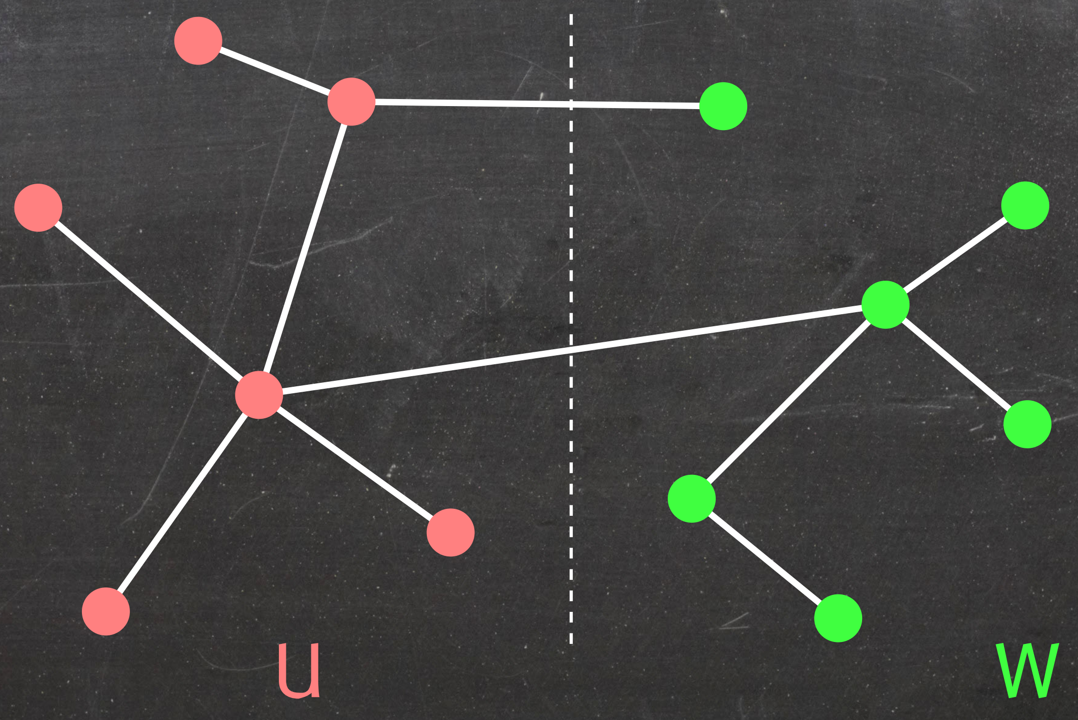
**Theorem:** Let T be a minimum spanning tree, let (U, W) be an arbitrary cut, and let e be the cheapest edge crossing the cut. Then there exists a minimum spanning tree that contains e and all edges of T that do not cross the cut.



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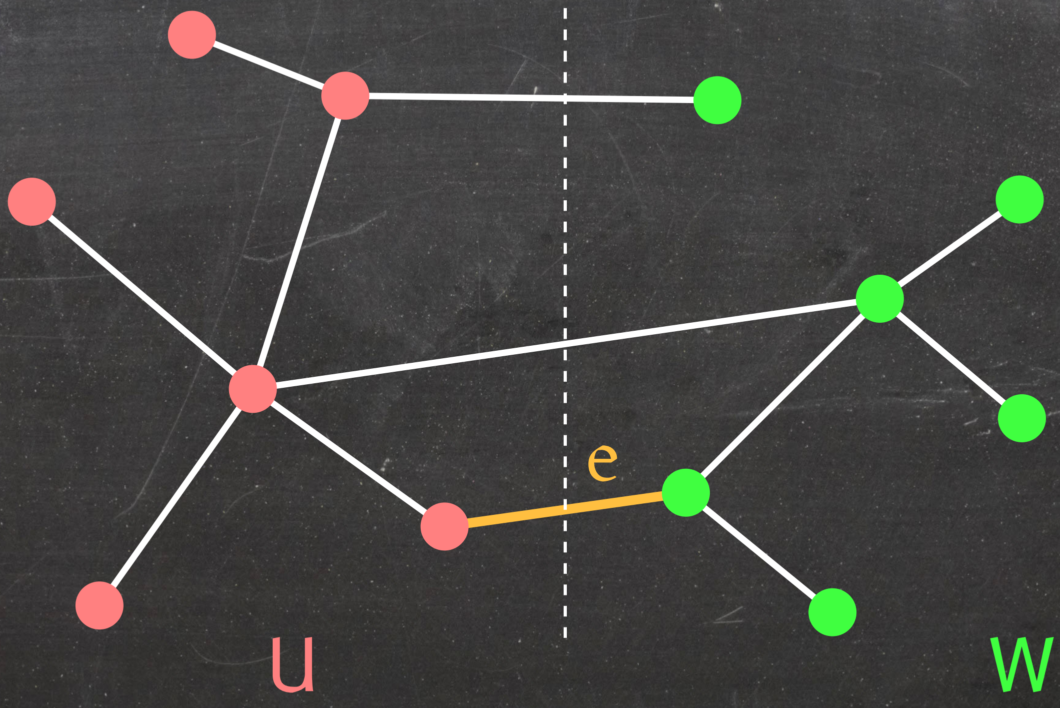

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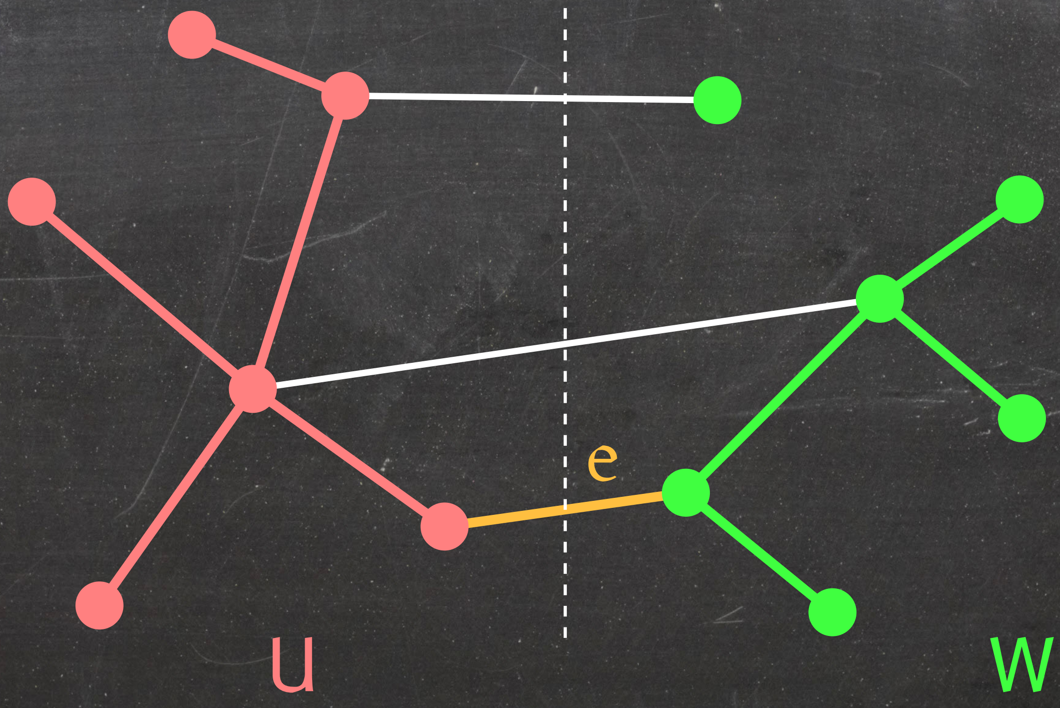

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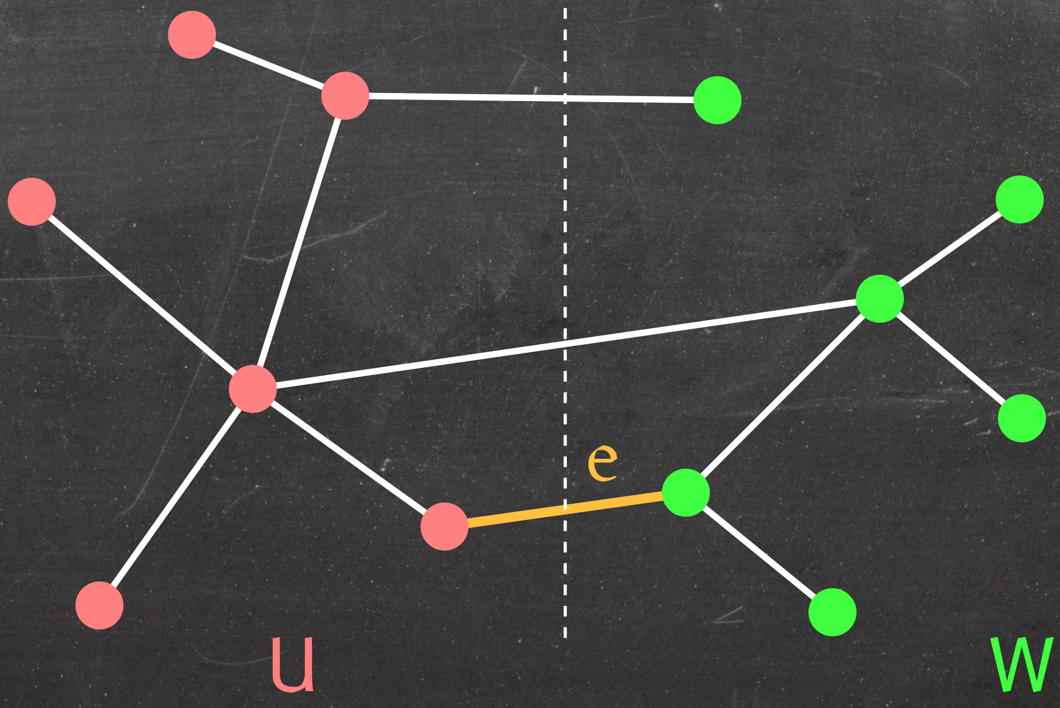

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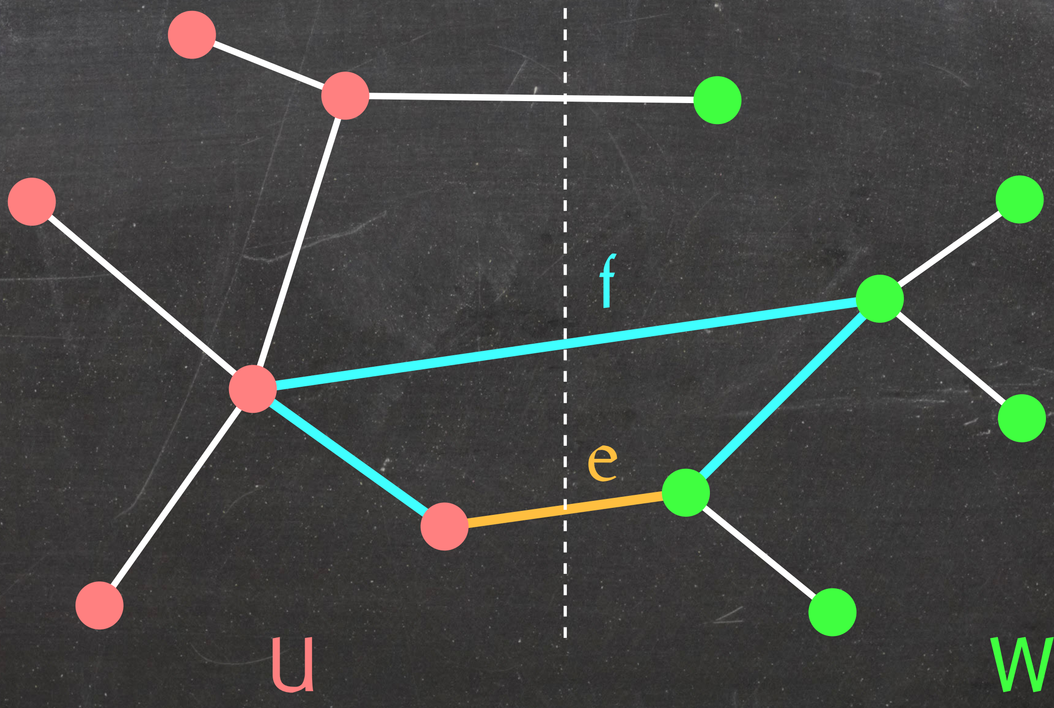

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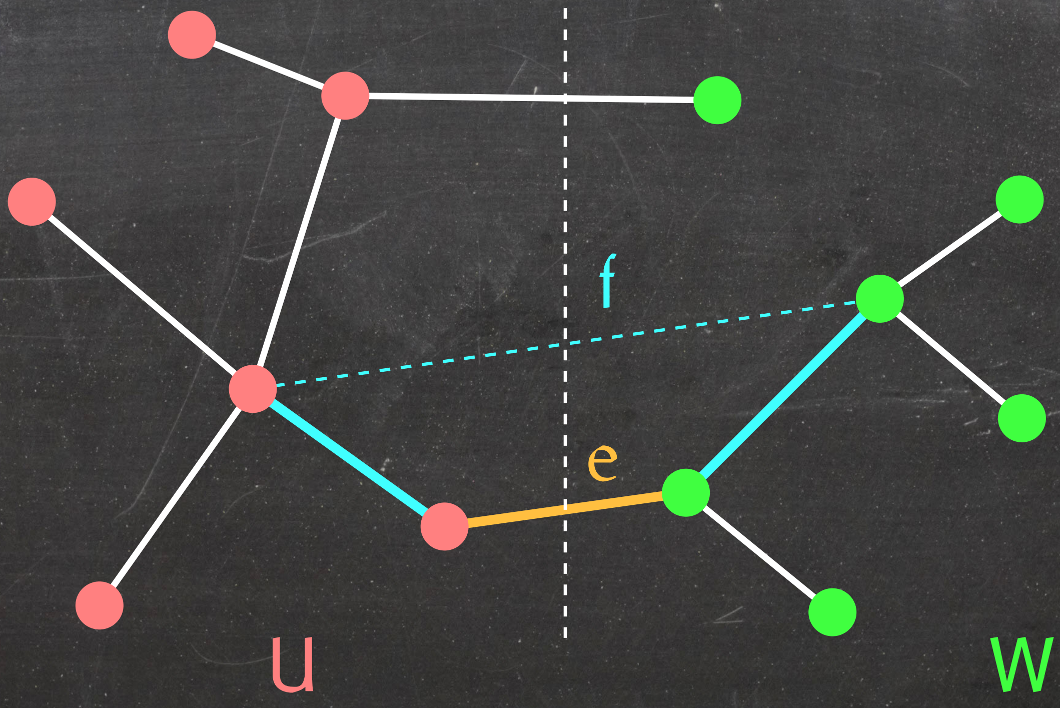

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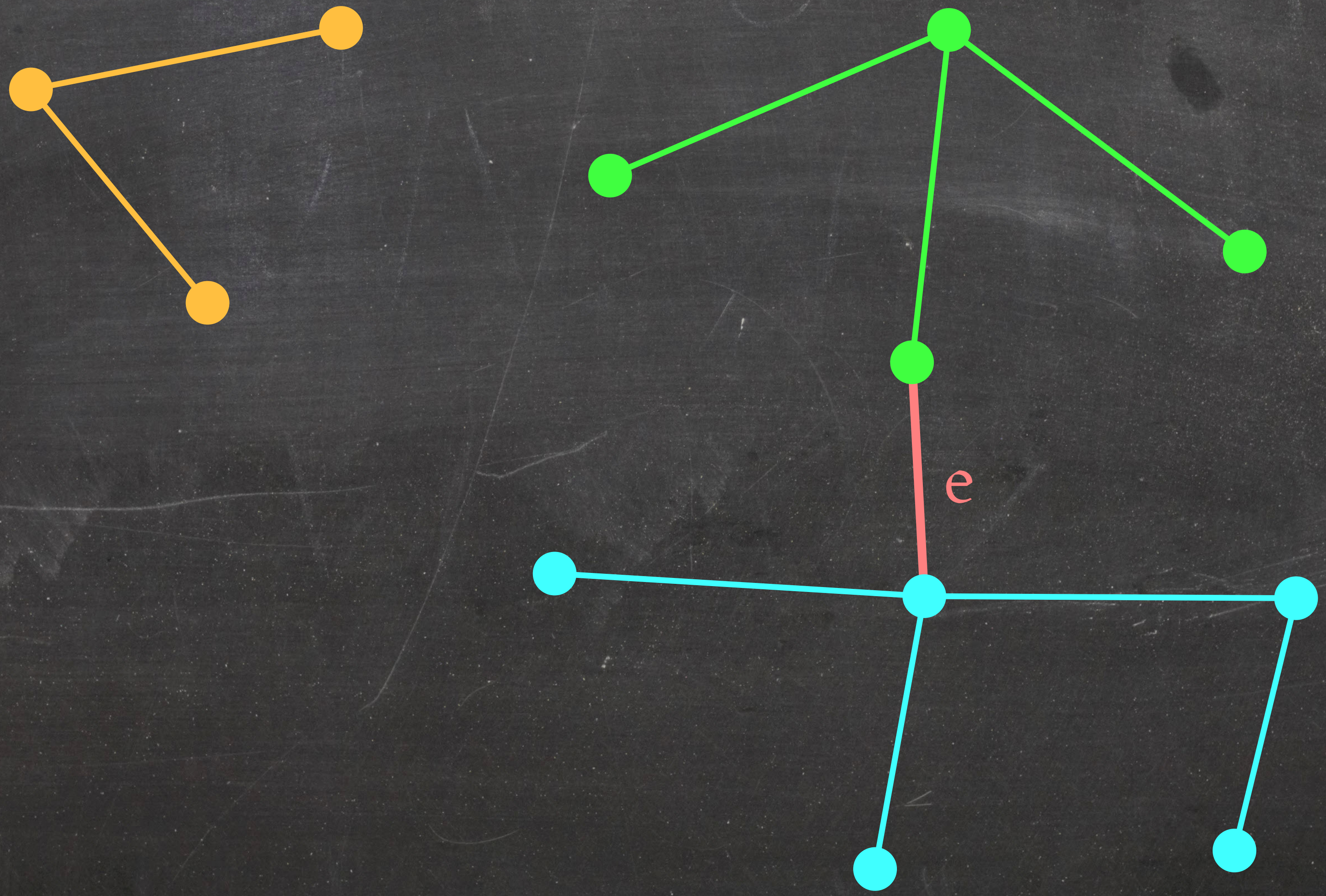

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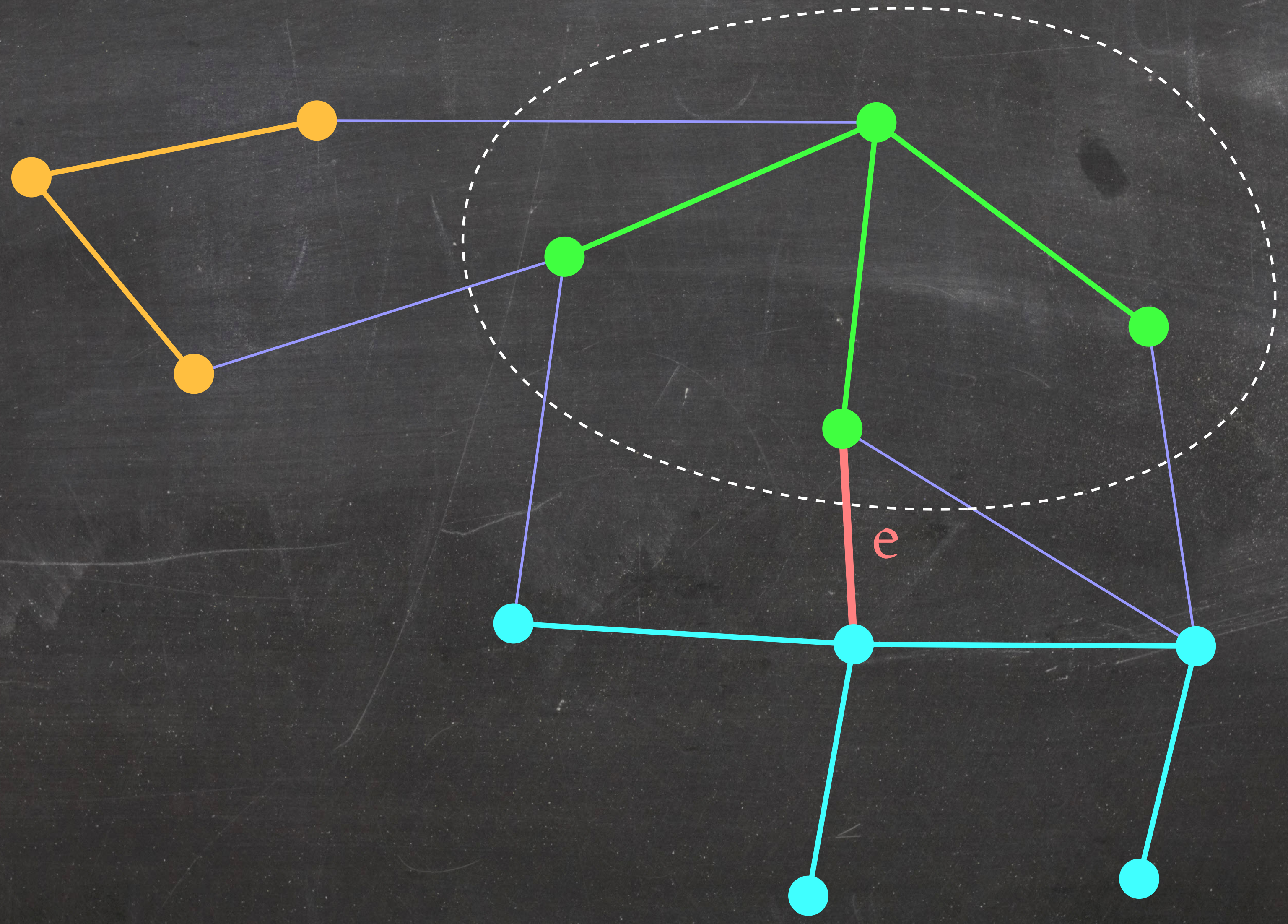

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for every edge $(v, w)$ of $G$, in sorted order
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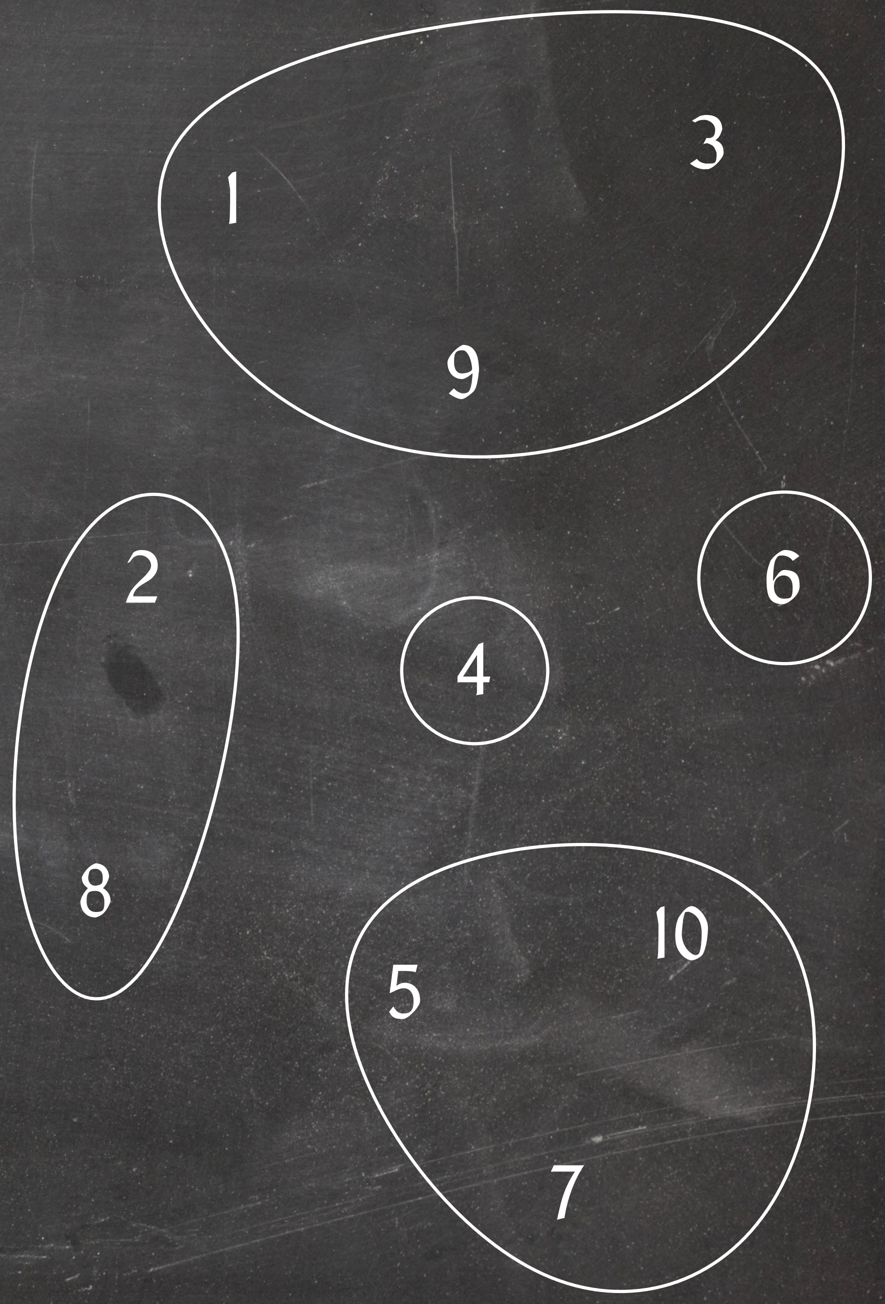
# A Union-Find Data Structure

Given a set $S$ of elements, maintain a partition of $S$ into subsets $S_1, S_2, \ldots, S_k$.



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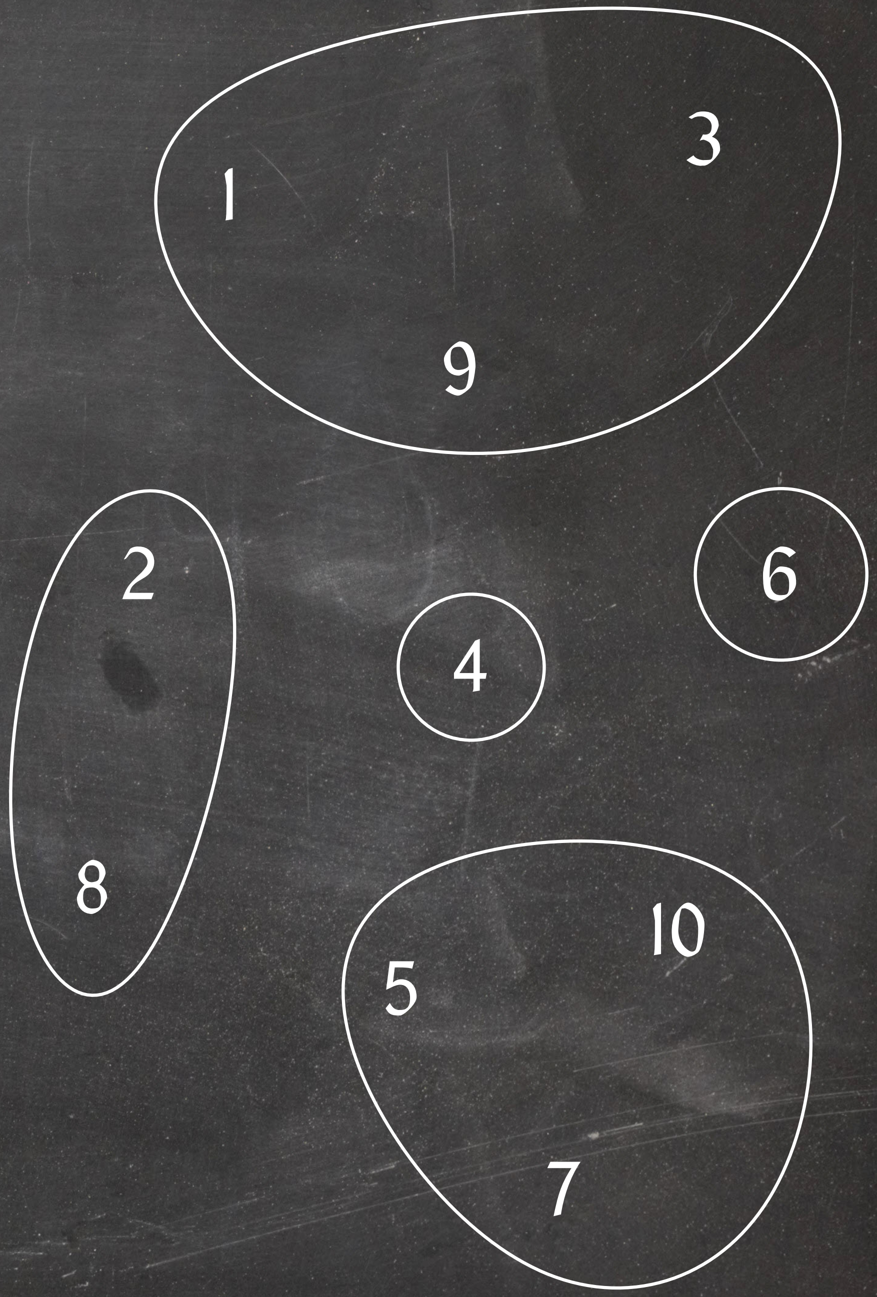

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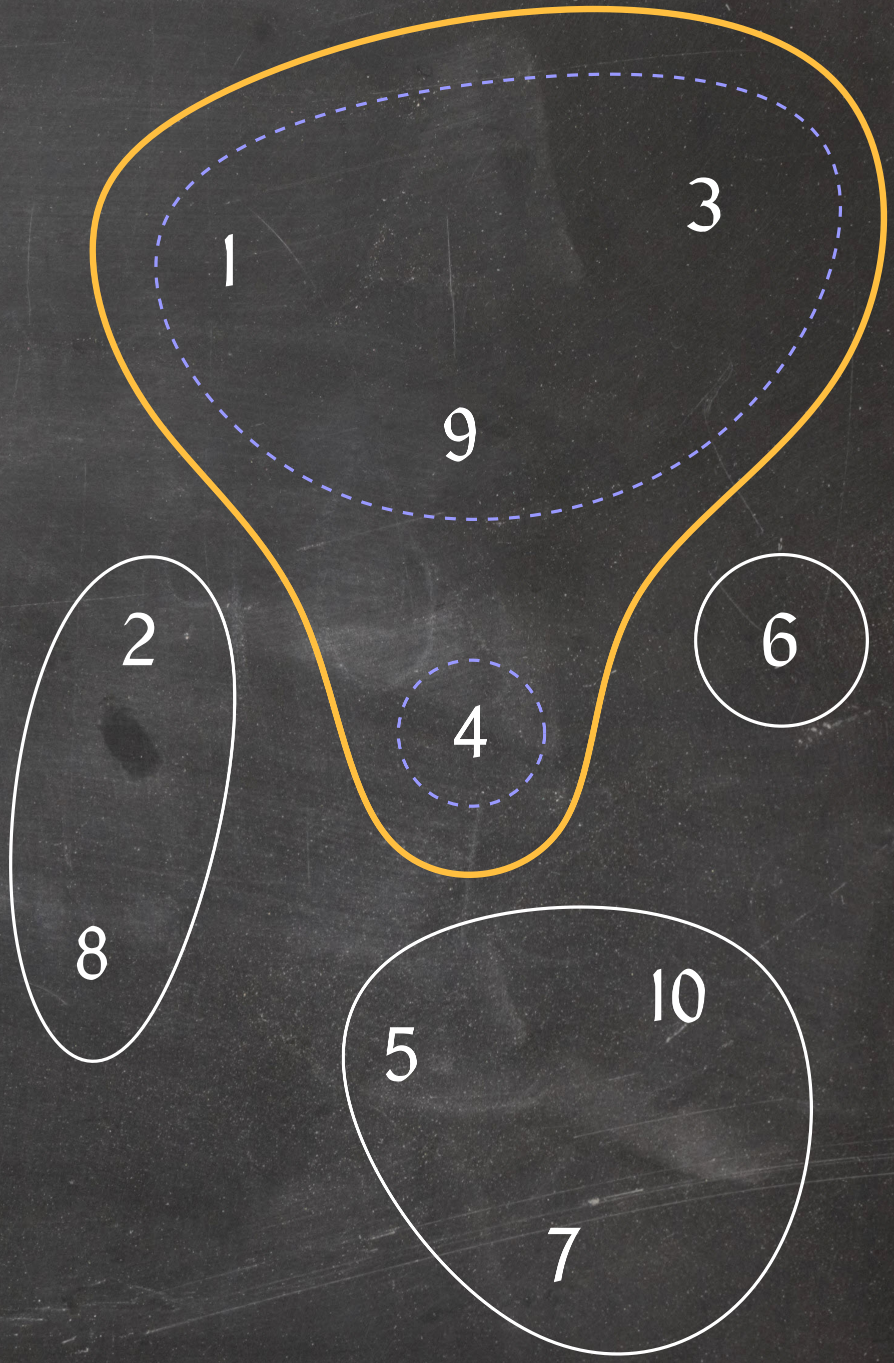

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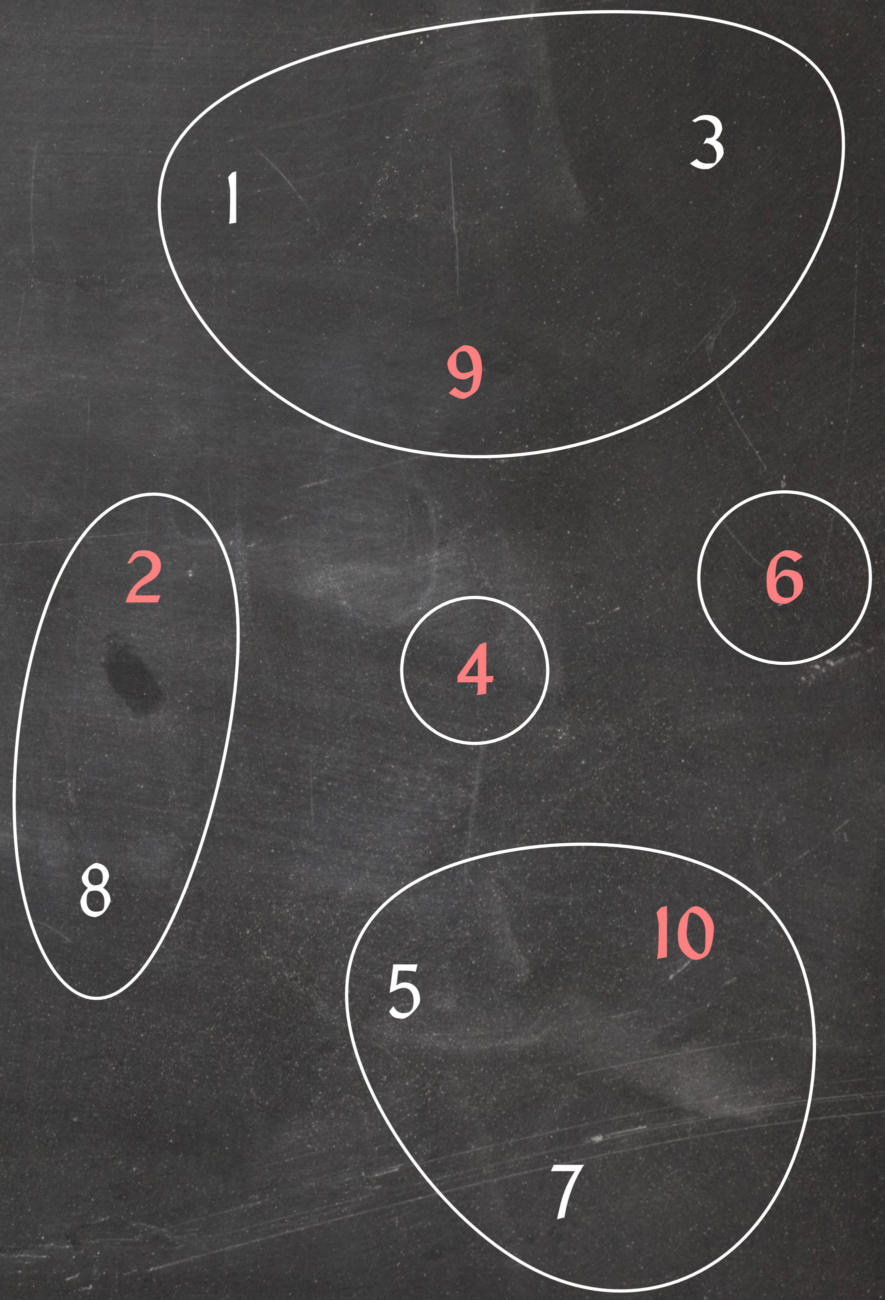

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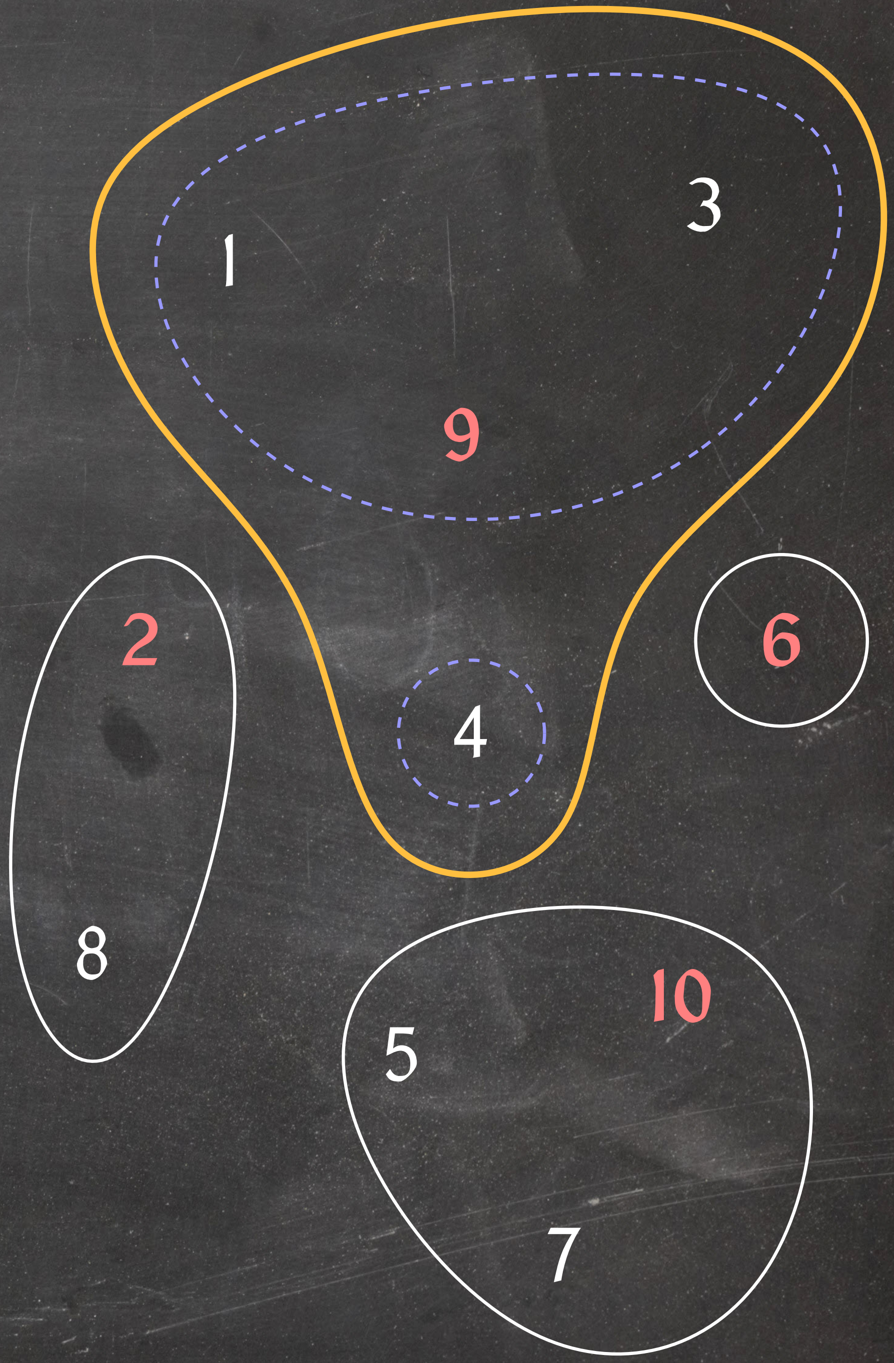


# A Union-Find Data Structure

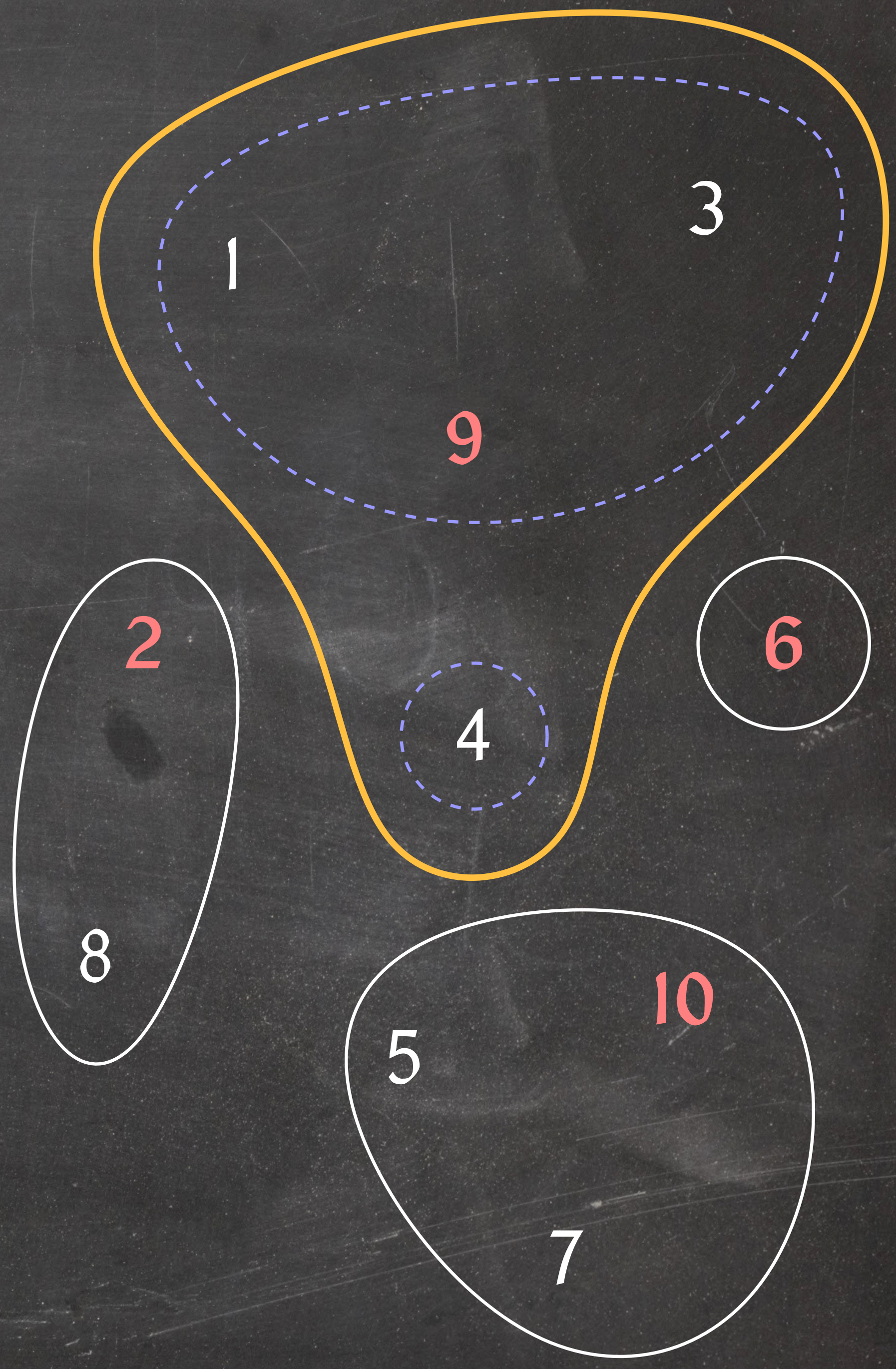
Given a set $S$ of elements, maintain a partition of $S$ into subsets $S_1, S_2, \ldots, S_k$.

Support the following operations:

**Union(x, y):** Replace sets $S_i$ and $S_j$ in the partition with $S_i \cup S_j$, where $x \in S_i$ and $y \in S_j$.

**Find(x):** Return a representative $r(S_i) \in S_i$ of the set $S_i$ that contains x.

In particular, Find(x) = Find(y) if and only if x and y belong to the same set.



# Kruskal's Algorithm Using Union-Find

**Idea:** Maintain a partition of V into the vertex sets of the connected components of T.

```
Kruskal(G)
T = (V, $\emptyset$)
initialize a union-find structure D for V with every vertex v $\in$ V in its own set
sort the edges in G by increasing weight
for every edge (v, w) of G, in sorted order
    do if D.find(v) $\neq$ D.find(w)
          then add (v, w) to T
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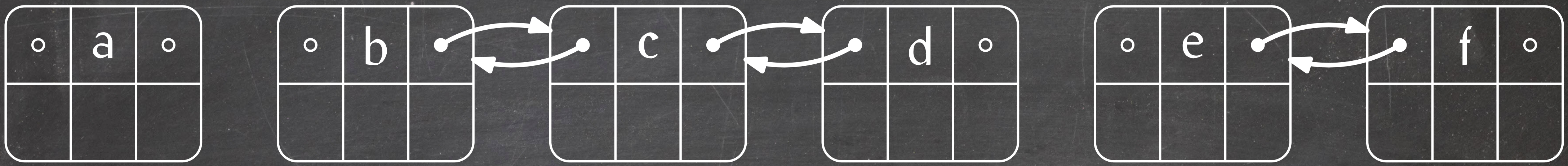
**Lemma:** Kruskal's algorithm takes $O(m \lg m)$ time plus the cost of 2m Find and $n - 1$ Union operations.

# A Simple Union-Find Structure



**List node:**

- A set element
- Pointers to predecessor and successor
- Pointer to head of the list
- Pointer to tail of the list (only valid for head node)
- Size of the list (only valid for head node)

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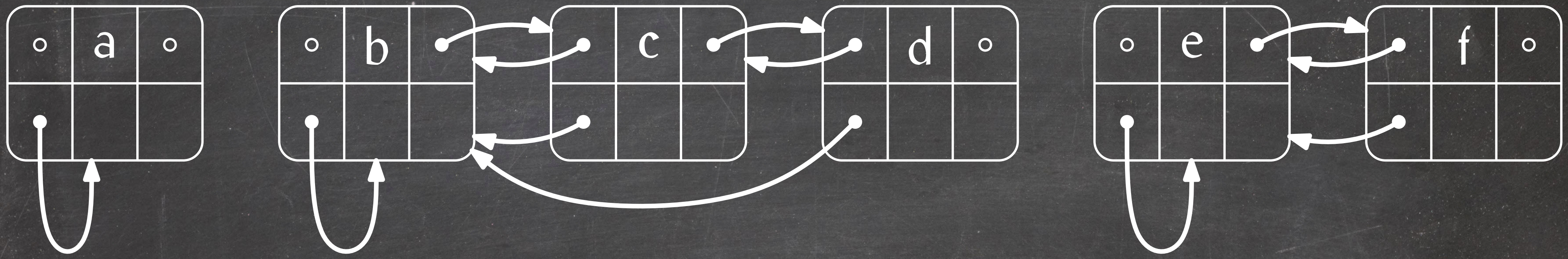

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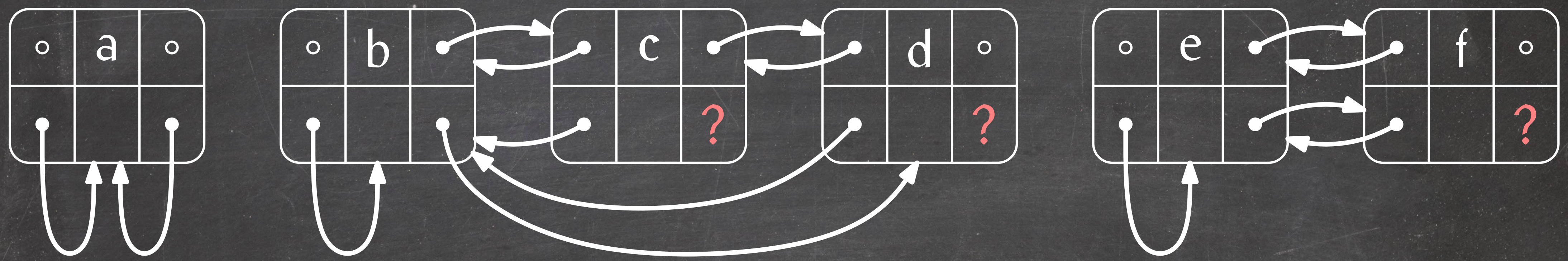

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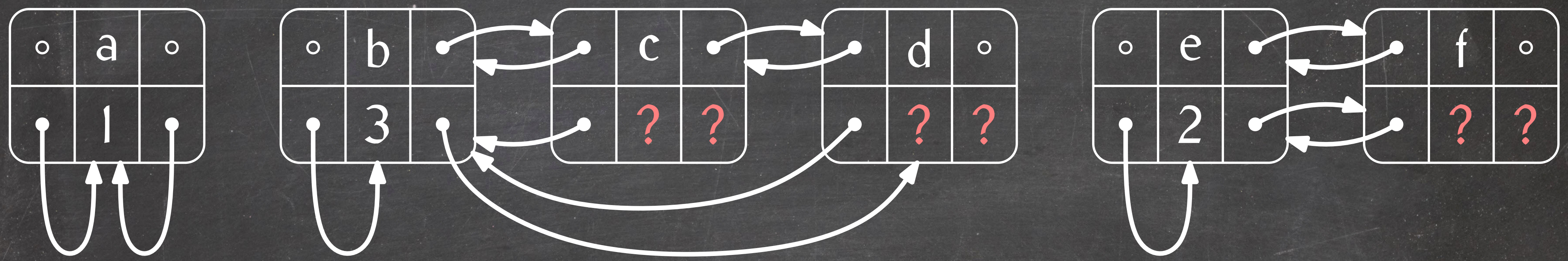

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# Find

```
D.find(x)
    return x.head.key
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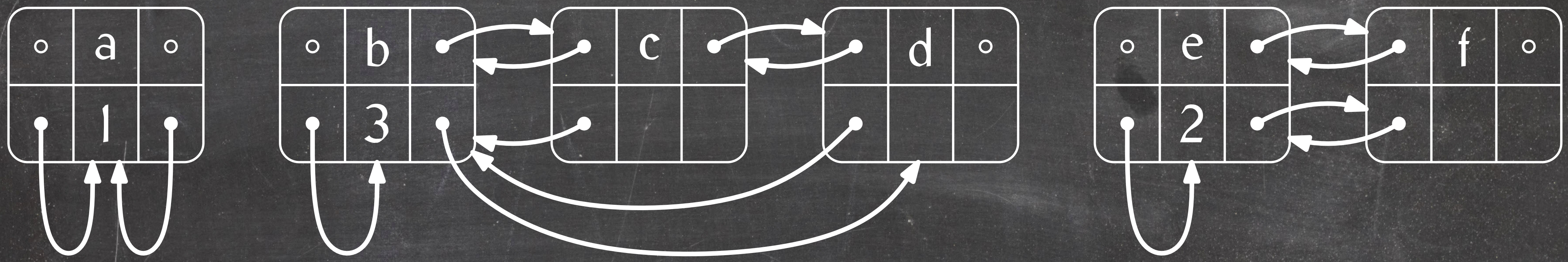

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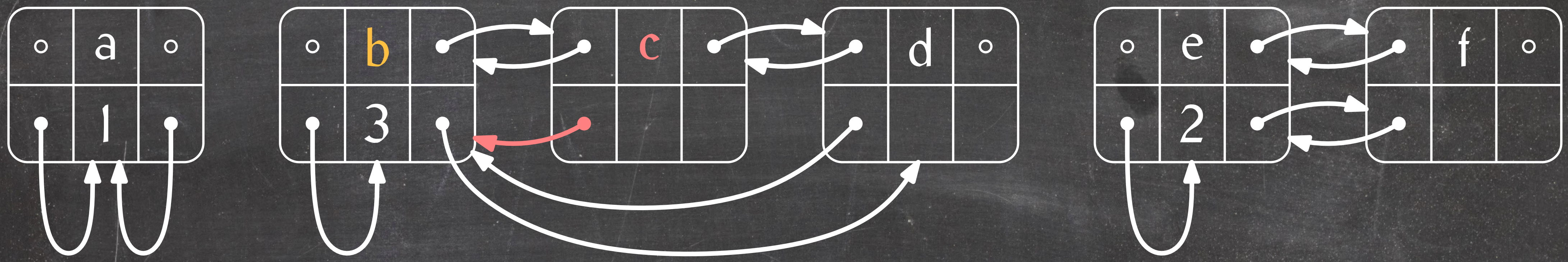

D.find(c) = b

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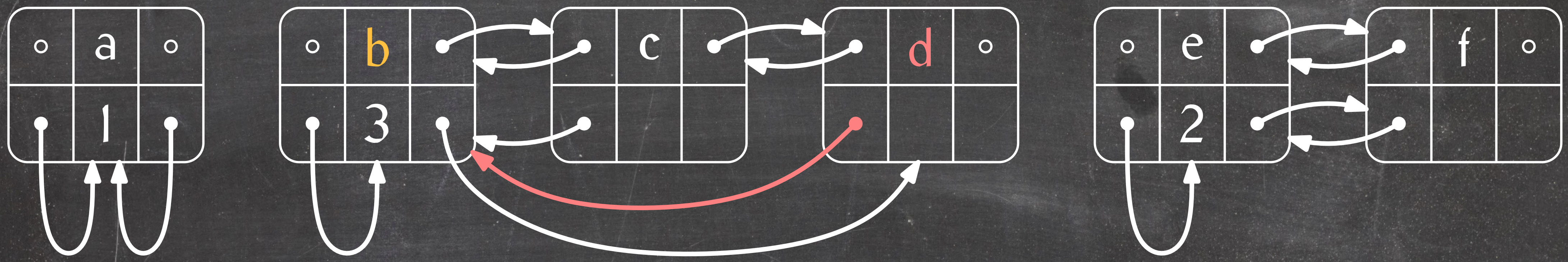

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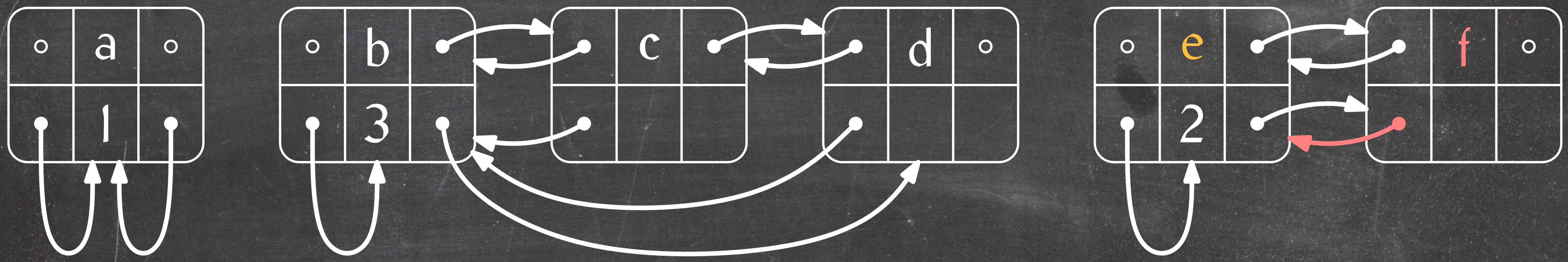

D.find(c) = b

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D.find(e) = e

# Union

## D.union(x, y)

```
if x.head.listSize < y.head.listSize
    then swap x and y
y.head.pred = x.head.tail
x.head.tail.succ = y.head
x.head.listSize = x.head.listSize + y.head.listSize
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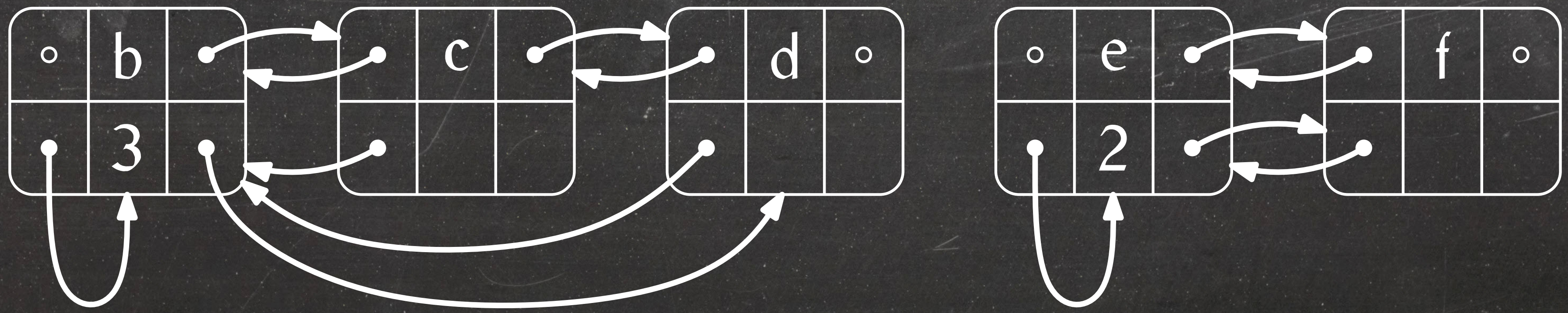

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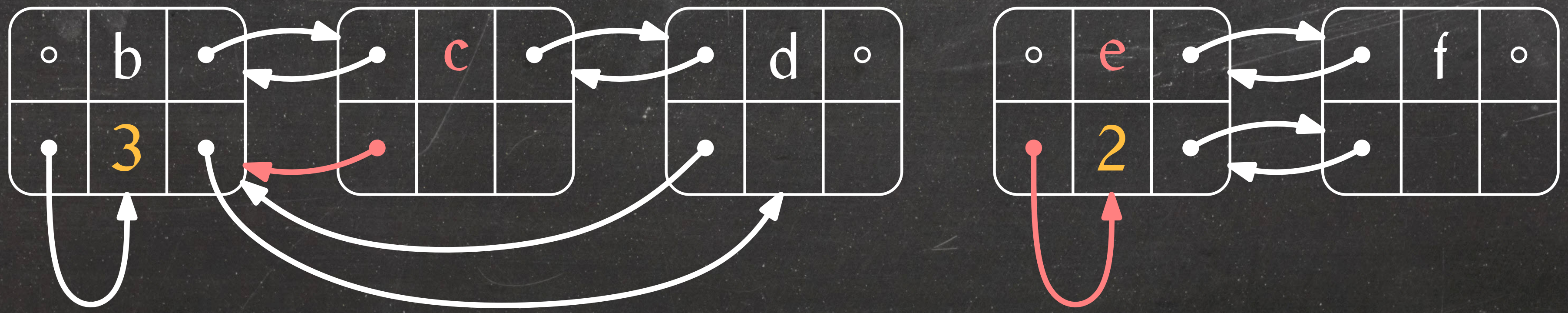

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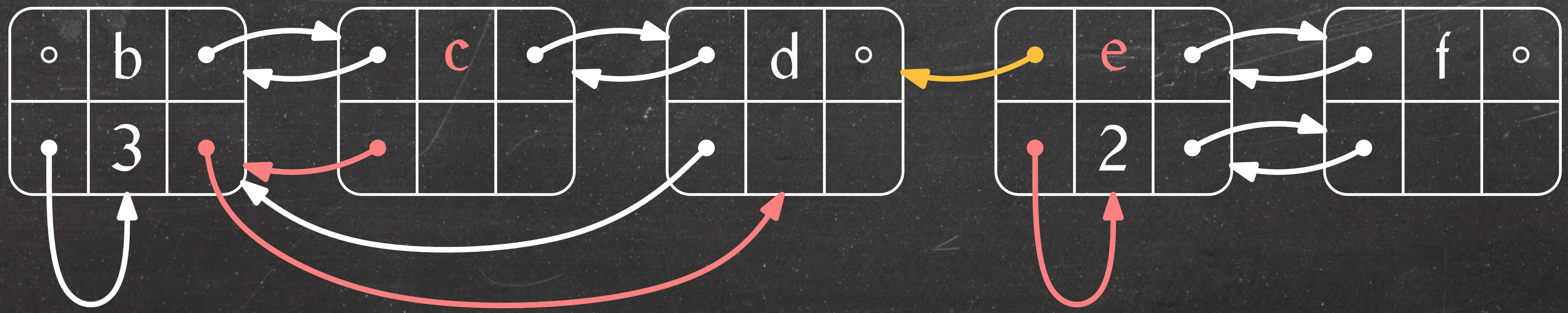

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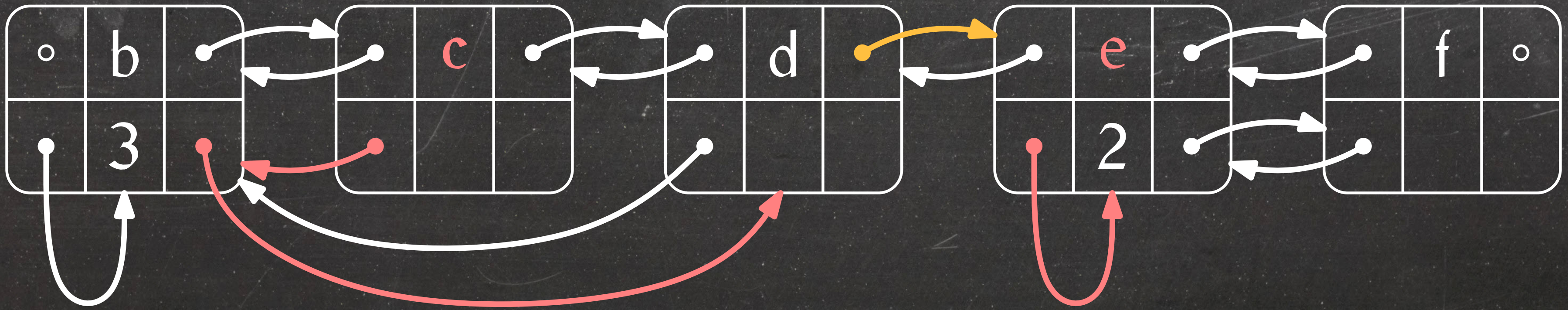

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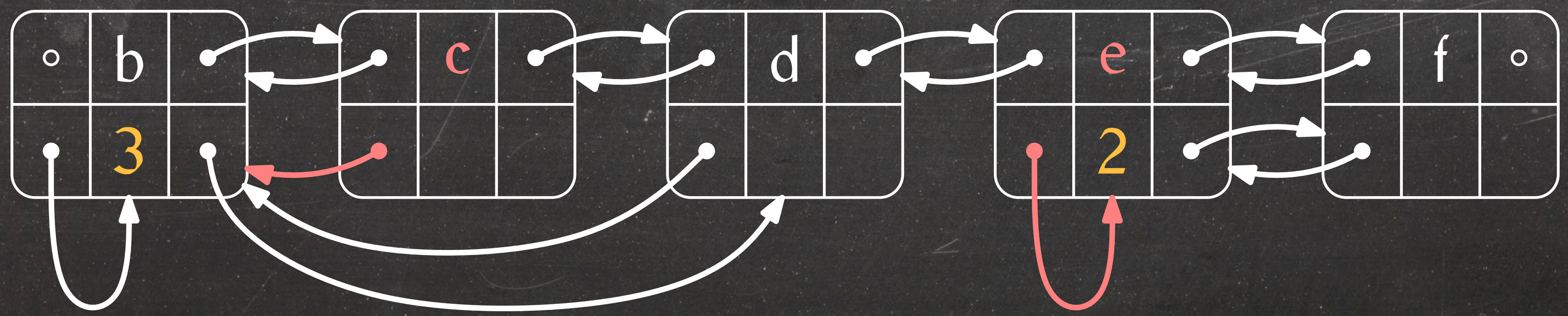

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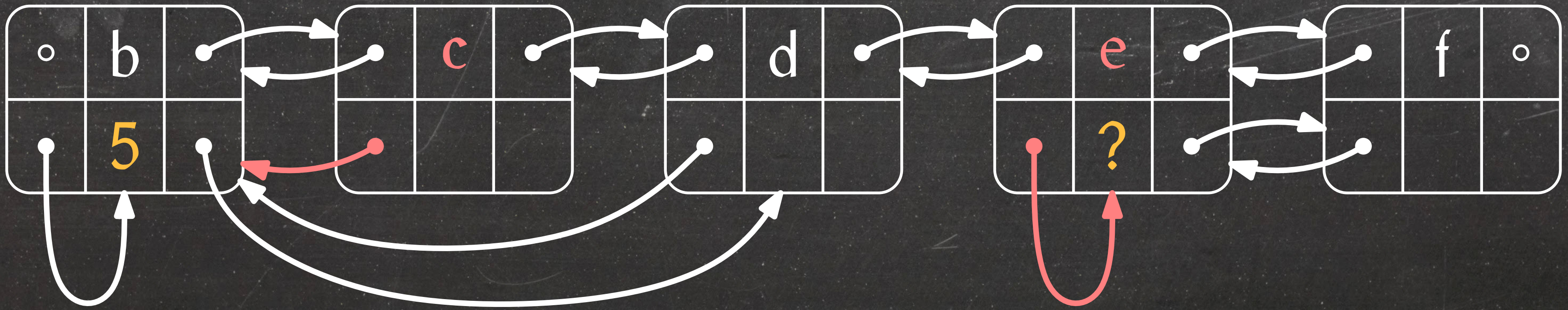

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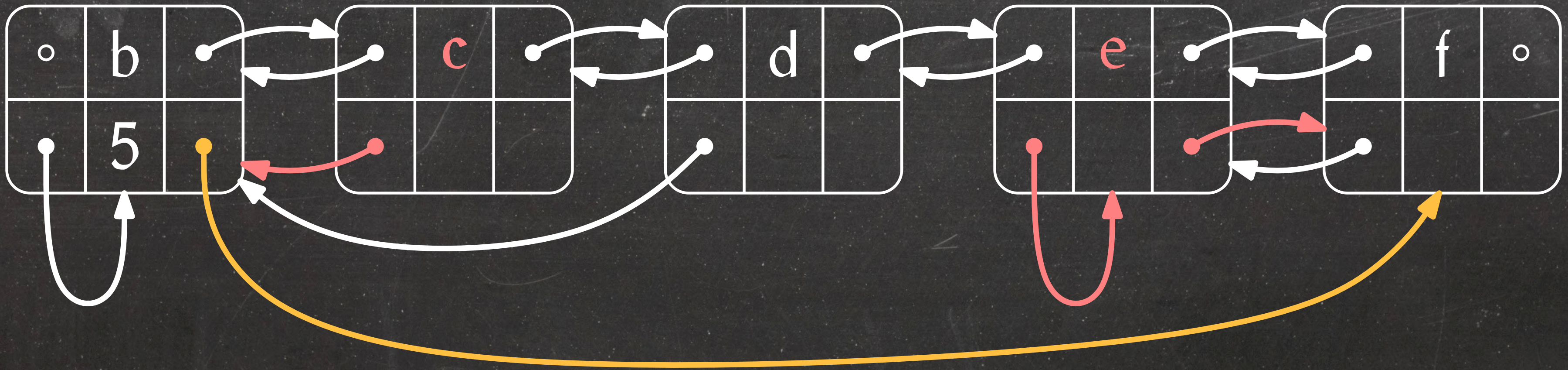

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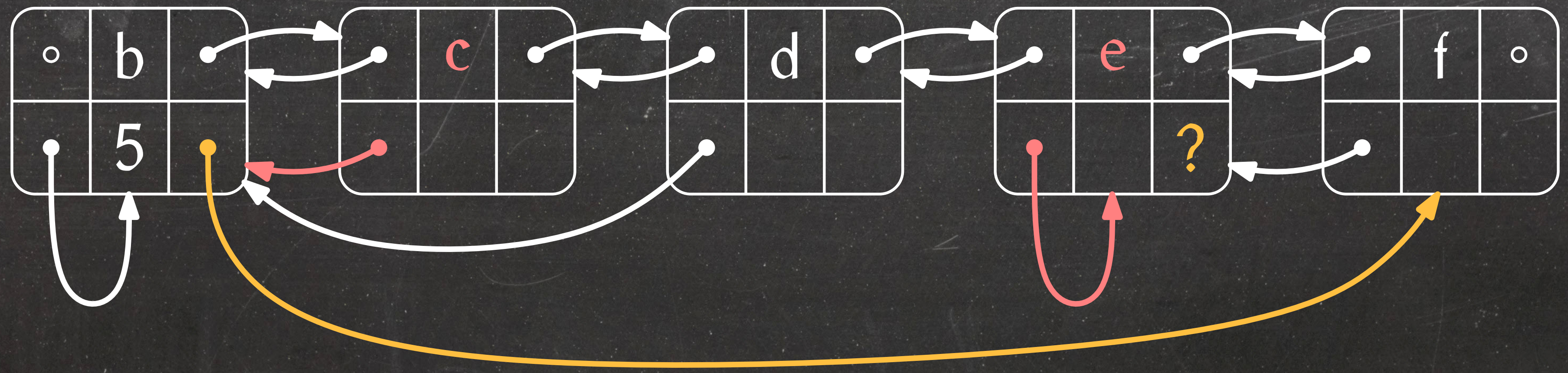

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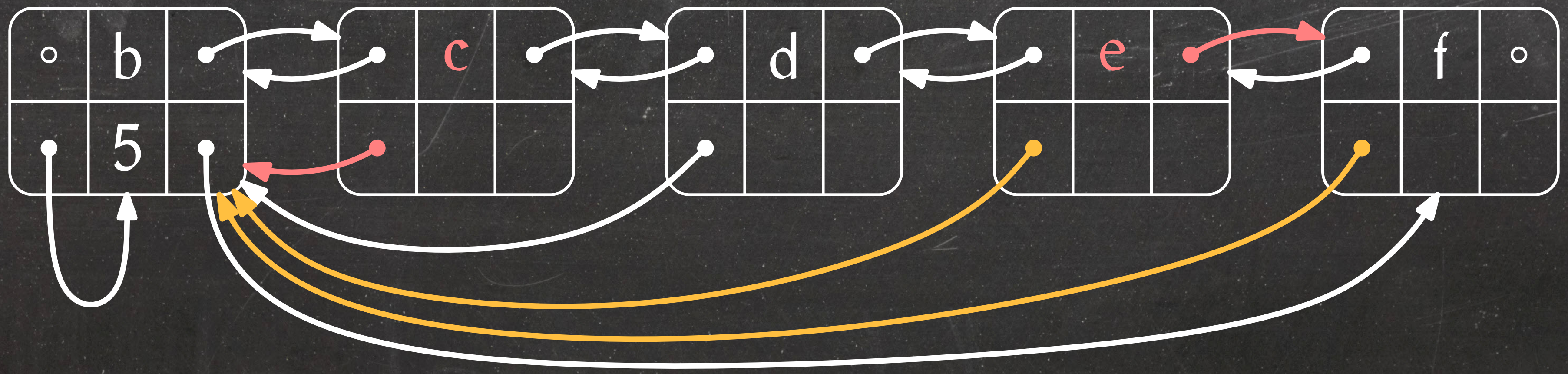

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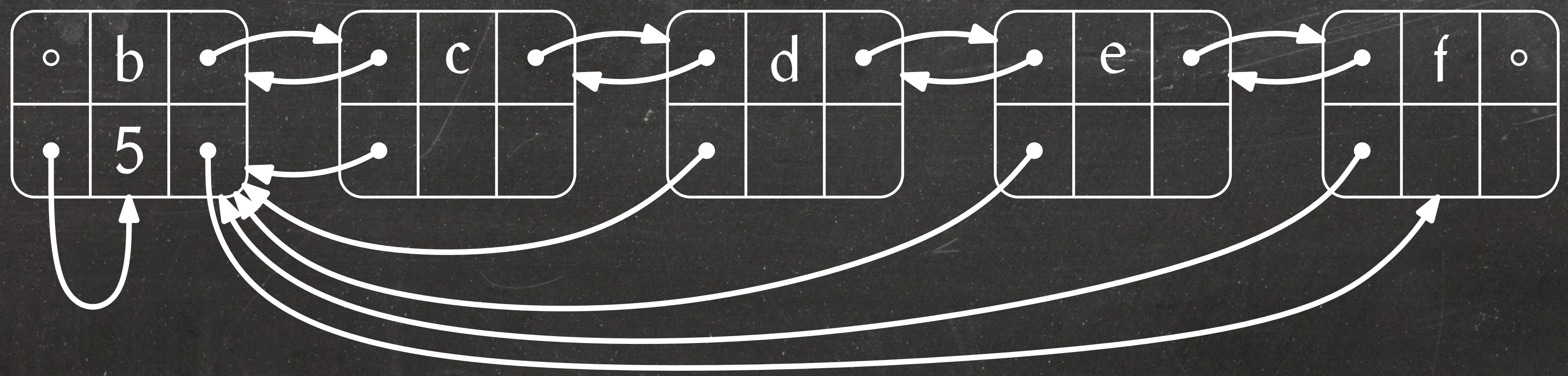

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**Corollary:** $c(x) \leq \lg n$ for all $x \in S$.

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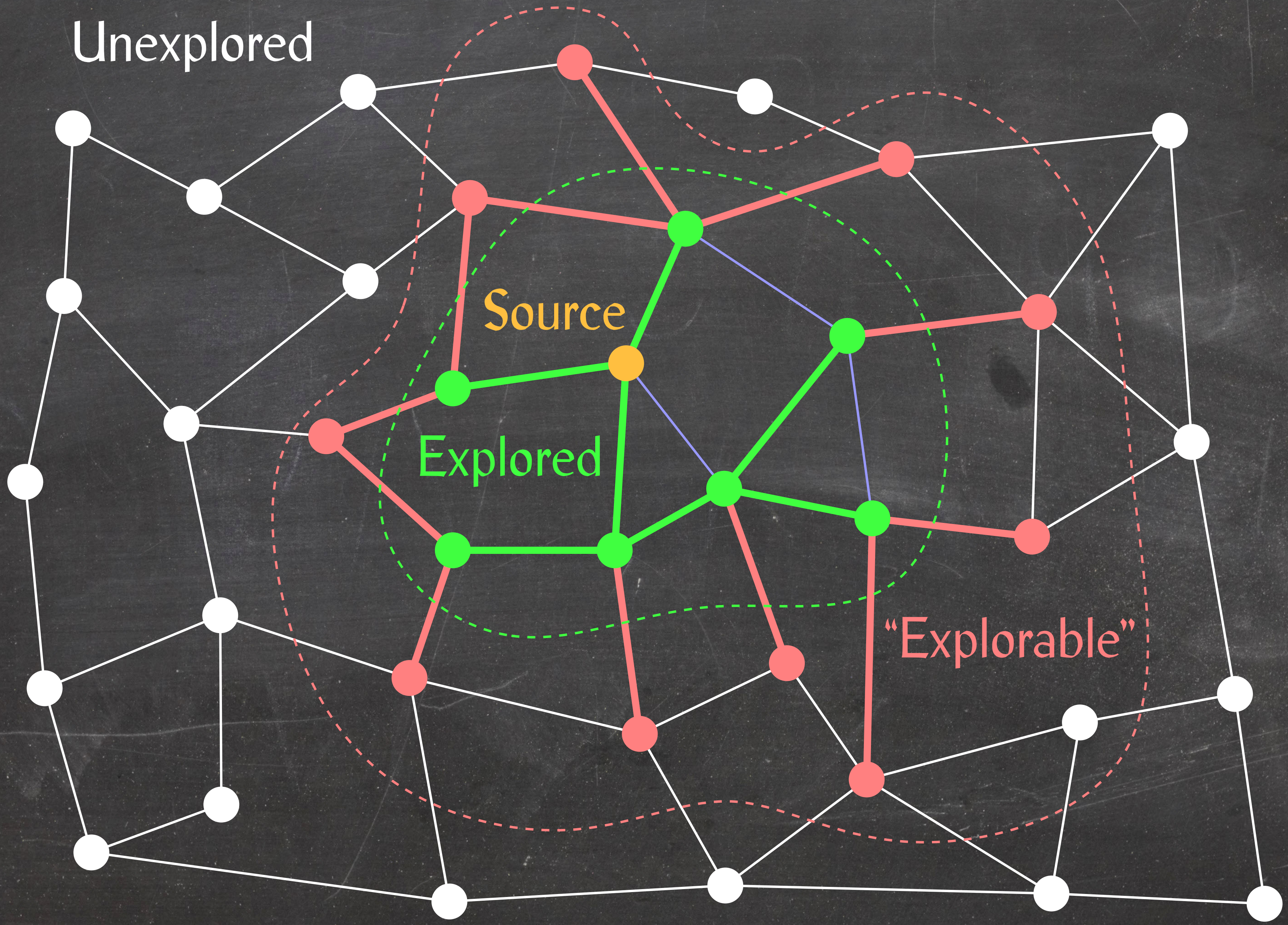
**Corollary:** Kruskal's algorithm takes $O(n \lg n + m \lg m)$ time.

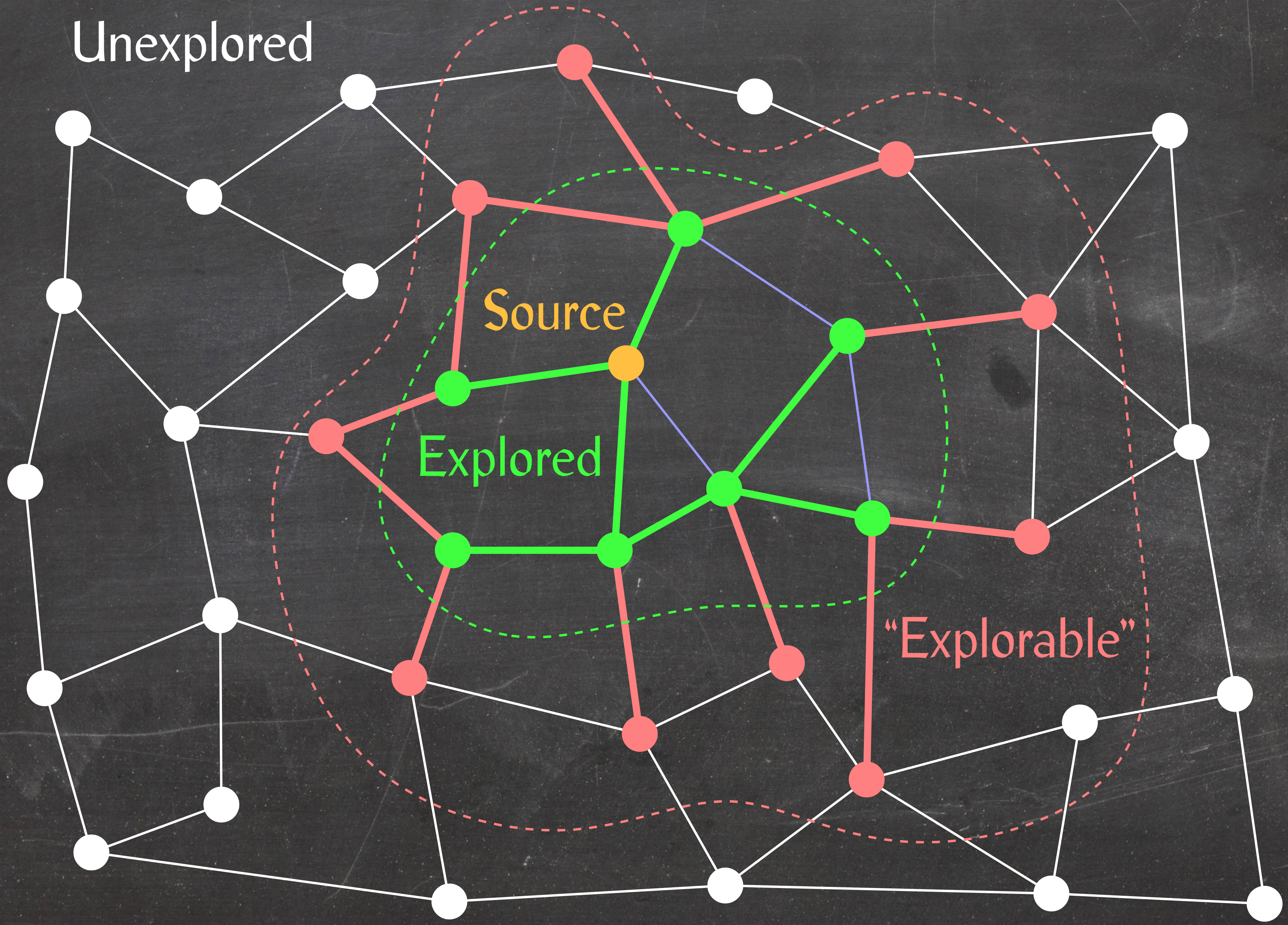
If the graph is connected, then $m \geq n - 1$, so the running time simplifies to $O(m \lg m)$.

# The Cut Theorem And Graph Traversal



# The Cut Theorem And Graph Traversal

If there exists an MST containing all green edges, then there exists an MST containing all green edges and the cheapest red edge.



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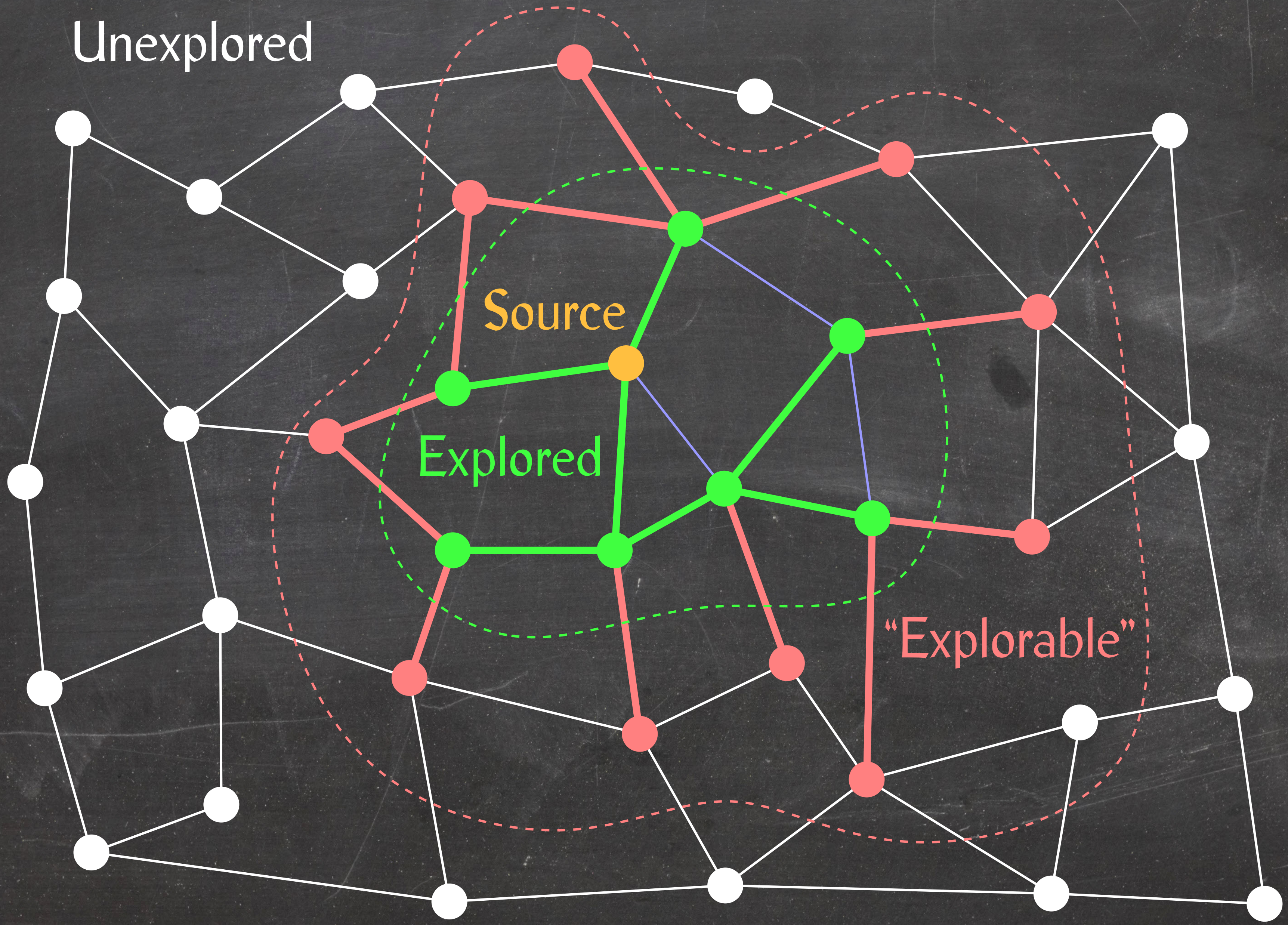

**Cut:** U = explored vertices, W = V \ U

# Prim's Algorithm

**Prim($G$)**

```
T = (V, ∅)
mark all vertices of G as unexplored
mark an arbitrary vertex s as explored
while not all vertices are explored
    do pick the cheapest edge e with exactly one unexplored endpoint v
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Once T is connected, we have $T^* = T$.

# The Abstract Data Type Priority Queue

**Operations:**

| | |
|---|---|
| Q.insert(x, p): | Insert element x with priority p |
| Q.delete(x): | Delete element x |
| Q.findMin(): | Find and return the element with minimum priority |
| Q.deleteMin(): | Delete the element with minimum priority and return it |
| Q.decreaseKey(x, p): | Change the priority $p_x$ of x to min(p, $p_x$) |

Delete and DecreaseKey assume they're given a pointer to the place in Q where x is stored.

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**Example:** A binary heap is a priority queue supporting all operations in $O(\lg |Q|)$ time.

# Prim's Algorithm Using A Priority Queue

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Prim(G)
    T = (V, ∅)
    mark every vertex of G as unexplored
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    Q = an empty priority queue
    for every edge (s, v) incident to s
        do Q.insert((s, v), w(s, v))
    while not Q.isEmpty()
        do (u, v) = Q.deleteMin()
            if v is unexplored
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This version of Prim's algorithm takes $O(m \lg m)$ time:

# Prim's Algorithm Using A Priority Queue

**Prim(G)**

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mark every vertex of G as unexplored
mark an arbitrary vertex s as explored
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while not Q.isEmpty()
    do (u, v) = Q.deleteMin()
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            then mark v as explored
                add edge (u, v) to T
                for every edge (v, w) incident to v
                    do Q.insert((v, w), w(v, w))
return T
```

**Invariant:** Q contains all edges with exactly one unexplored endpoint.

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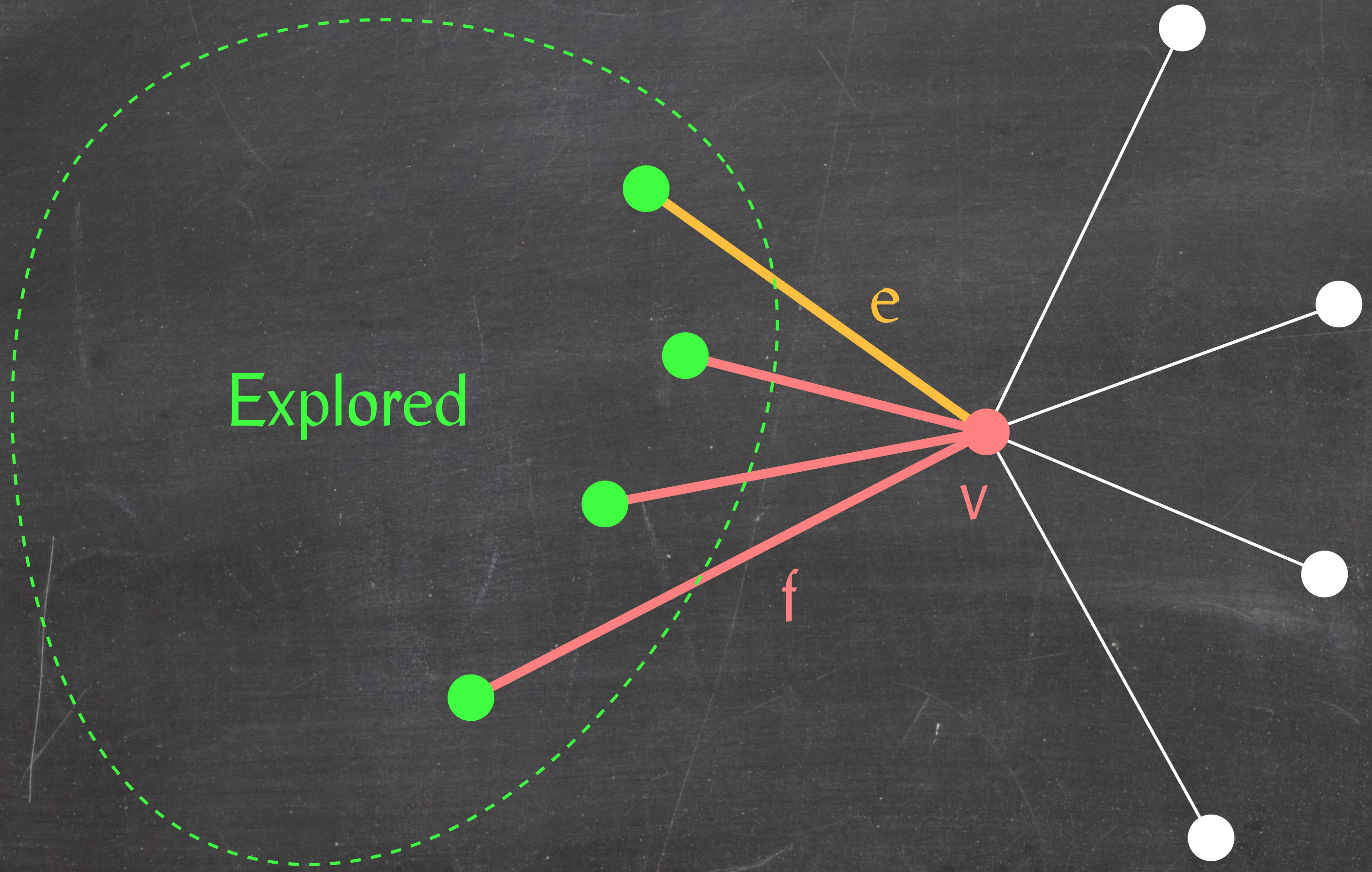
⇒ 2m priority queue operations.

# Most Edges In Q Are Useless

**Observation:** Of all the edges connecting an unexplored vertex to explored vertices only the cheapest has a chance of being added to the MST.



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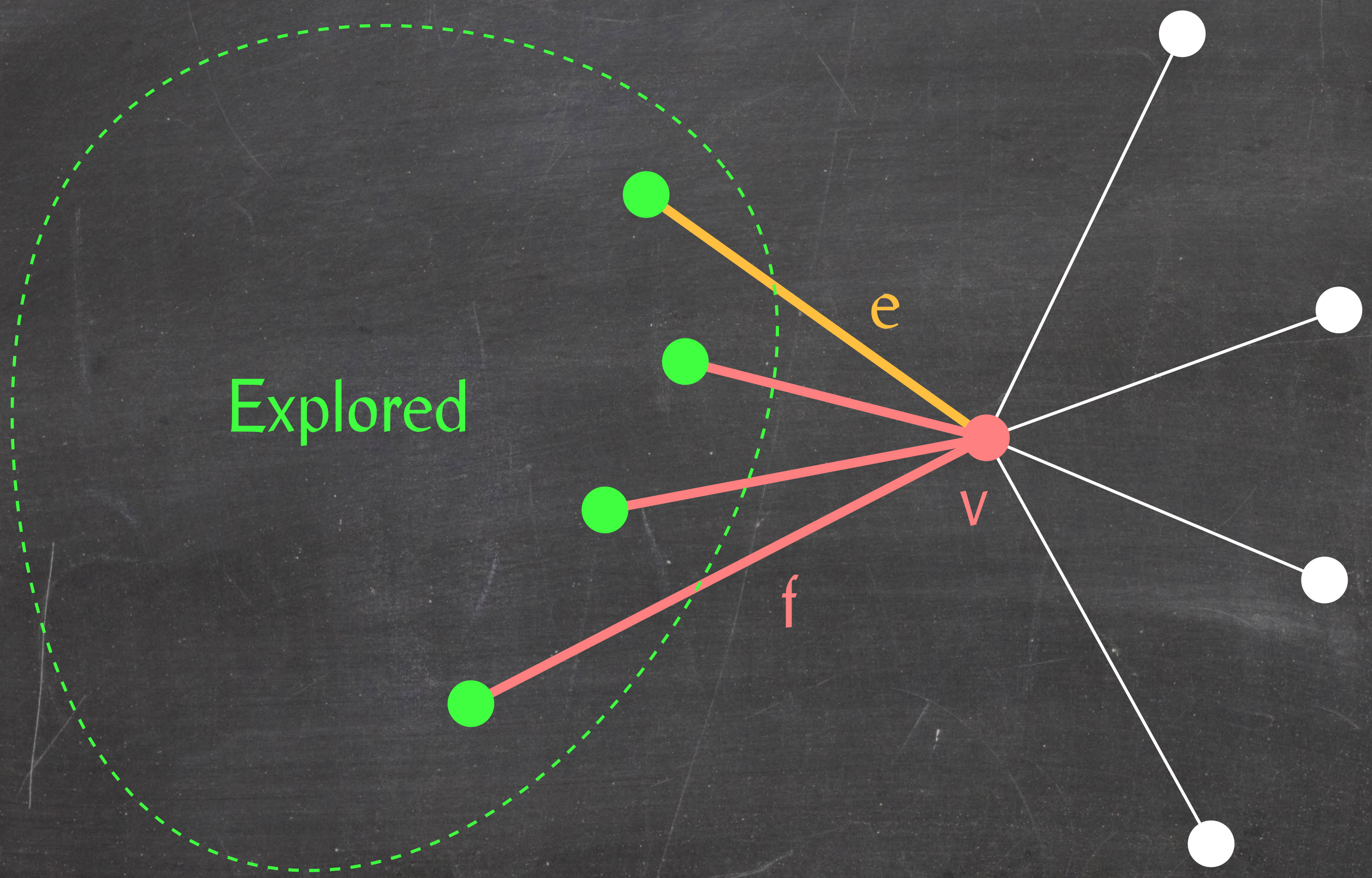

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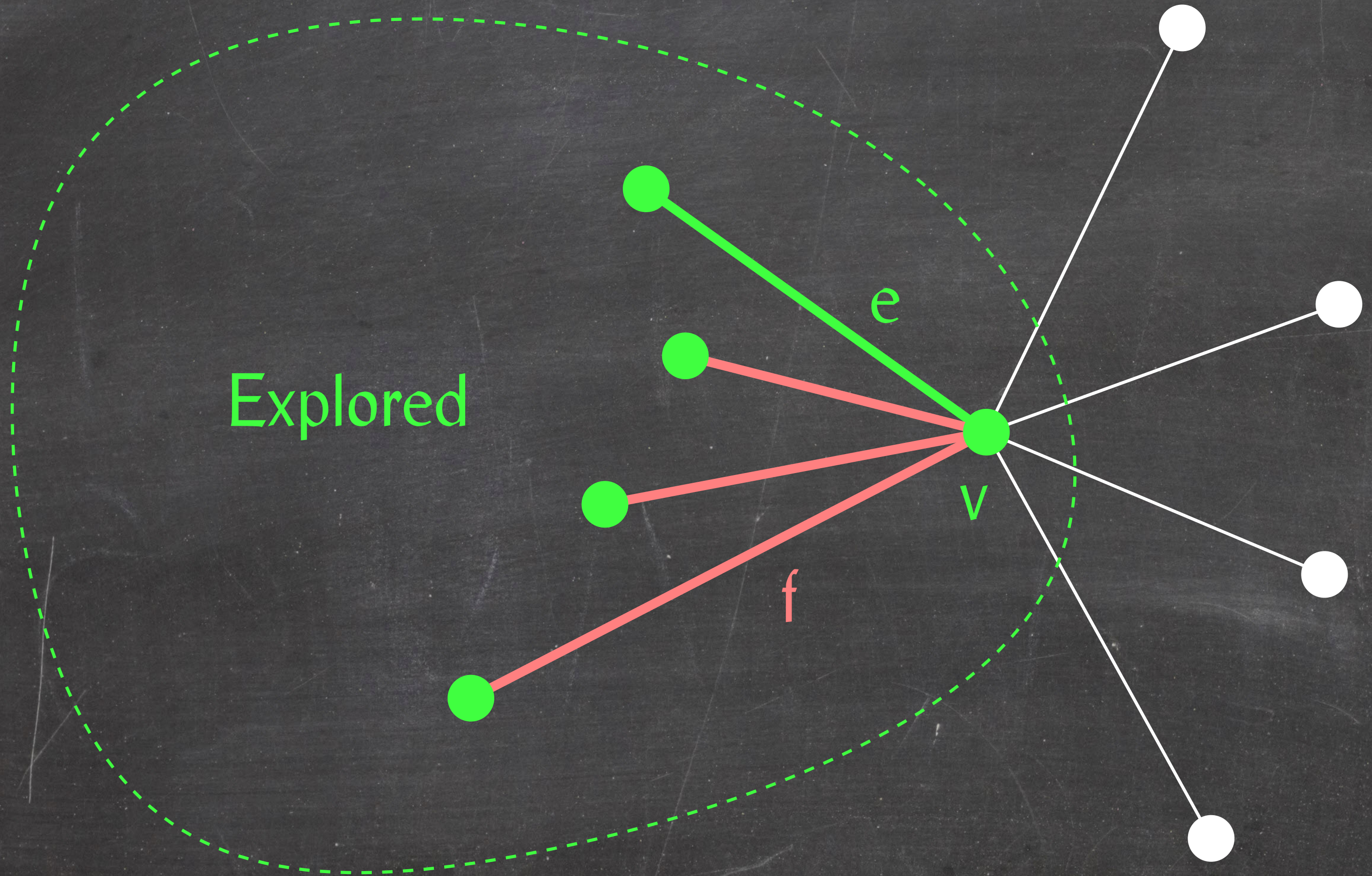

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While v is unexplored, all red and orange edges are in Q, so none of the red edges can be the first edge to be removed from Q.

After marking v as explored, both endpoints of red edges are explored, so they cannot be added to T either.

# A Faster Version Of Prim's Algorithm

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Prim(G)
  T = (V, ∅)
  mark every vertex of G as unexplored
  set e(v) = nil for every vertex $v \in G$
  mark an arbitrary vertex s as explored
  Q = an empty priority queue
  for every edge (s, v) incident to s
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         e(v) = (s, v)
  while not Q.isEmpty()
      do u = Q.deleteMin()
         mark u as explored
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         for every edge (u, v) incident to u
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This version of Prim's algorithm also takes $O(m \lg m)$ time:

- n Insert operations

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    do $u$ = Q.deleteMin()
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       add $e(u)$ to $T$
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                 then if $v \notin Q$
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                         else Q.decreaseKey($v$, $w(u, v)$)
                      $e(v) = (u, v)$
return $T$
```

This version of Prim's algorithm also takes $O(m \lg m)$ time:

- $n$ Insert operations
- $m - n$ DecreaseKey operations
- $n$ DeleteMin operations

$\Rightarrow$ $n + m$ priority queue operations.

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The Thin Heap is a priority queue which supports

- Insert, DecreaseKey, and FindMin in O(1) time and
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- A sequence of m operations, d of them DeleteMin or Delete operations, takes O(m + d lg n) time in the worst case.

Prim's algorithm performs n + m priority queue operations, n of which are DeleteMin operations.

**Lemma:** Prim's algorithm takes O(n lg n + m) time.

# Thin Tree

A Thin Heap is built from Thin Trees. Thin Trees are defined inductively.

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Thick node: $k$ children of ranks $k-1, k-2, \ldots, 0$ or
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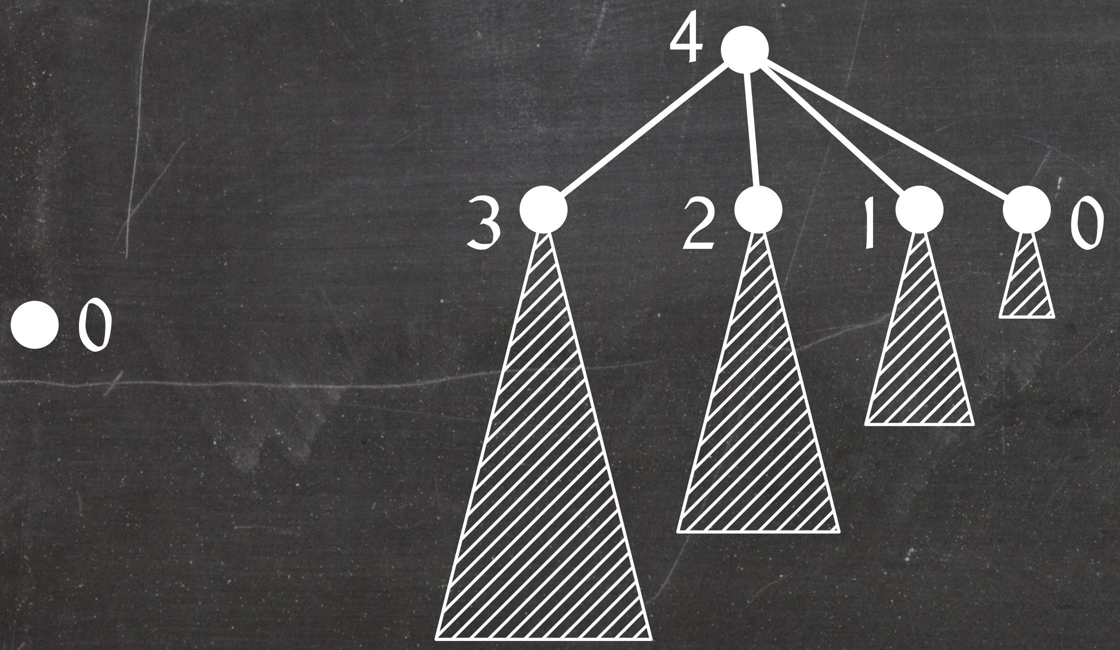

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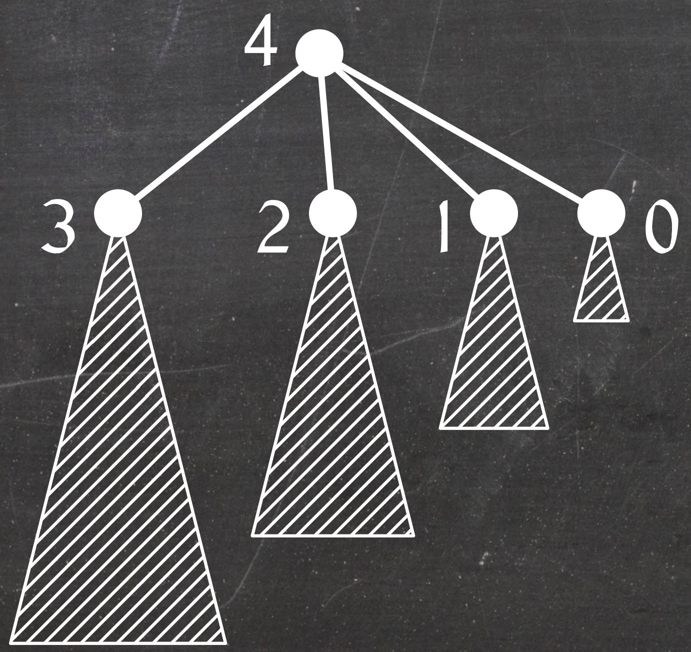
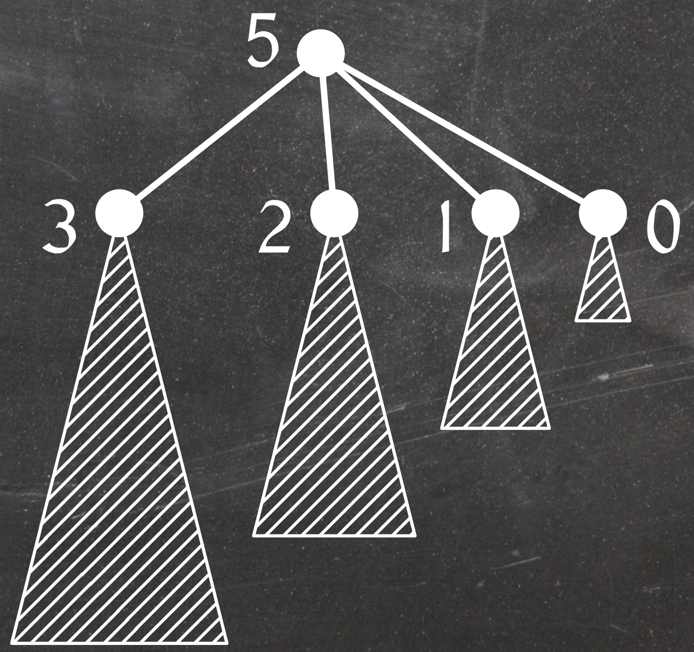


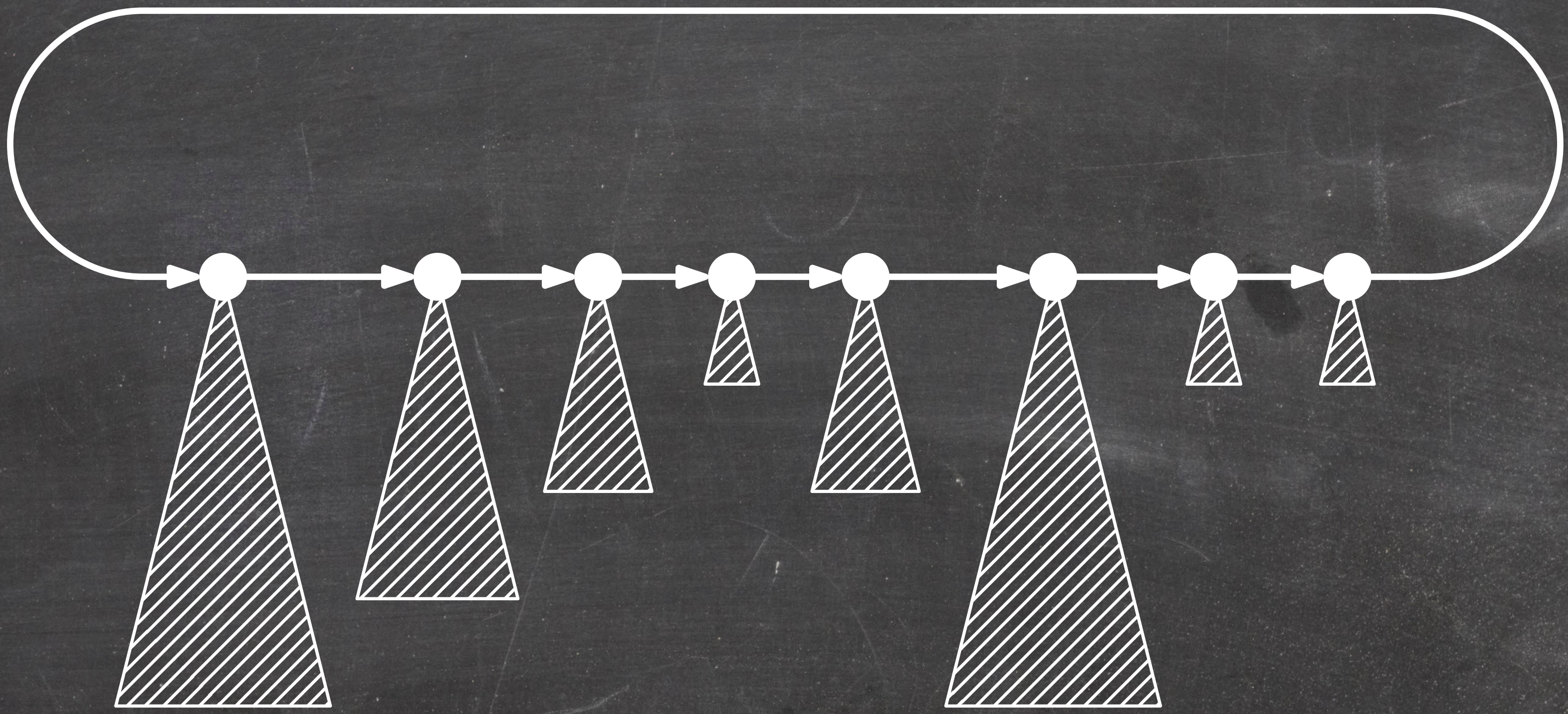
Rank 4, thick



Rank 5, thin

# Thin Heap

A Thin Heap is a circular list of heap-ordered Thin Trees.



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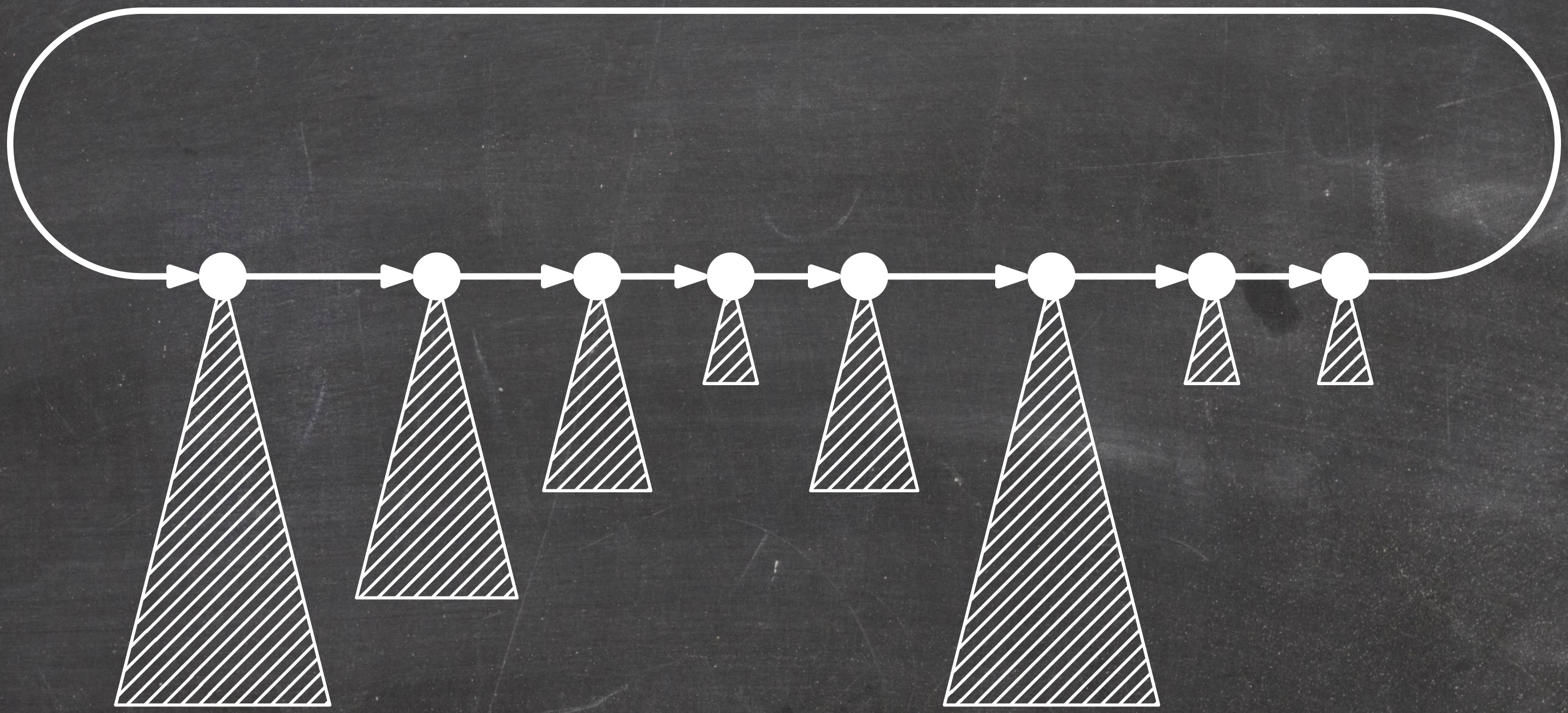

All roots are thick.

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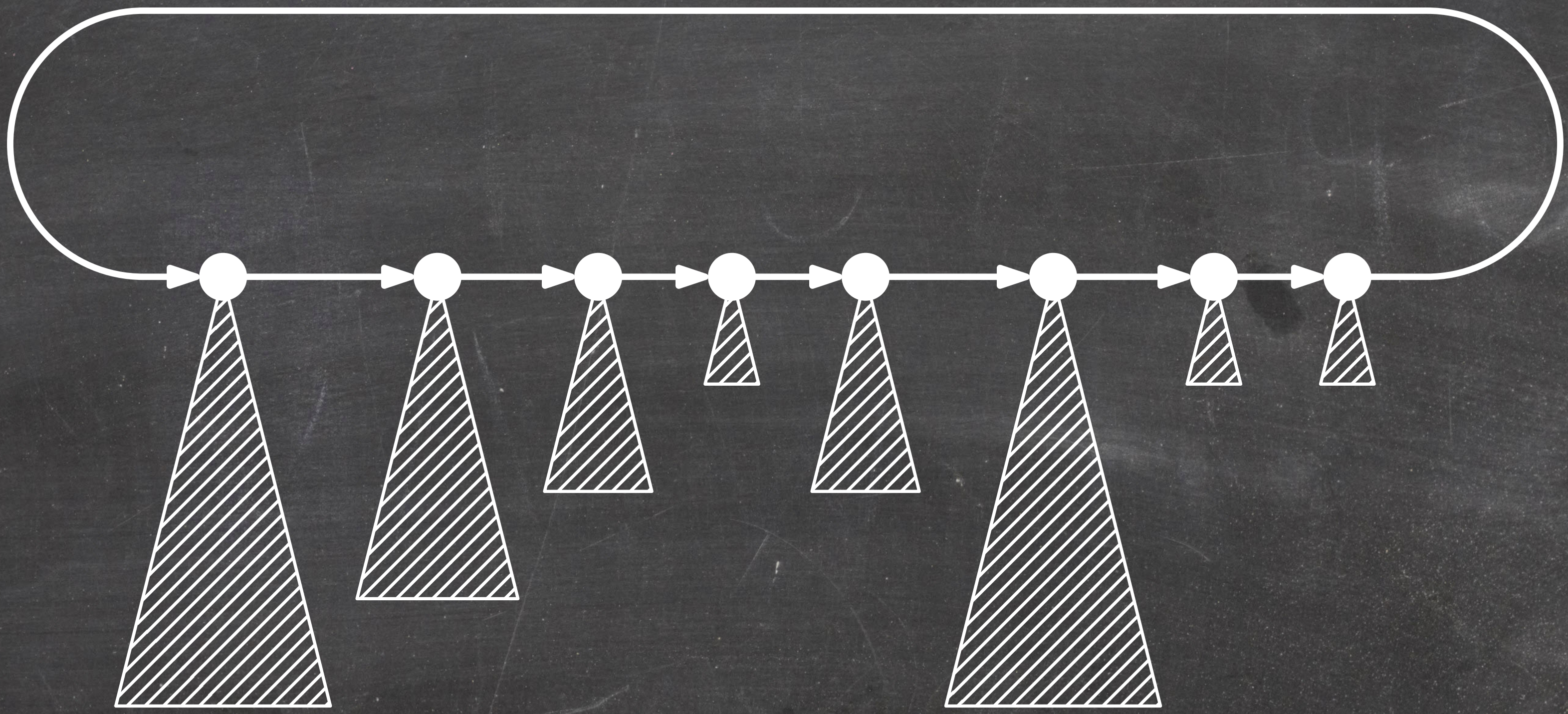
Heap-ordered: Every node stores an element no less than the element stored at its parent.



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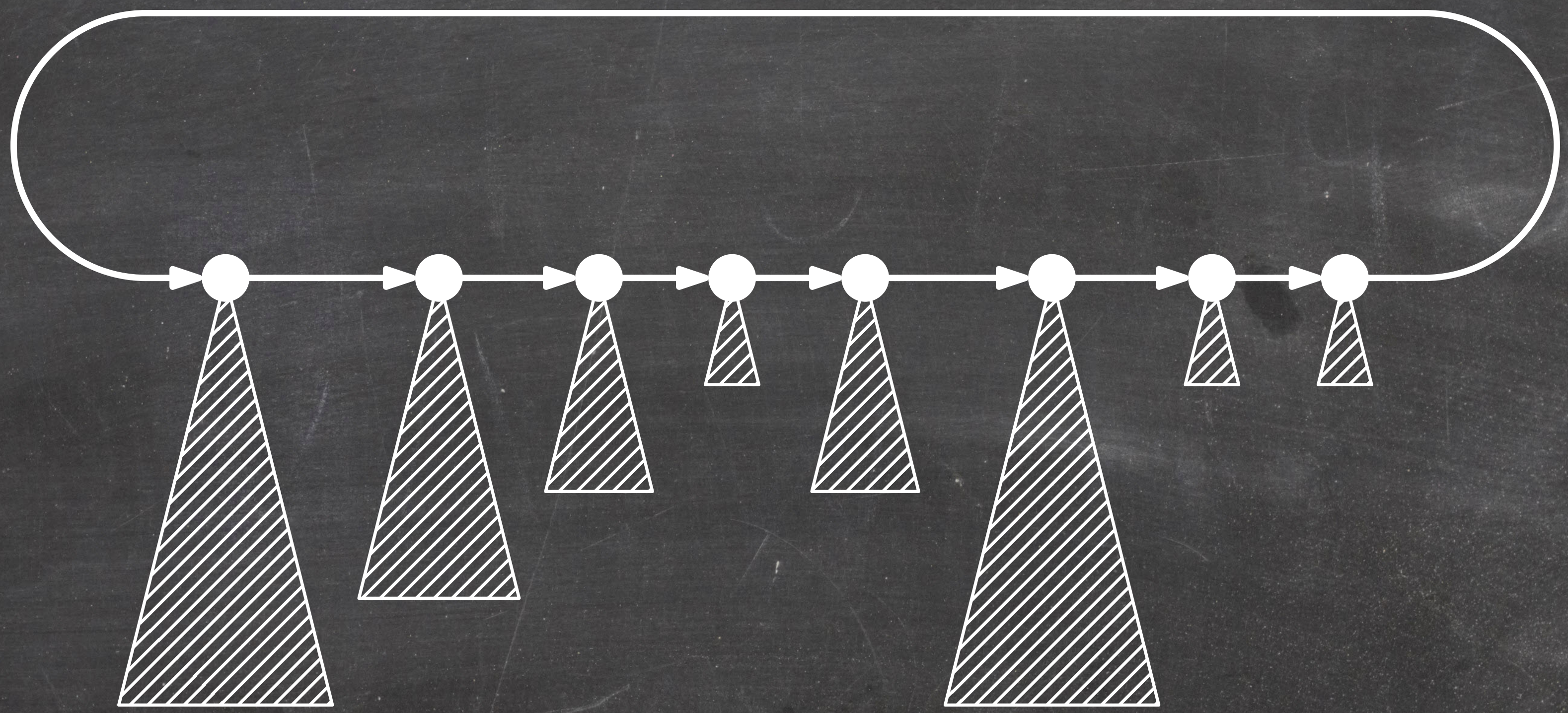

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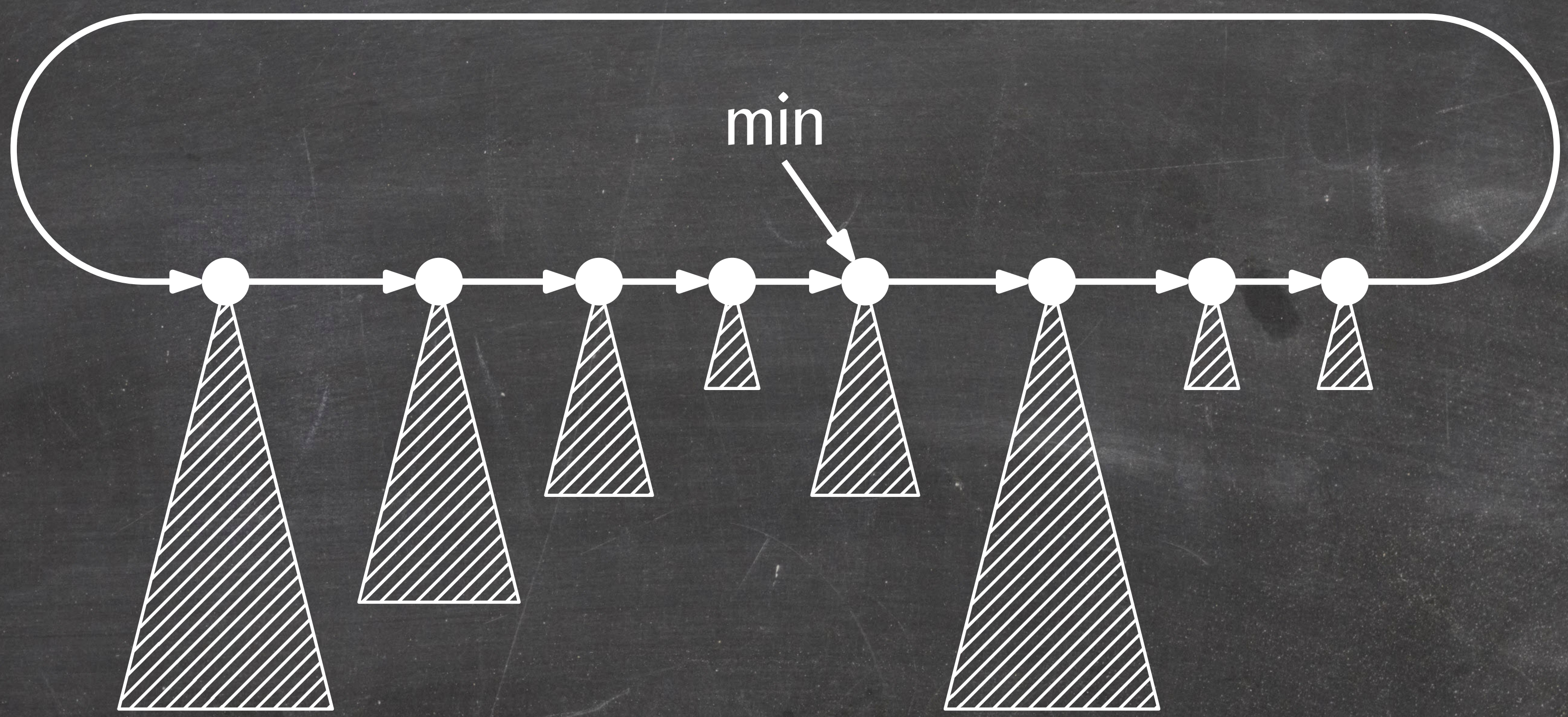


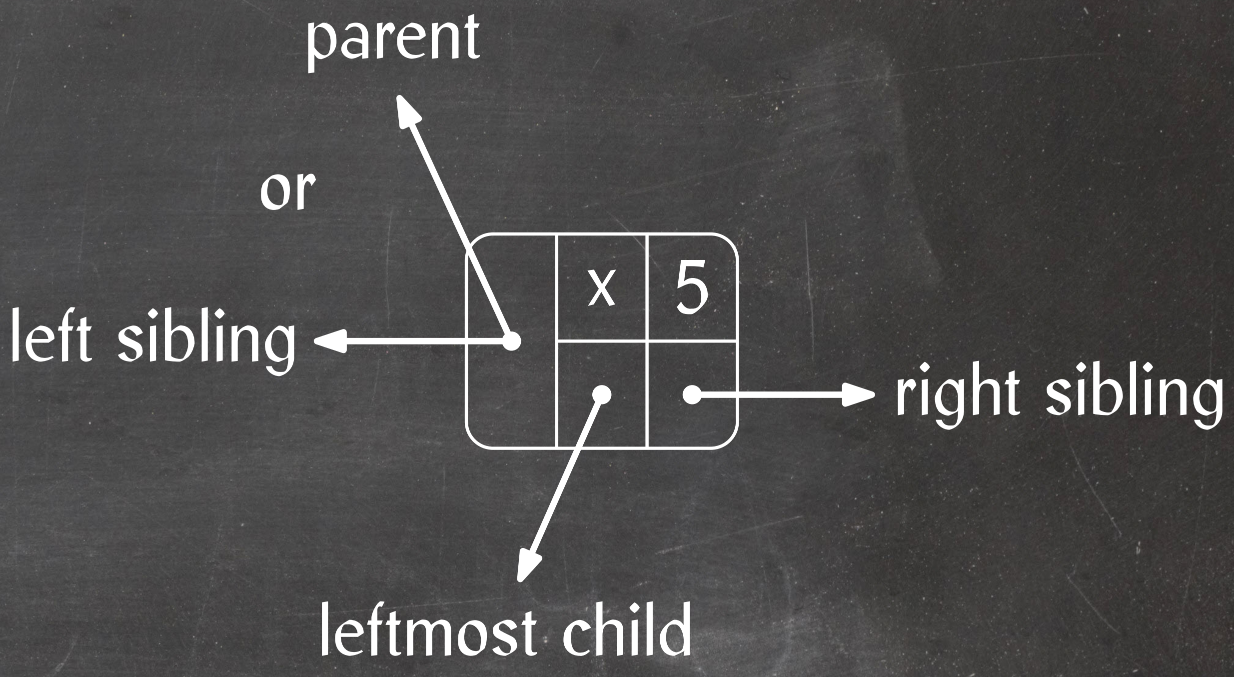
All roots are thick.

The minimum element is stored at one of the roots.

We store a pointer to this root.

# Node Representation

- Element stored at the node
- Rank
- Pointer to leftmost child
- Pointer to right sibling
- Pointer to left sibling or parent



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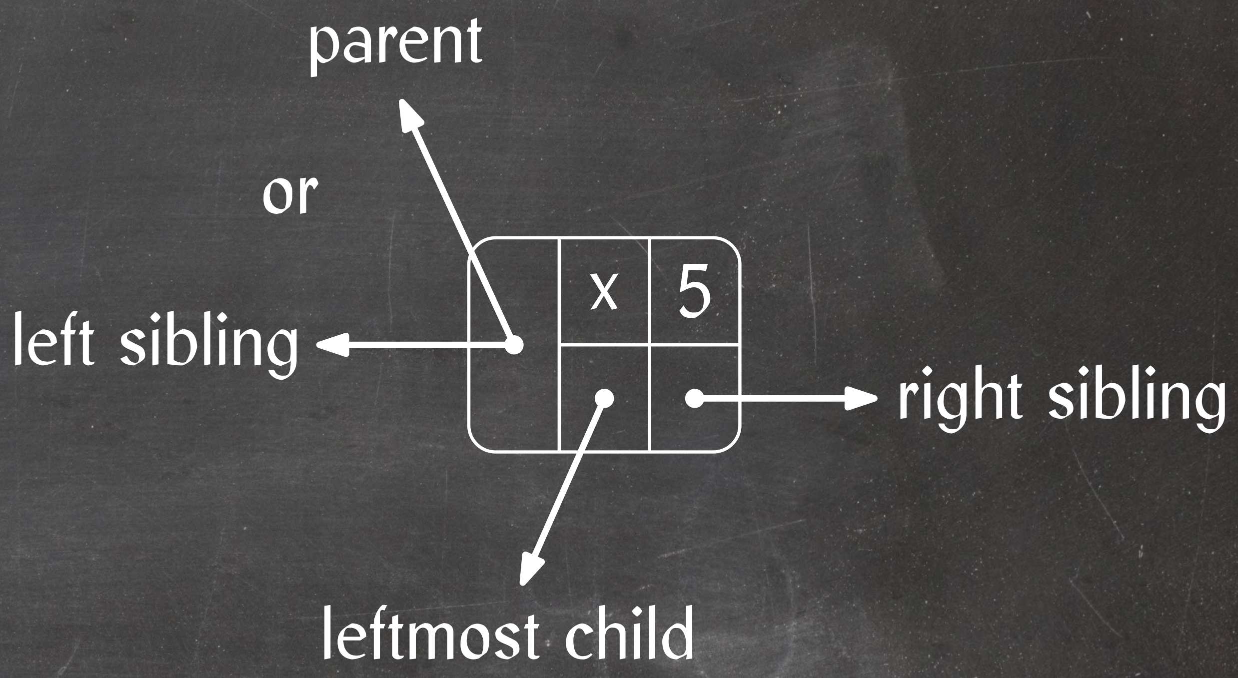
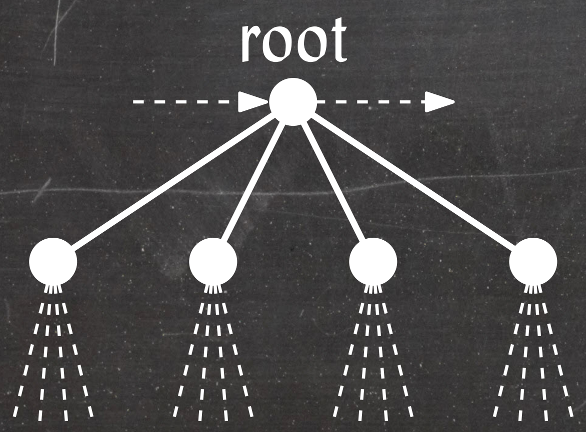
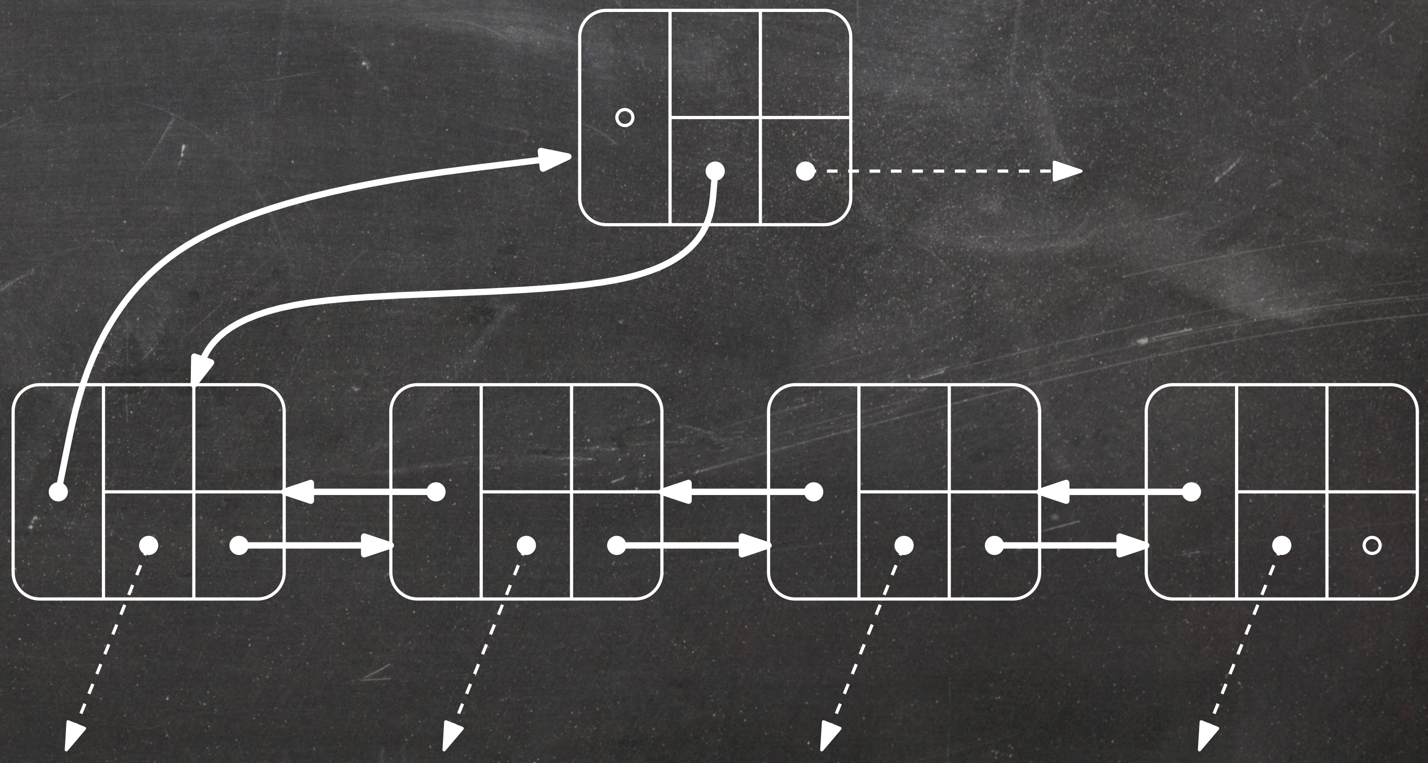
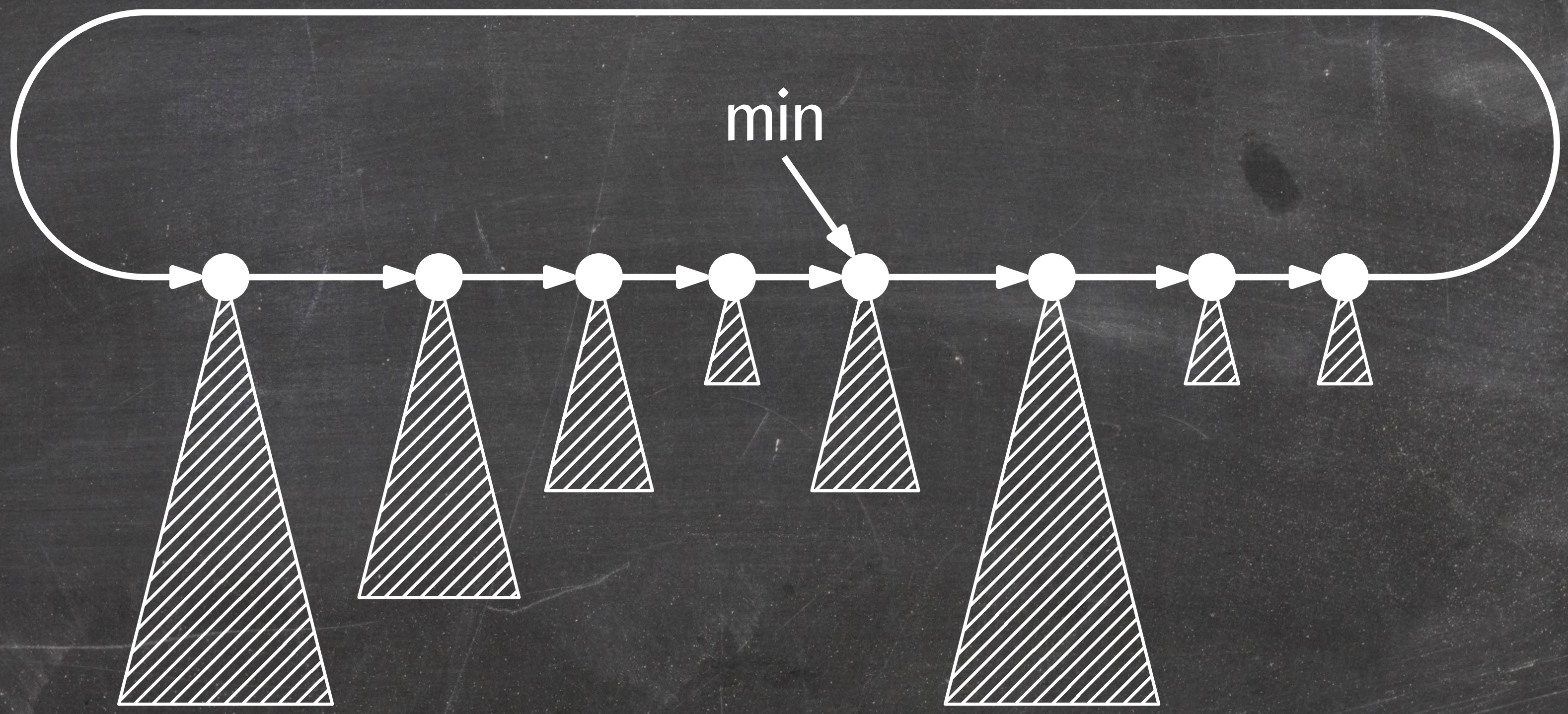






# FindMin

... is easy:



# Delete

… can be implemented using DecreaseKey and DeleteMin:

**Q.delete(x)**

```
Q.decreaseKey(x, $-\infty$)
Q.deleteMin()
```

# Insert

# Insert

If Q is empty:

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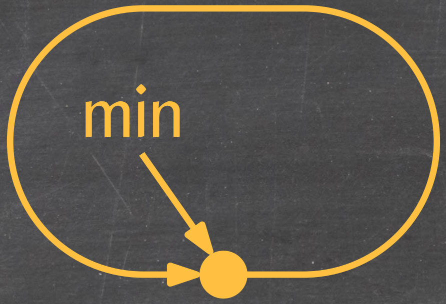

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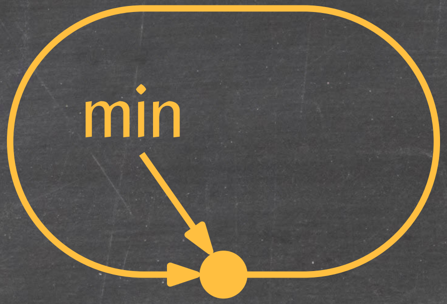

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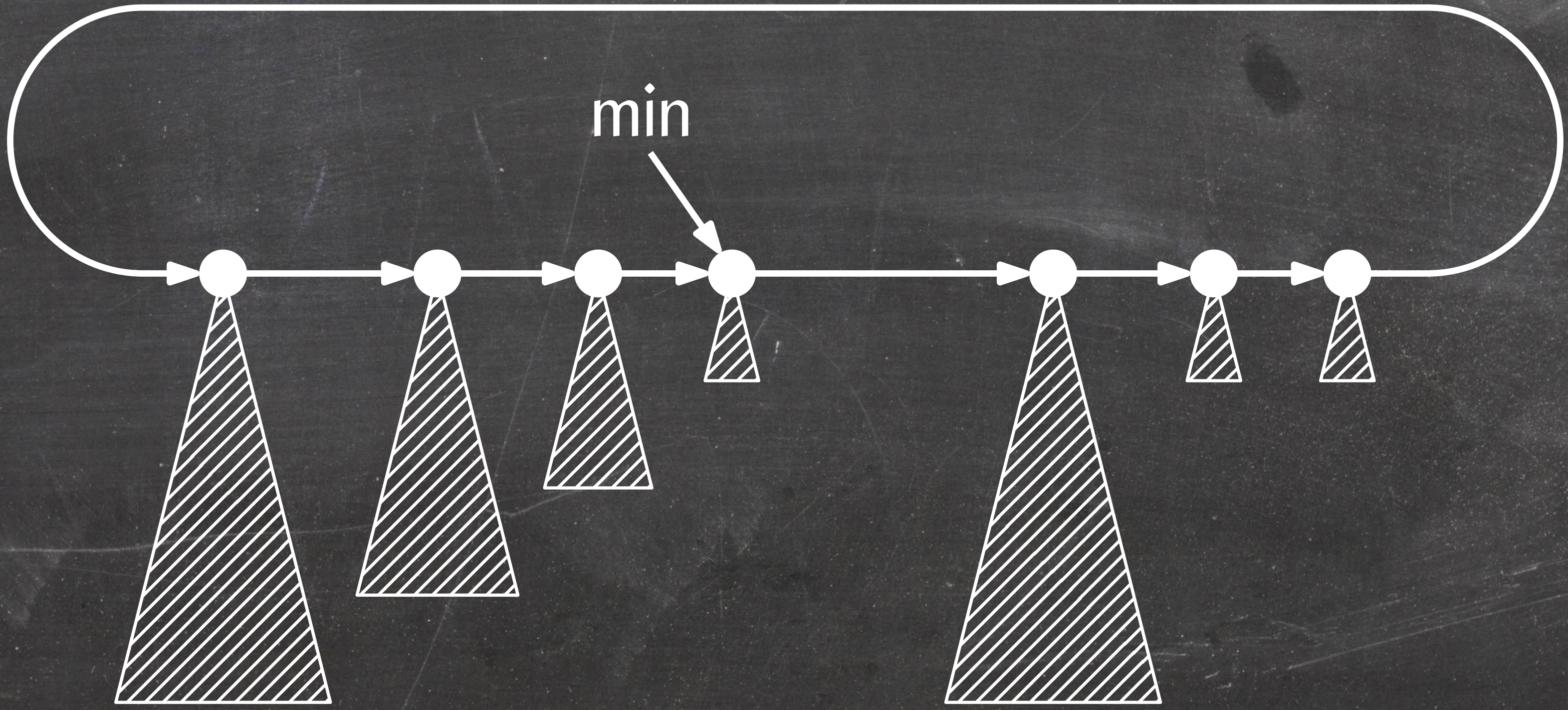

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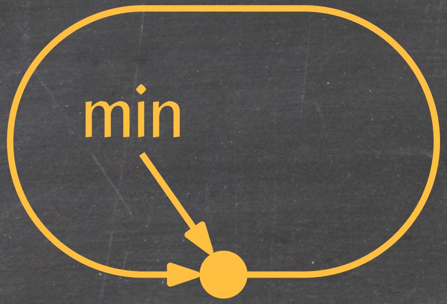

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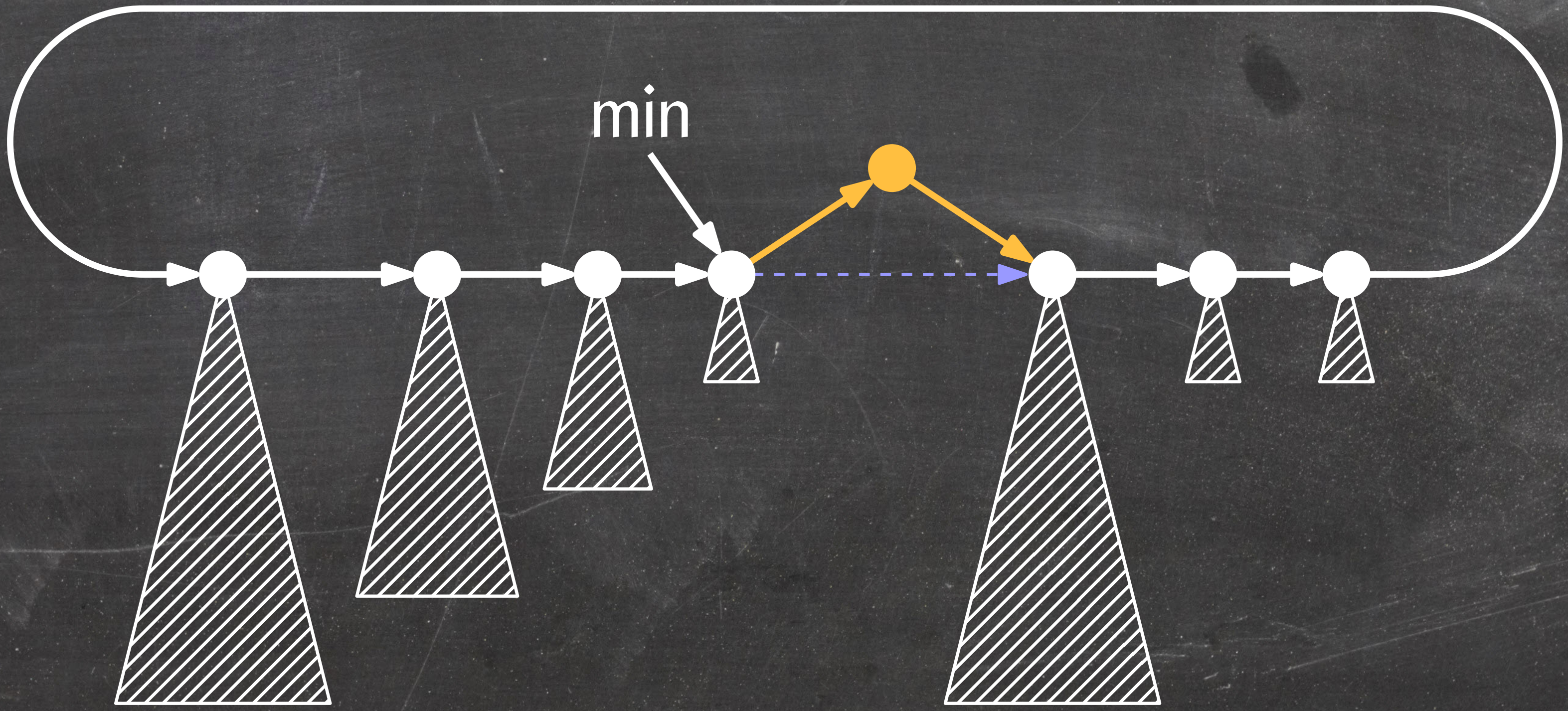

- Insert new element between min and its successor.

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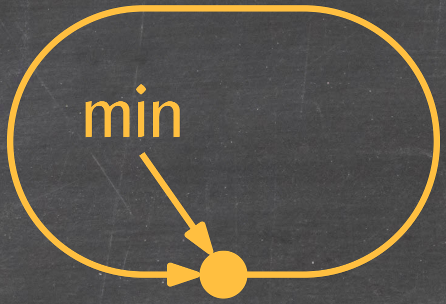

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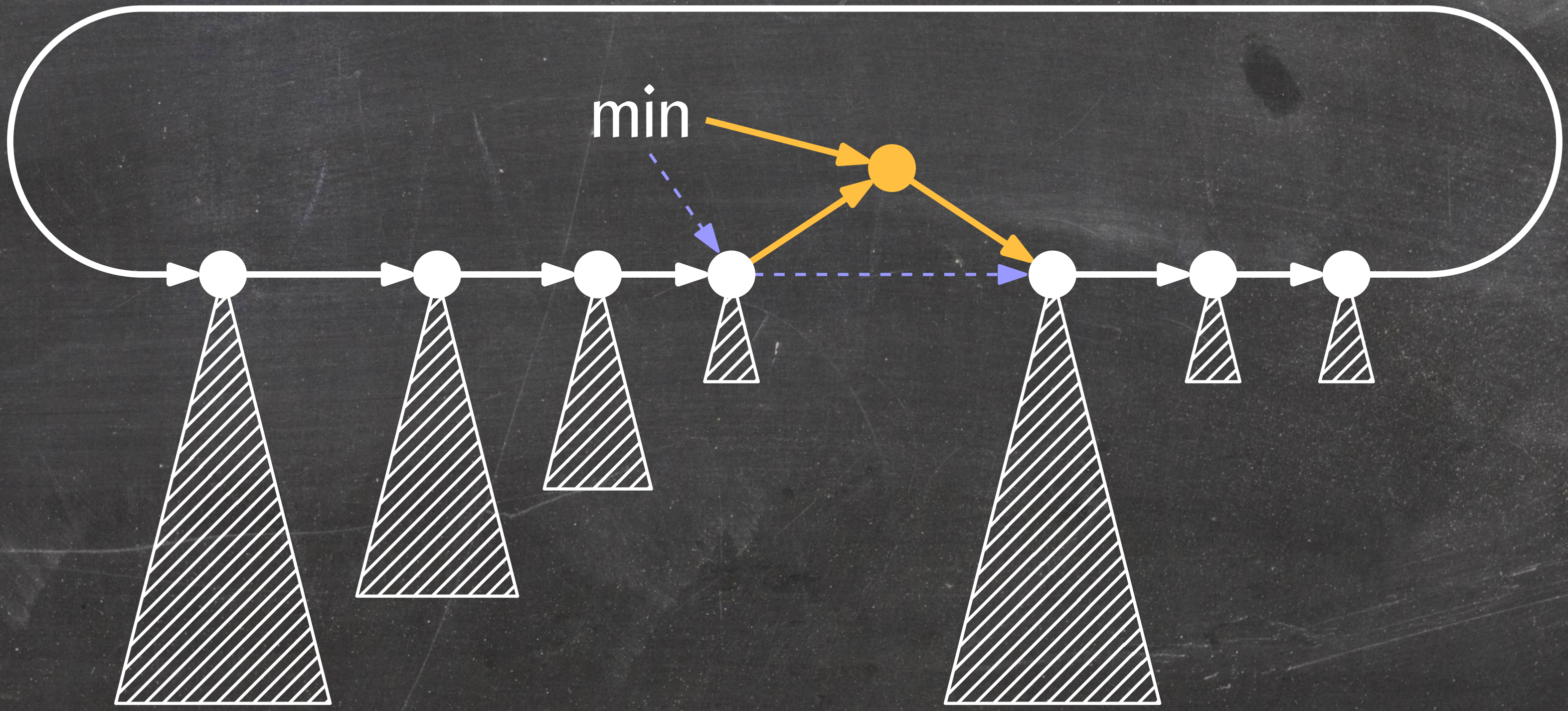
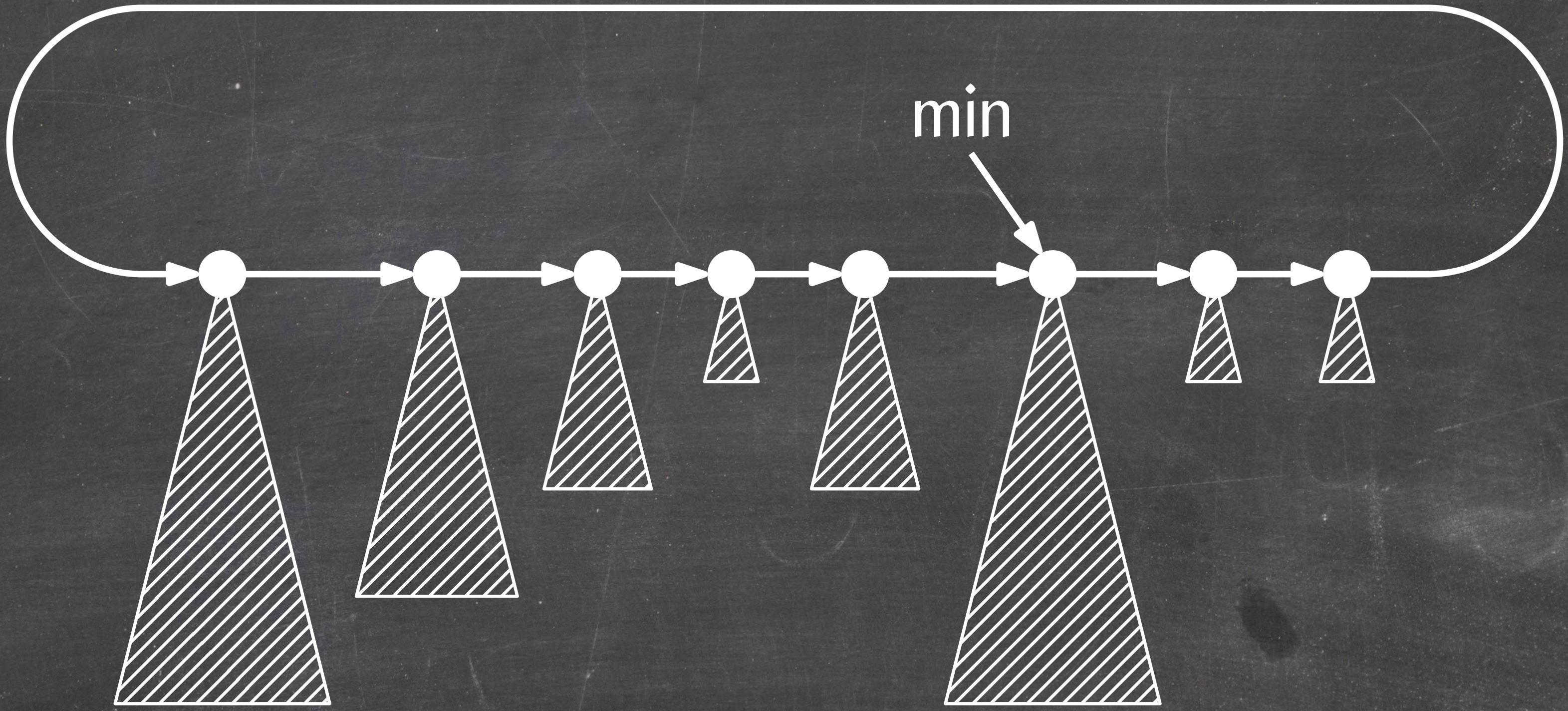


- Insert new element between min and its successor.
- Update min if the new element is the new smallest element.

# DeleteMin



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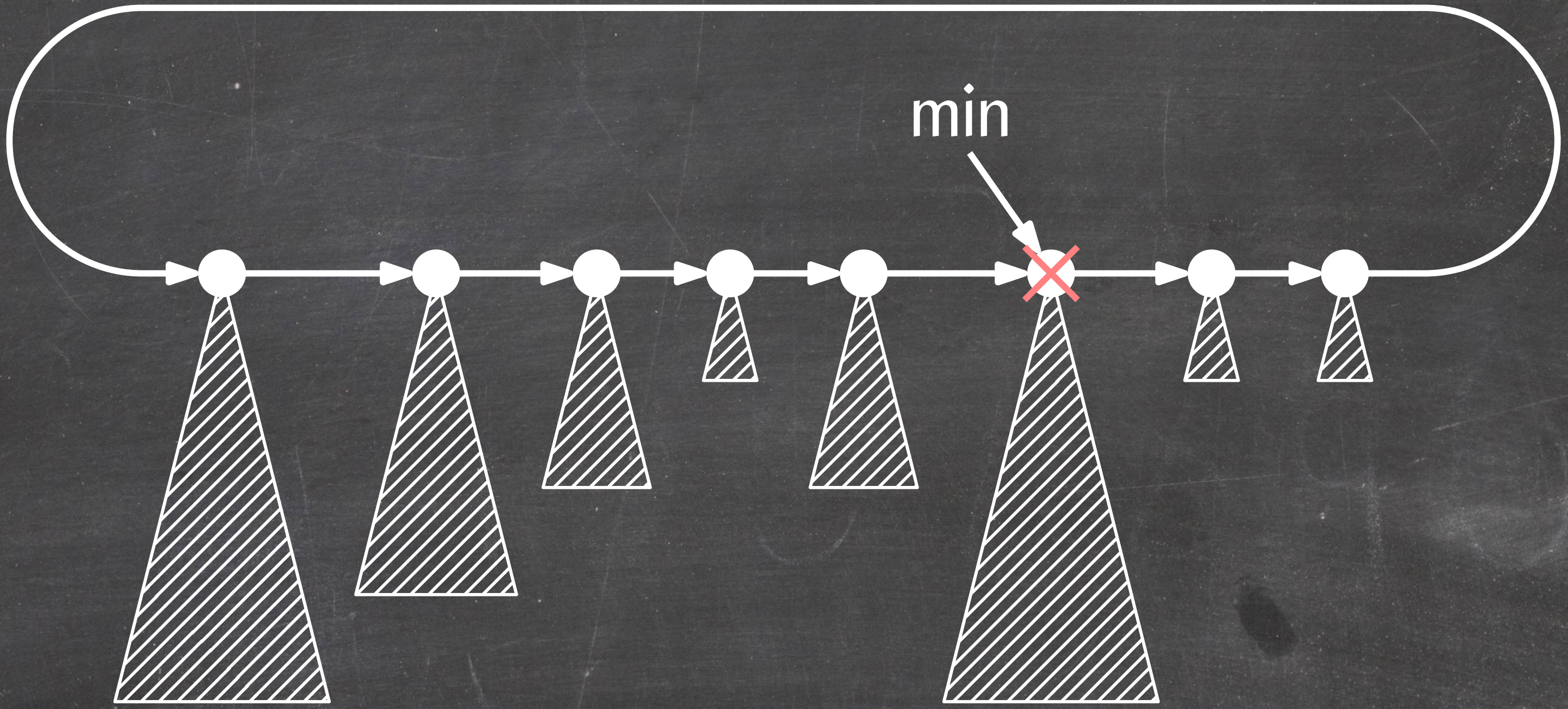

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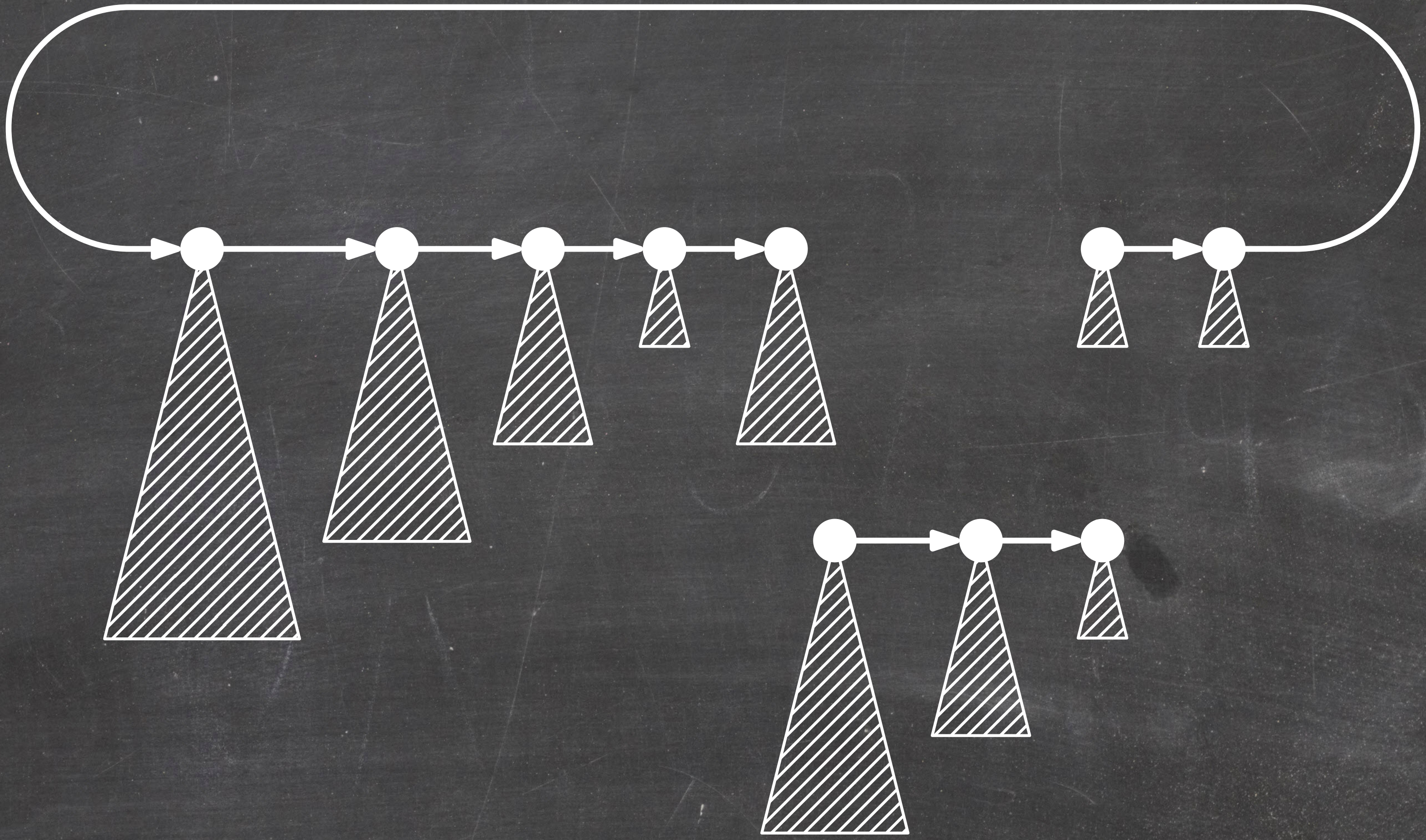

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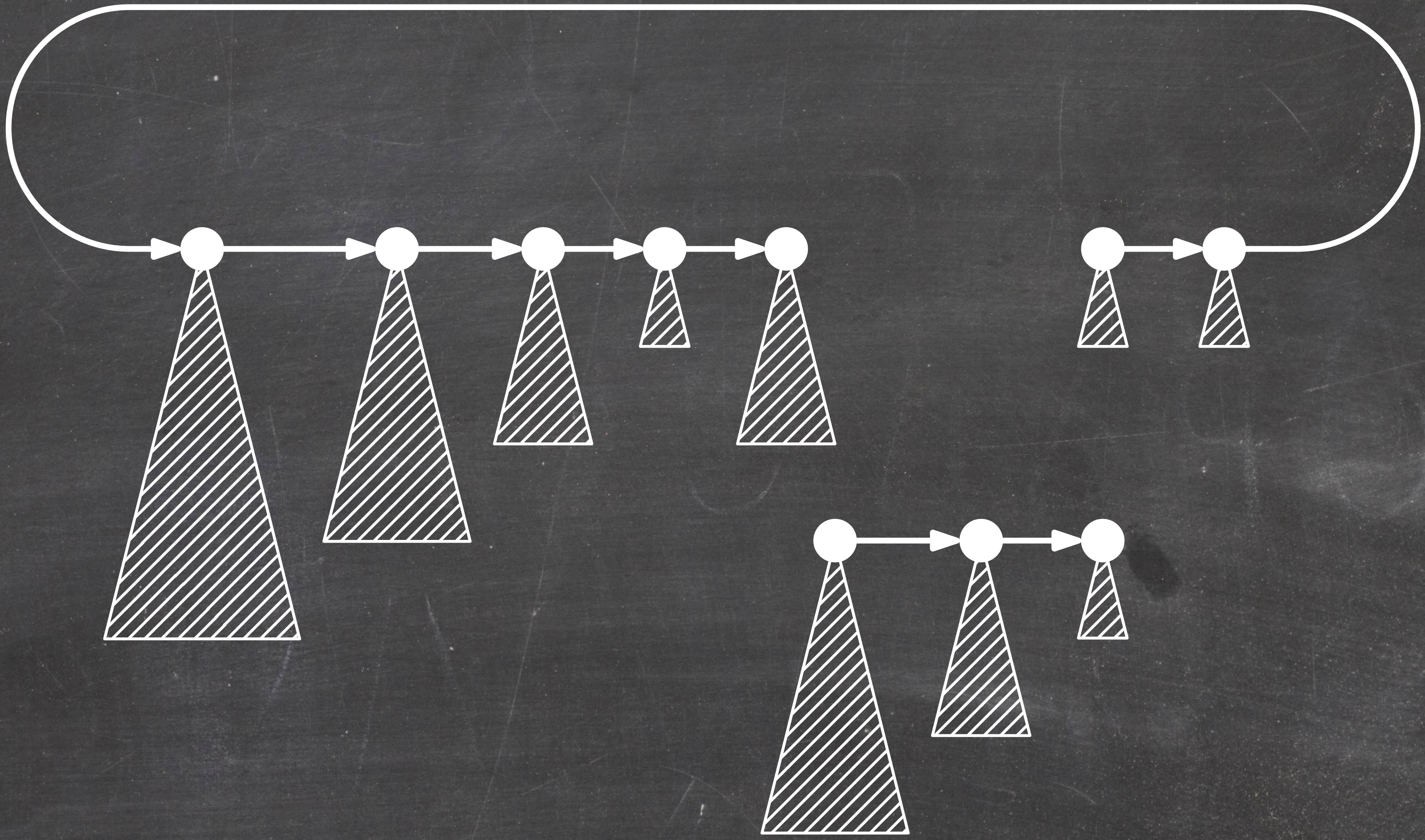
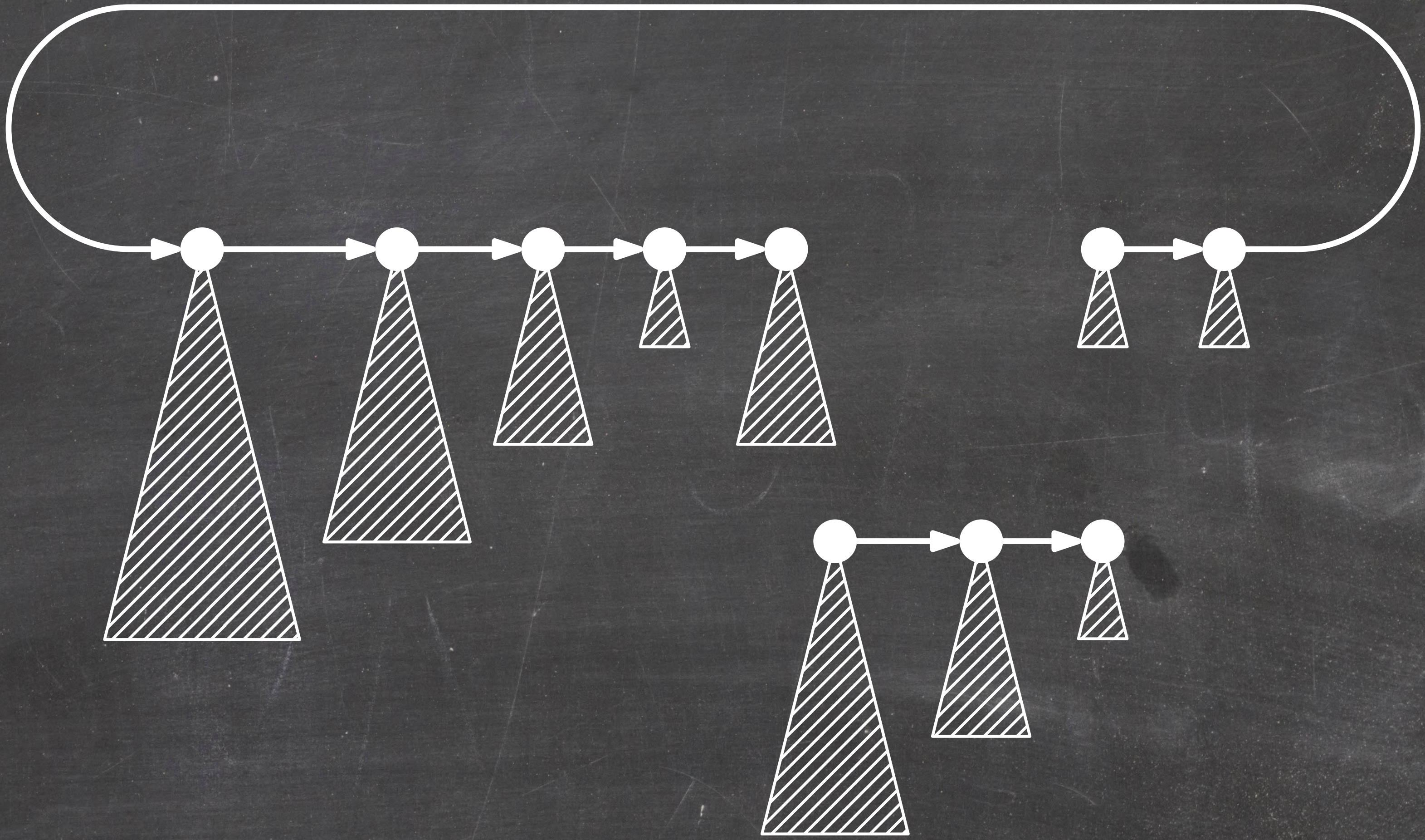


What do we do with the children?

How do we find the new minimum?

# DeleteMin

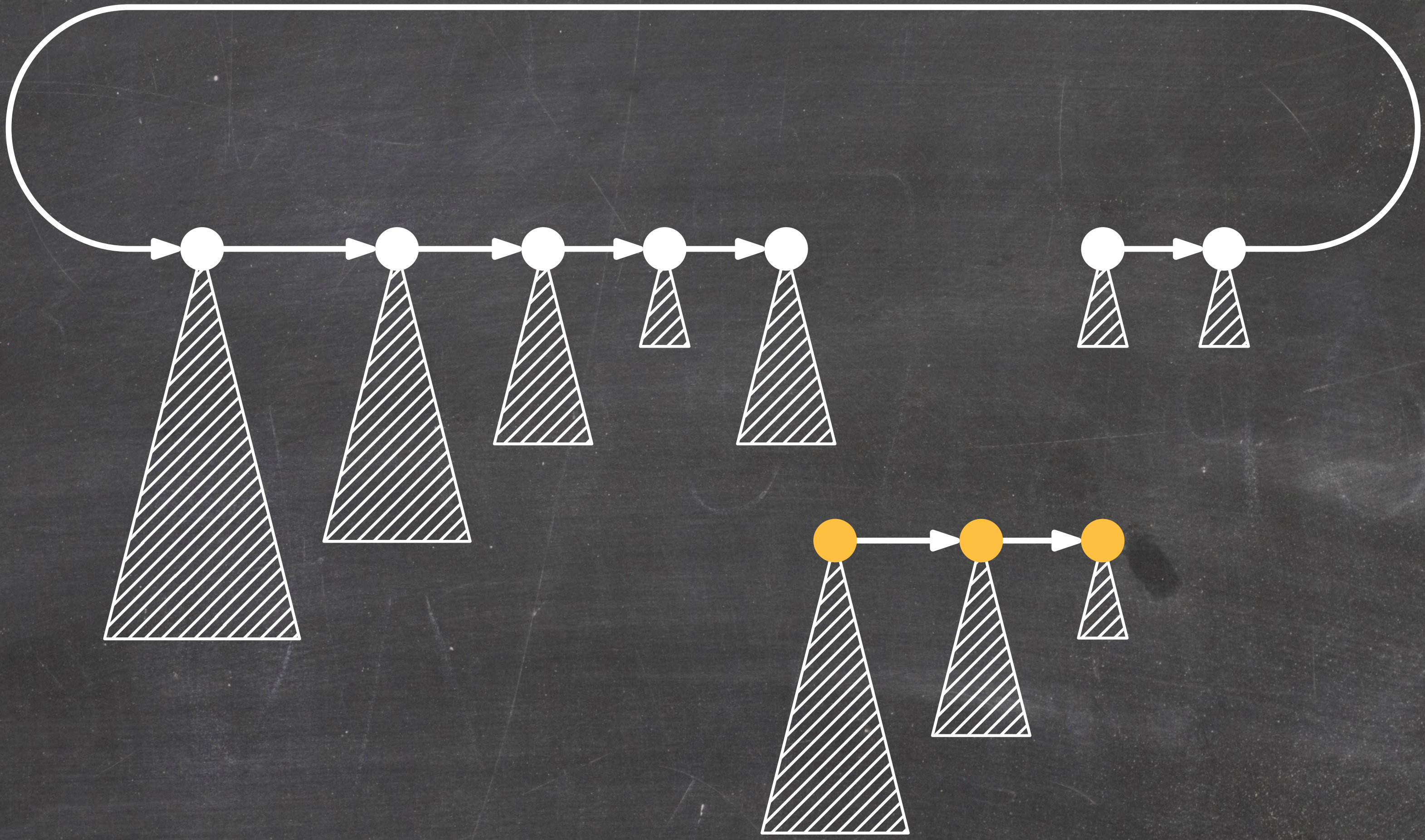


What do we do with the children?

How do we find the new minimum?

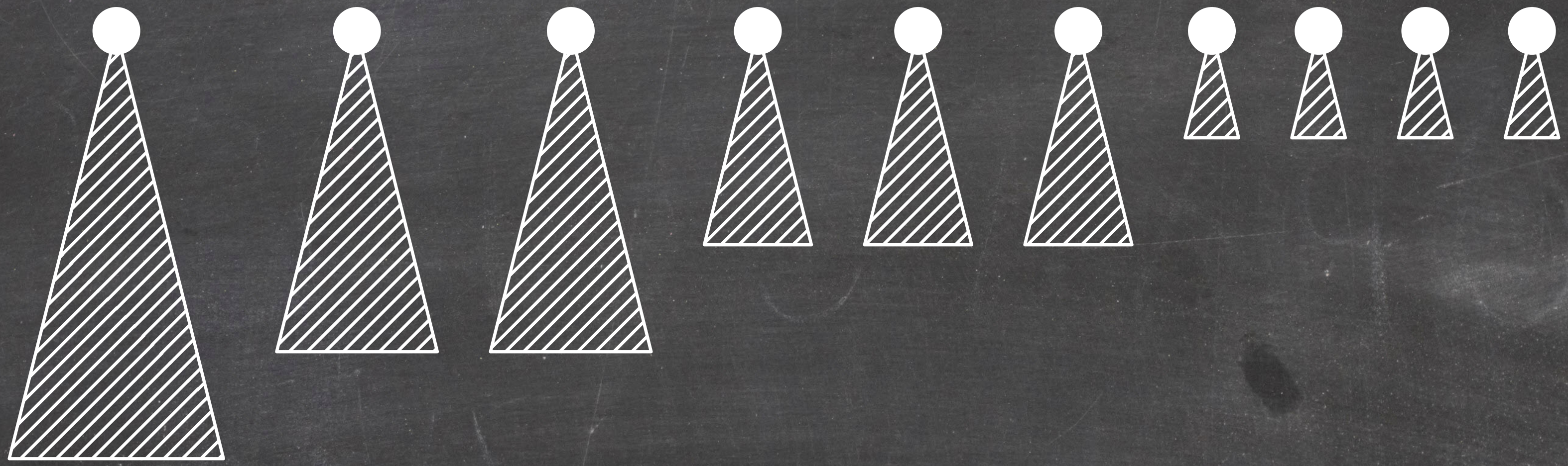
- Could be one of the children.
- Could be one of the other roots.

# DeleteMin



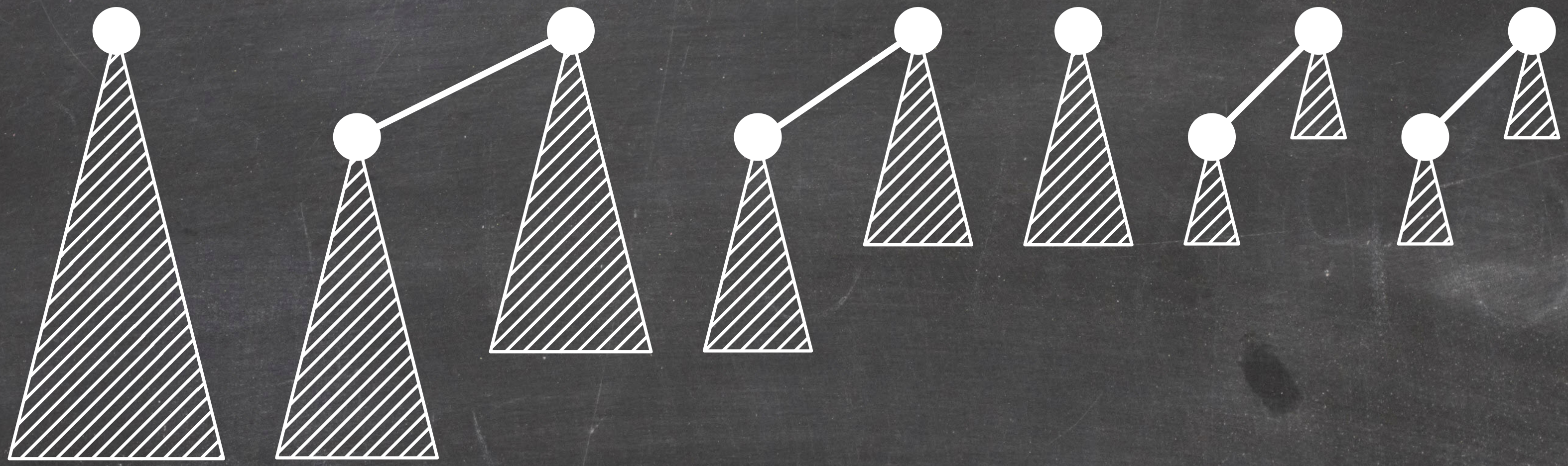
- Ensure all former children of min are thick. How?

# DeleteMin



- Ensure all former children of min are thick. How?
- Collect all roots and former children of min.

# DeleteMin



- Ensure all former children of min are thick. How?
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- Link trees of the same rank until at most one tree of each rank remains.

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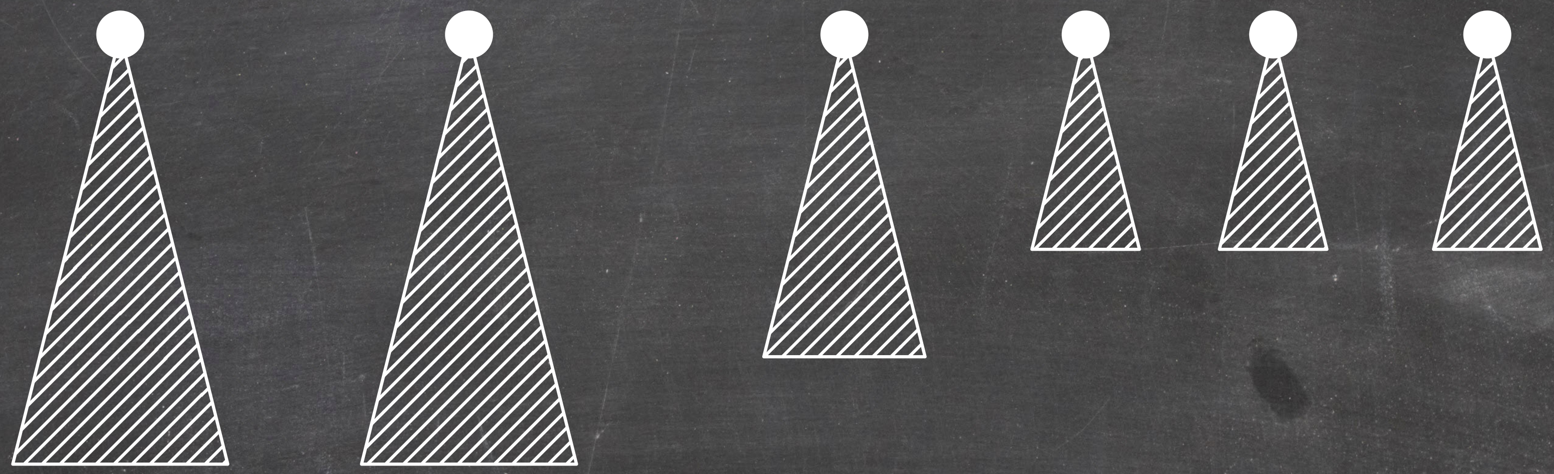

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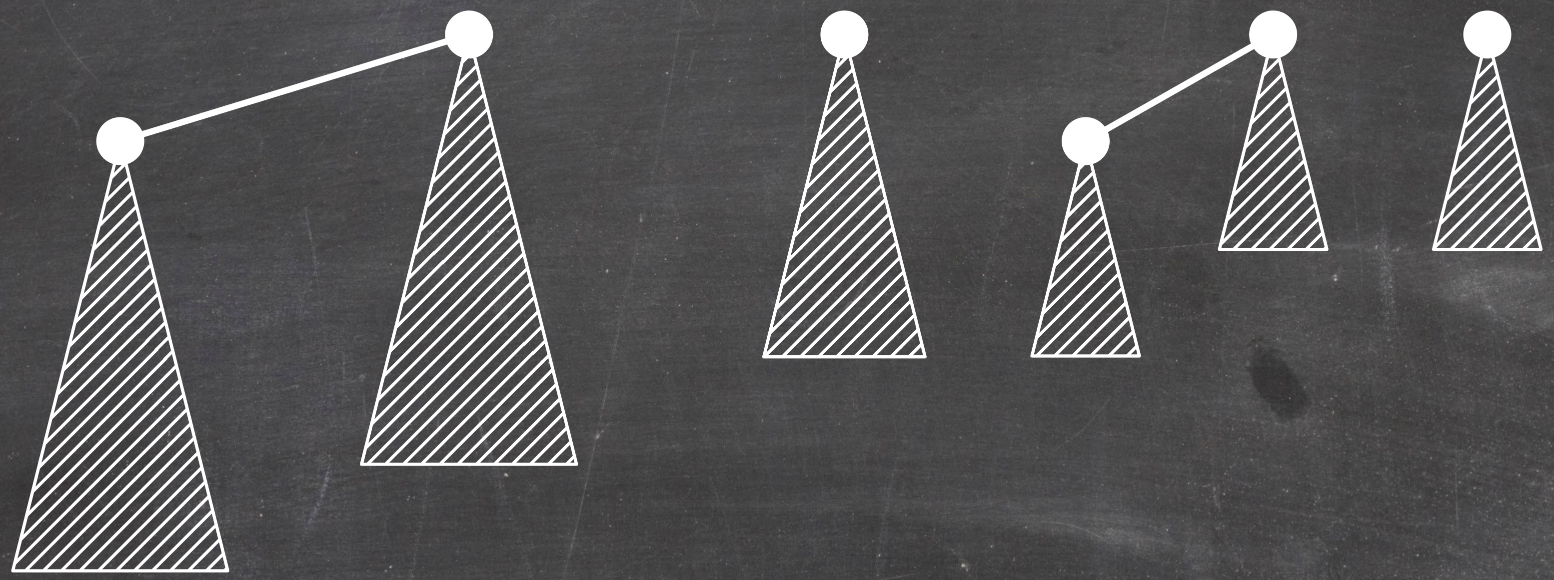

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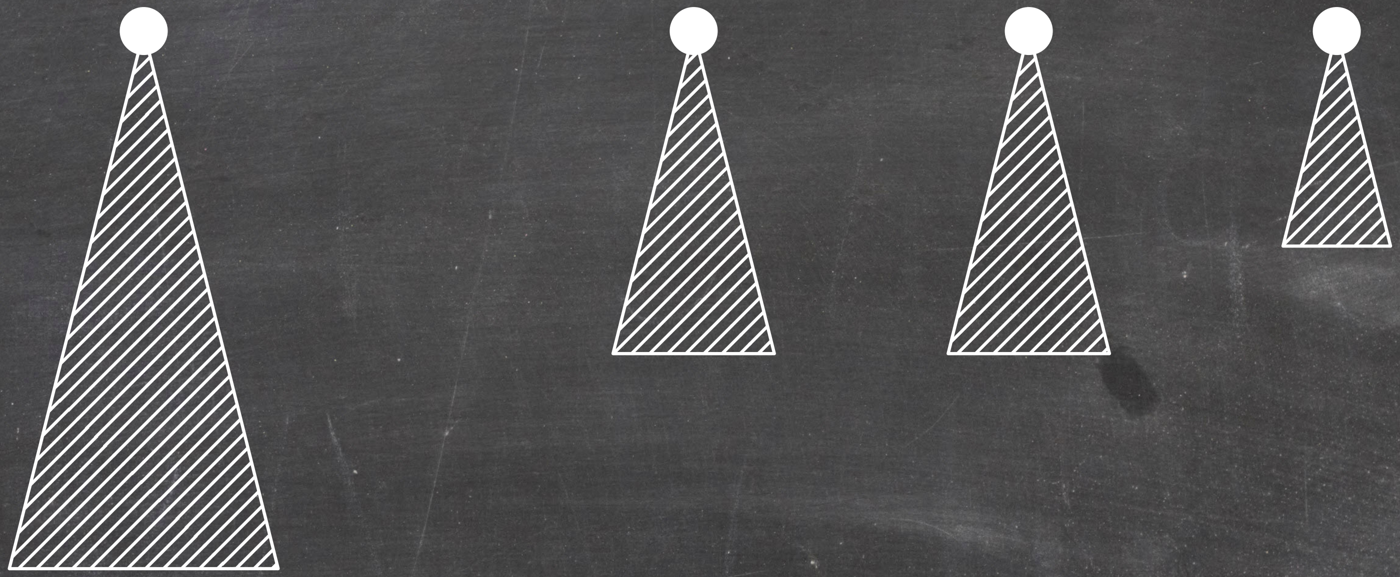

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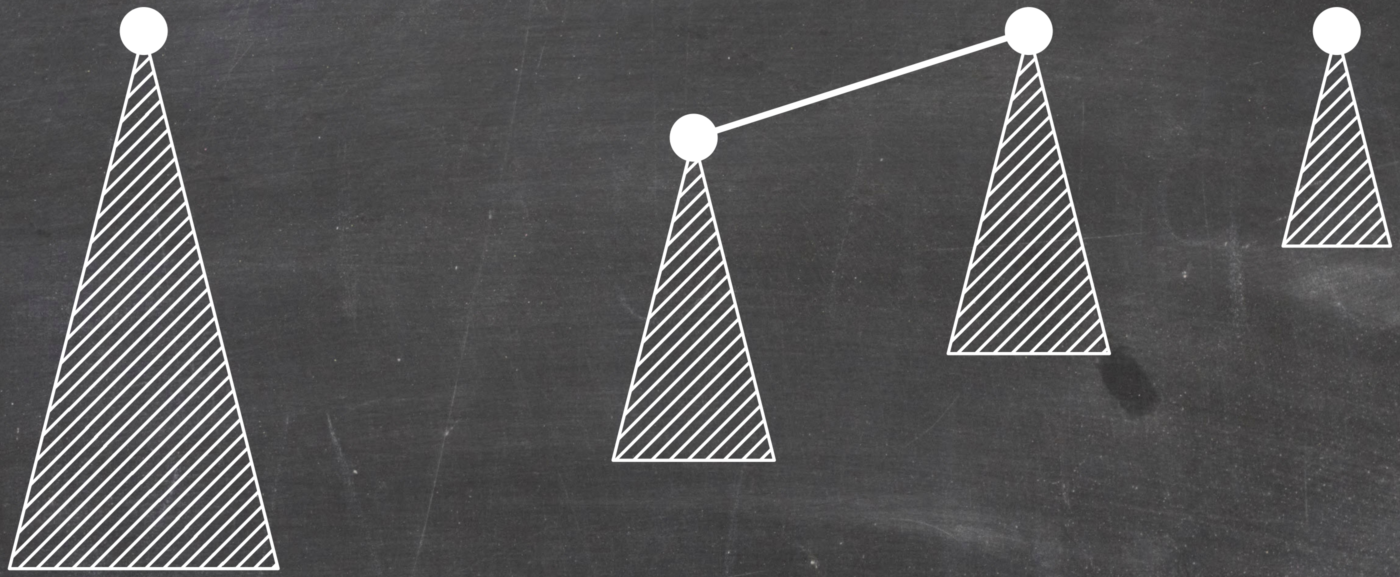

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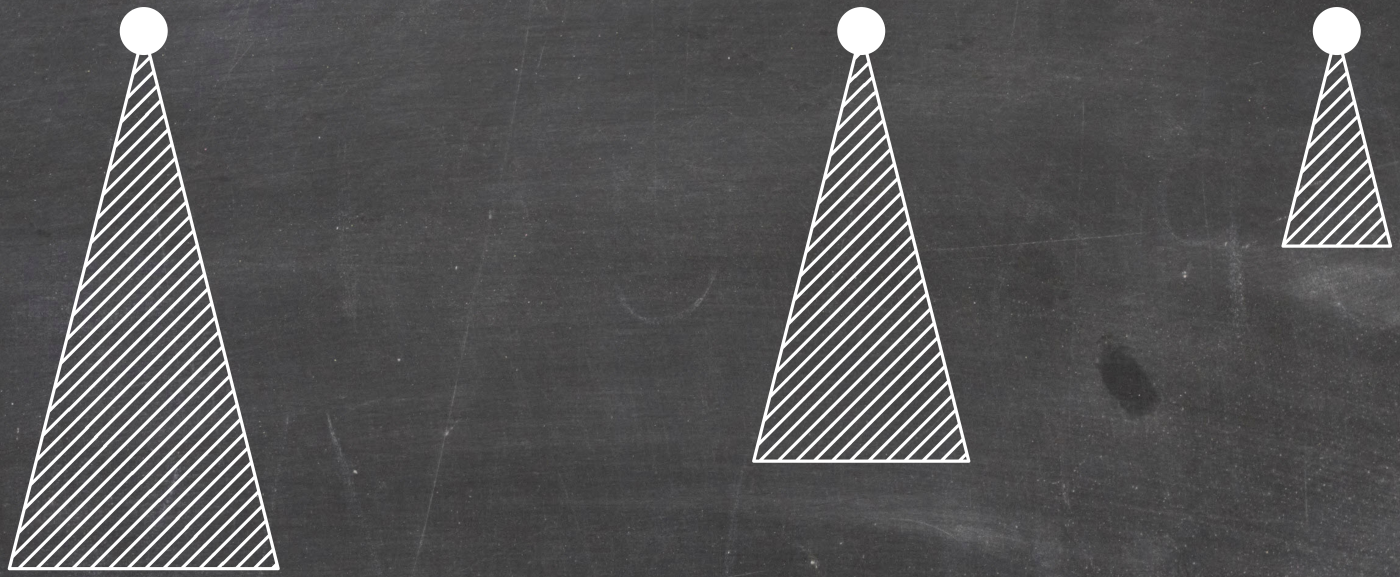

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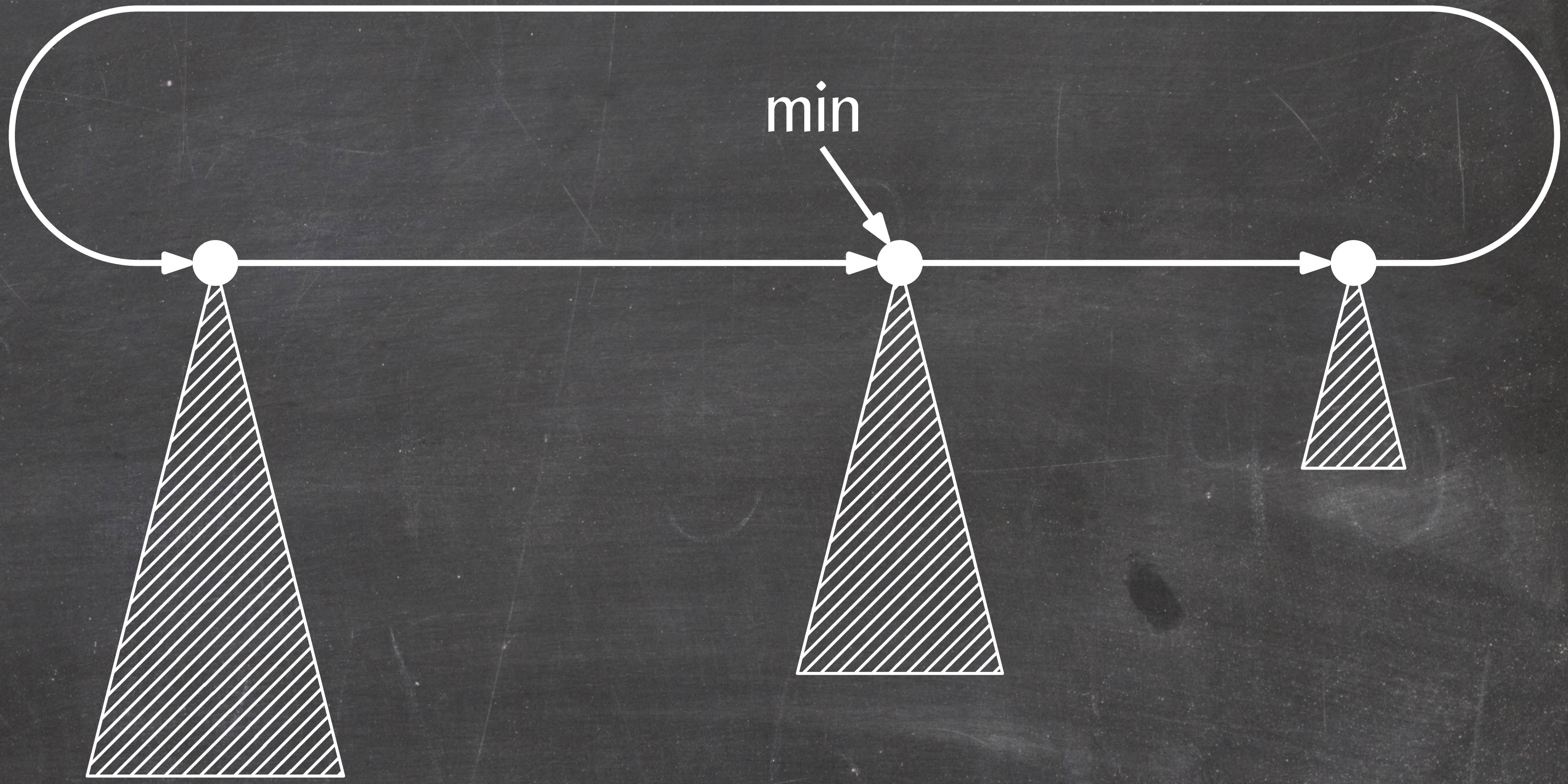

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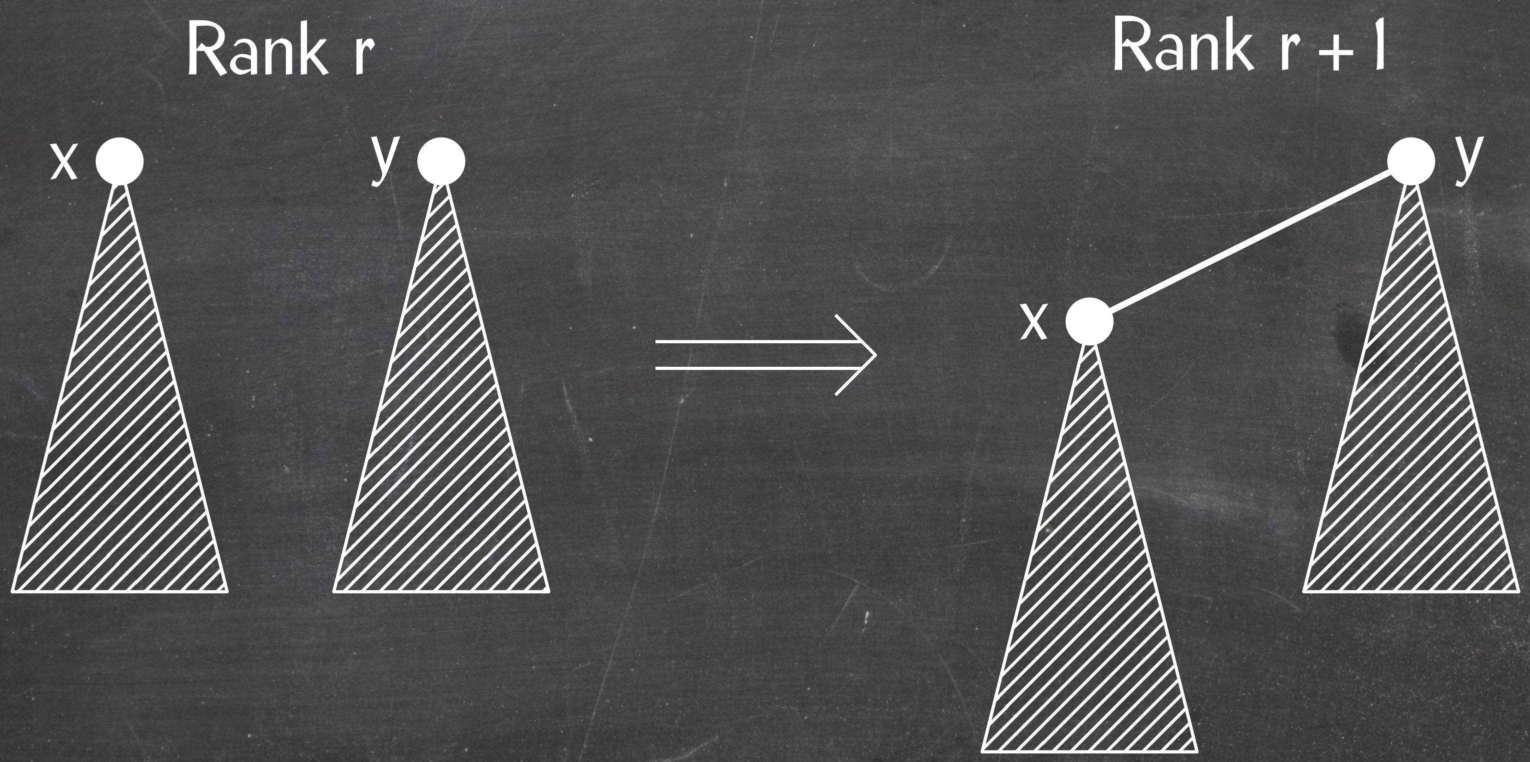
- Collect all roots and former children of min.
- Link trees of the same rank until at most one tree of each rank remains.
- Relink roots into circular list and make min point to the minimum root.

# Linking

Important: Both nodes need to be thick and of the same rank.

Assume $y < x$ (swap the two trees otherwise).



This produces a valid thin tree:

$y$ had $r$ children of ranks $r-1, r-2, \ldots, 0$ before.
$\Rightarrow$ $y$ has $r+1$ children of ranks $r, r-1, \ldots, 0$ after.

# Bounding the Maximum Rank

**Lemma:** A tree whose root has rank r has at least $F_r$ nodes, where $F_r$ is the rth Fibonacci number.

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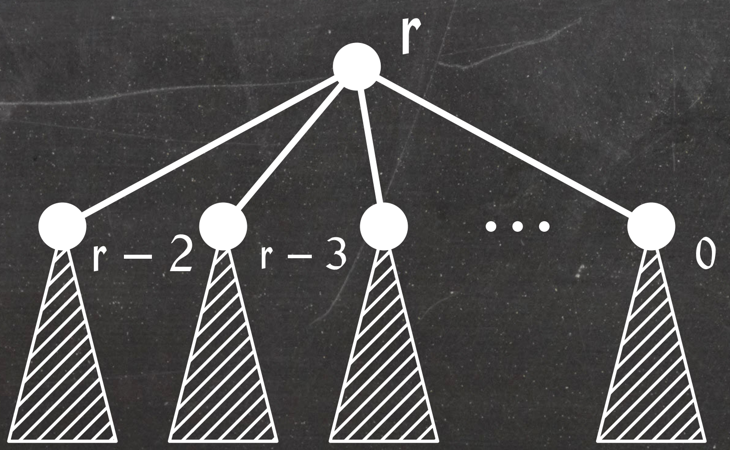

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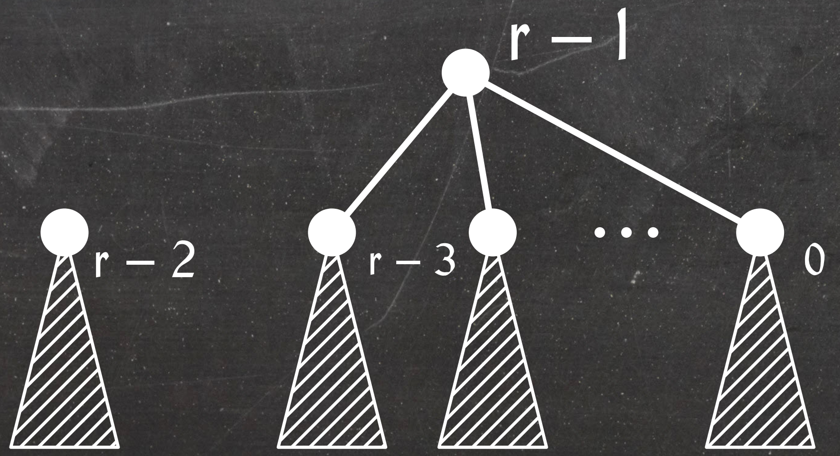

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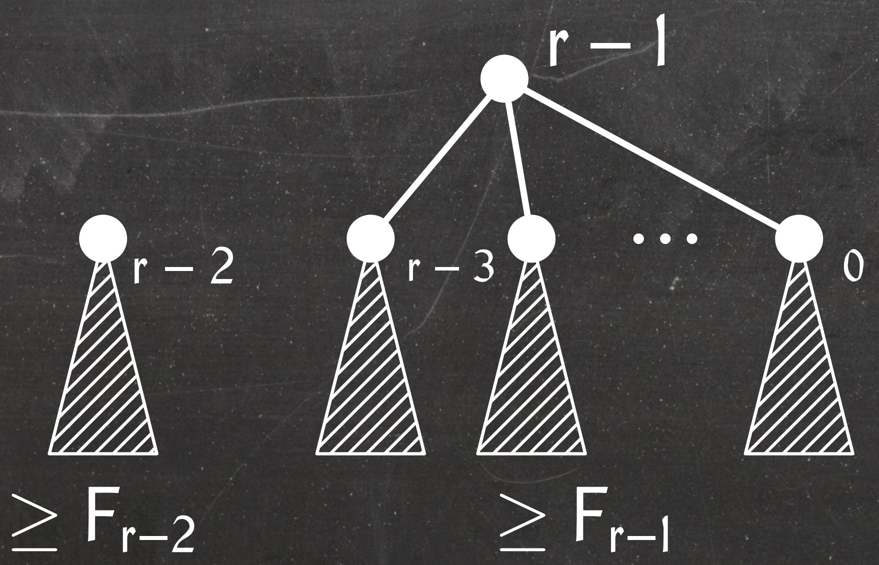

$$F_{r-1} + F_{r-2} = F_r$$

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$$F_r = F_{r-1} + F_{r-2} \geq \phi^{r-2} + \phi^{r-3}$$

$$= \left(\frac{1+\sqrt{5}}{2} + 1\right) \phi^{r-3} = \frac{3+\sqrt{5}}{2} \phi^{r-3}$$

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**Corollary:** The maximum rank in a Thin Heap storing n elements is $\log_\phi n < 2 \lg n$.

# Implementation of DeleteMin

**Q.deleteMin()**

```
x = Q.min
R = array of size $2 \lg n$ with all its entries initially null.
for every root r other than Q.min
    do LinkTrees(R, r)
for every child c of Q.min
    do decrease c's rank if necessary to make it thick
       LinkTrees(R, c)
Q.min = null
for $i = 0$ to $2 \lg n$
    do if $R[i] \neq$ null
          then R[i].leftSibOrParent = null
               if Q.min = null
                  then Q.min = R[i]
                       Q.min.rightSib = Q.min
                  else R[i].rightSib = Q.min.rightSib
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Collect pairs of trees while ensuring no two have the same rank.

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Collect pairs of trees while ensuring no two have the same rank.

**LinkTrees(R, x)**

```
r = x.rank
while R[r] $\neq$ null
    do x = Link(x, R[r])
       R[r] = null
       r = r + 1
R[r] = x
```

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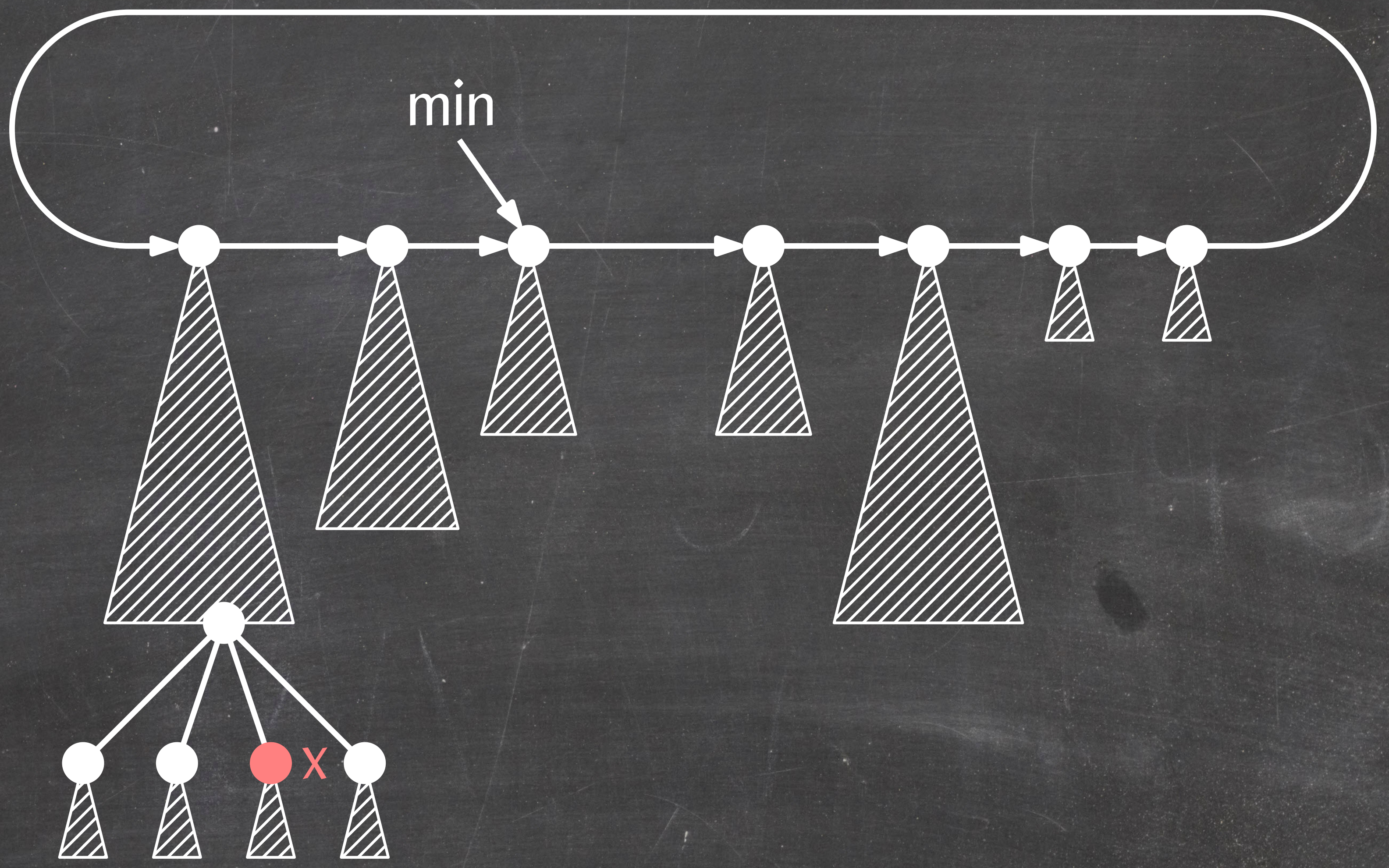
```

Collect remaining trees and form circular list.

# DecreaseKey



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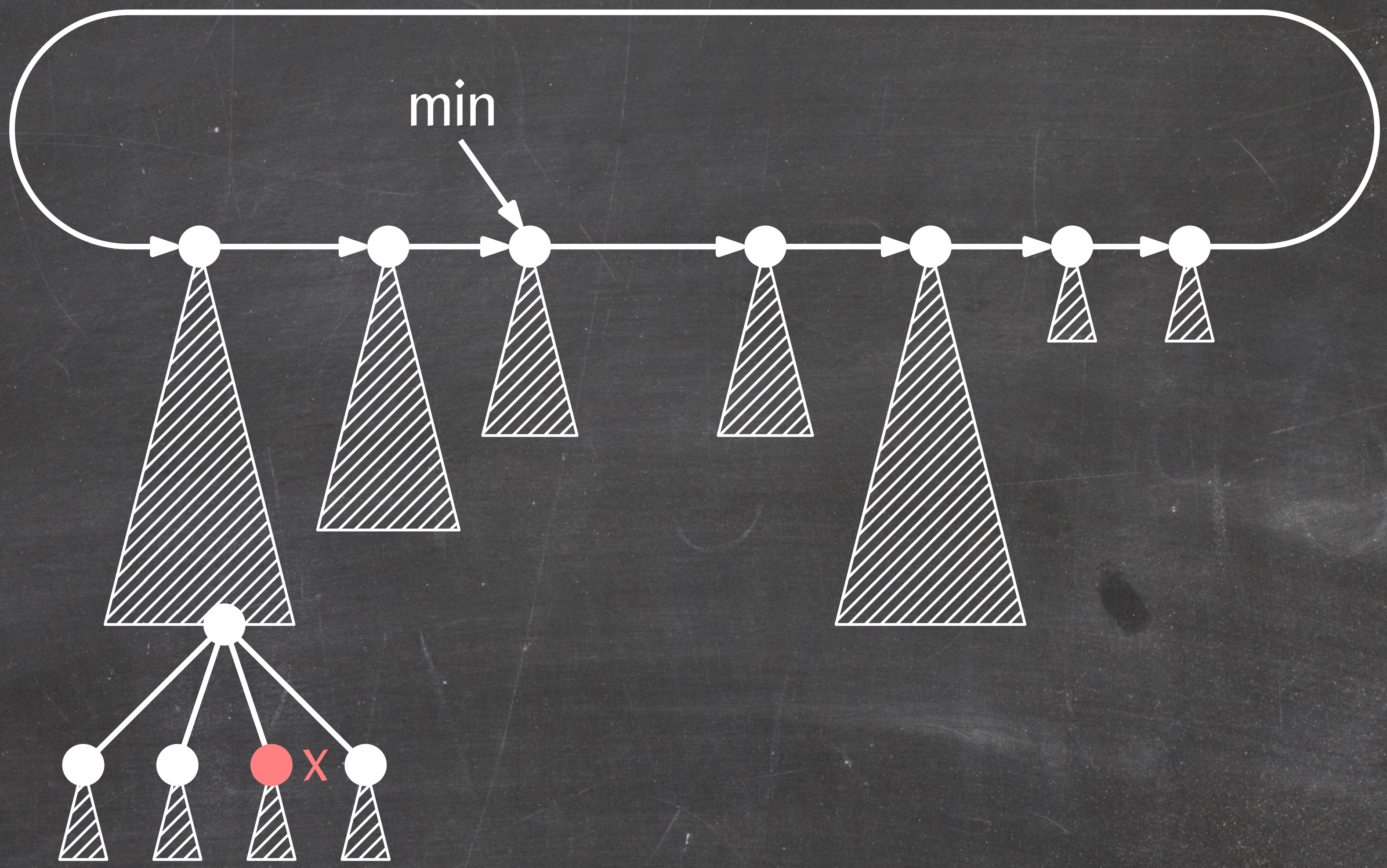

- Update x's priority

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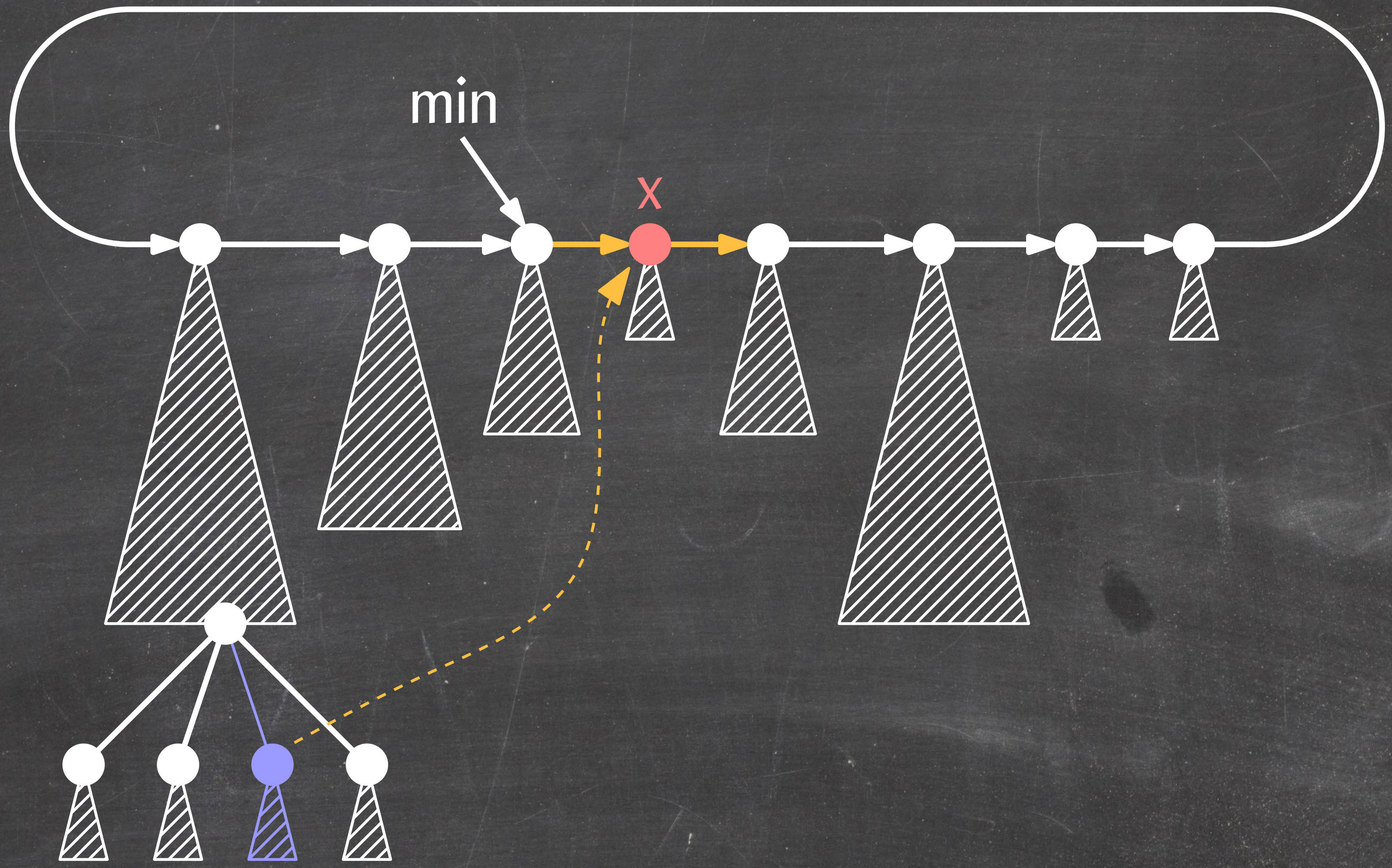

- Update x's priority
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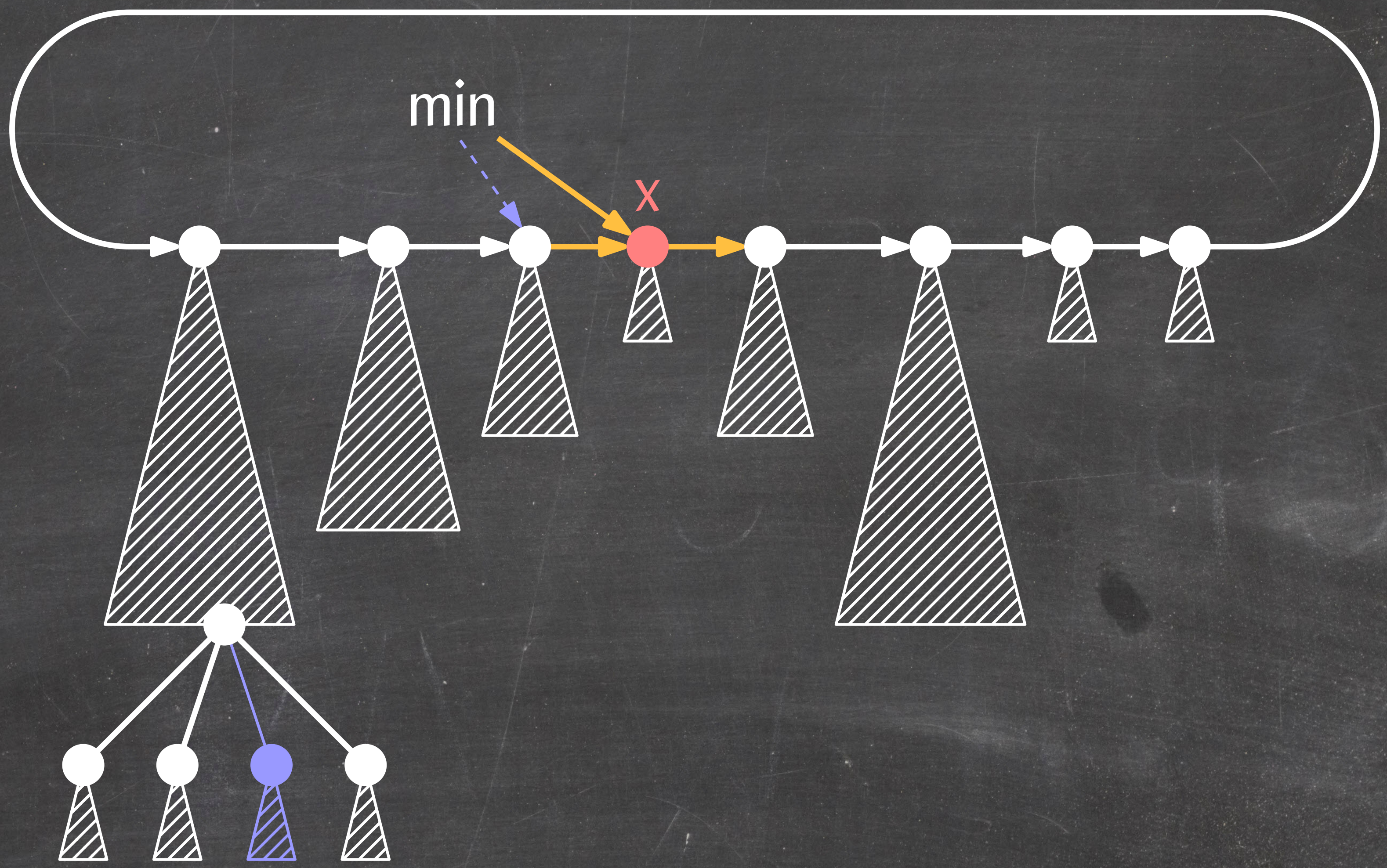

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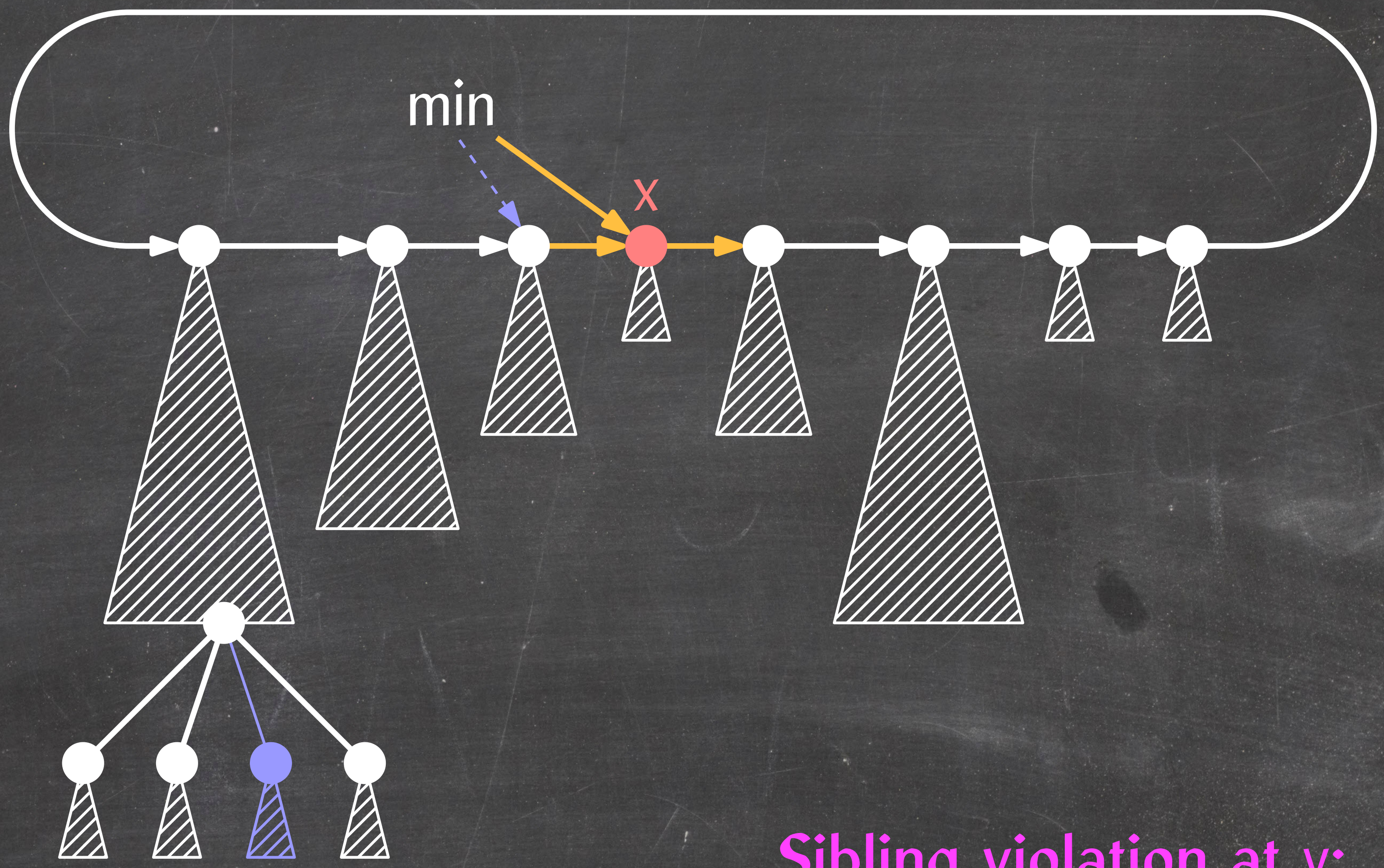

**Sibling violation at y:**

y.rank > 0 and y has no right sibling or y.rightSib.rank < y.rank − 1.

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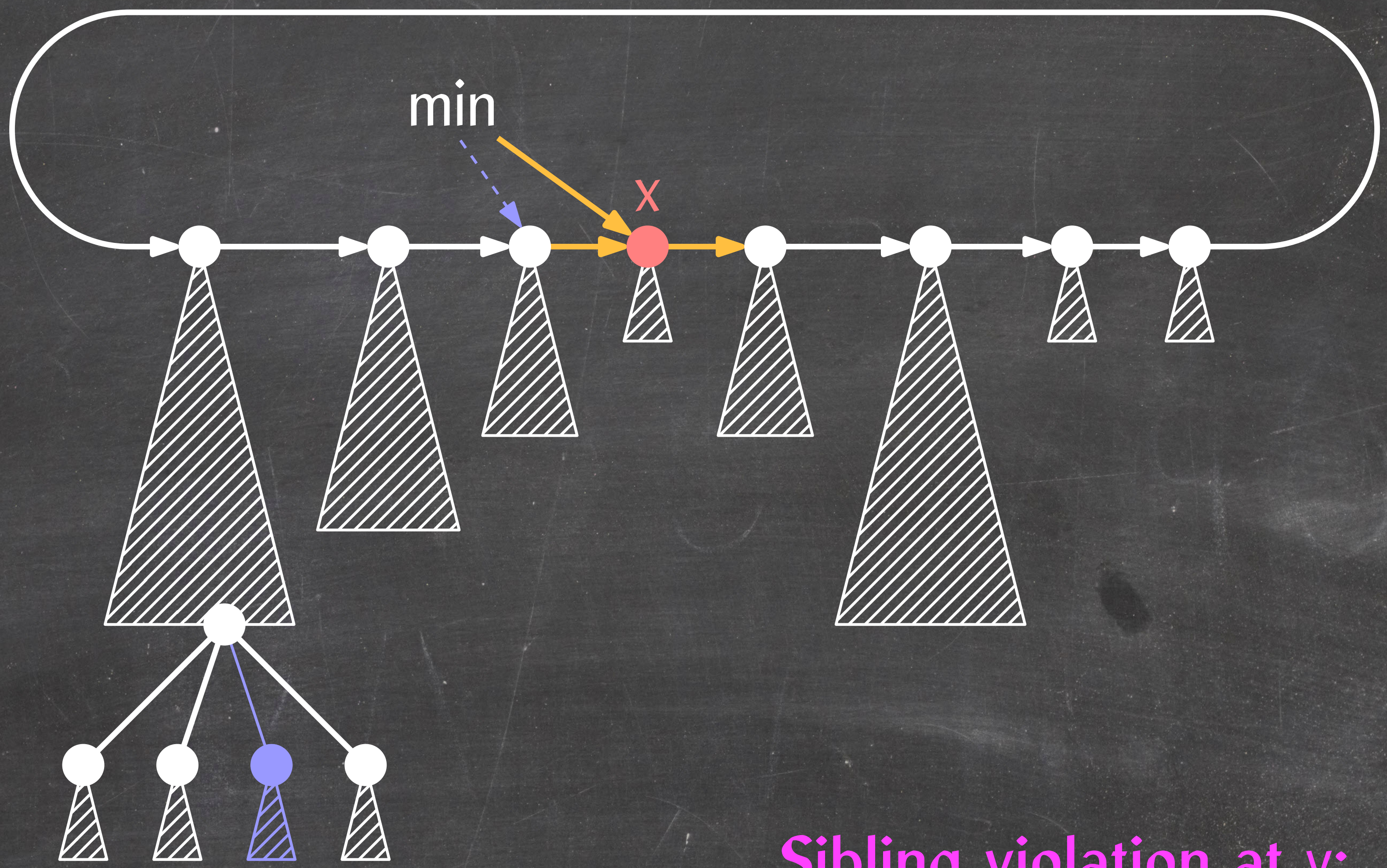

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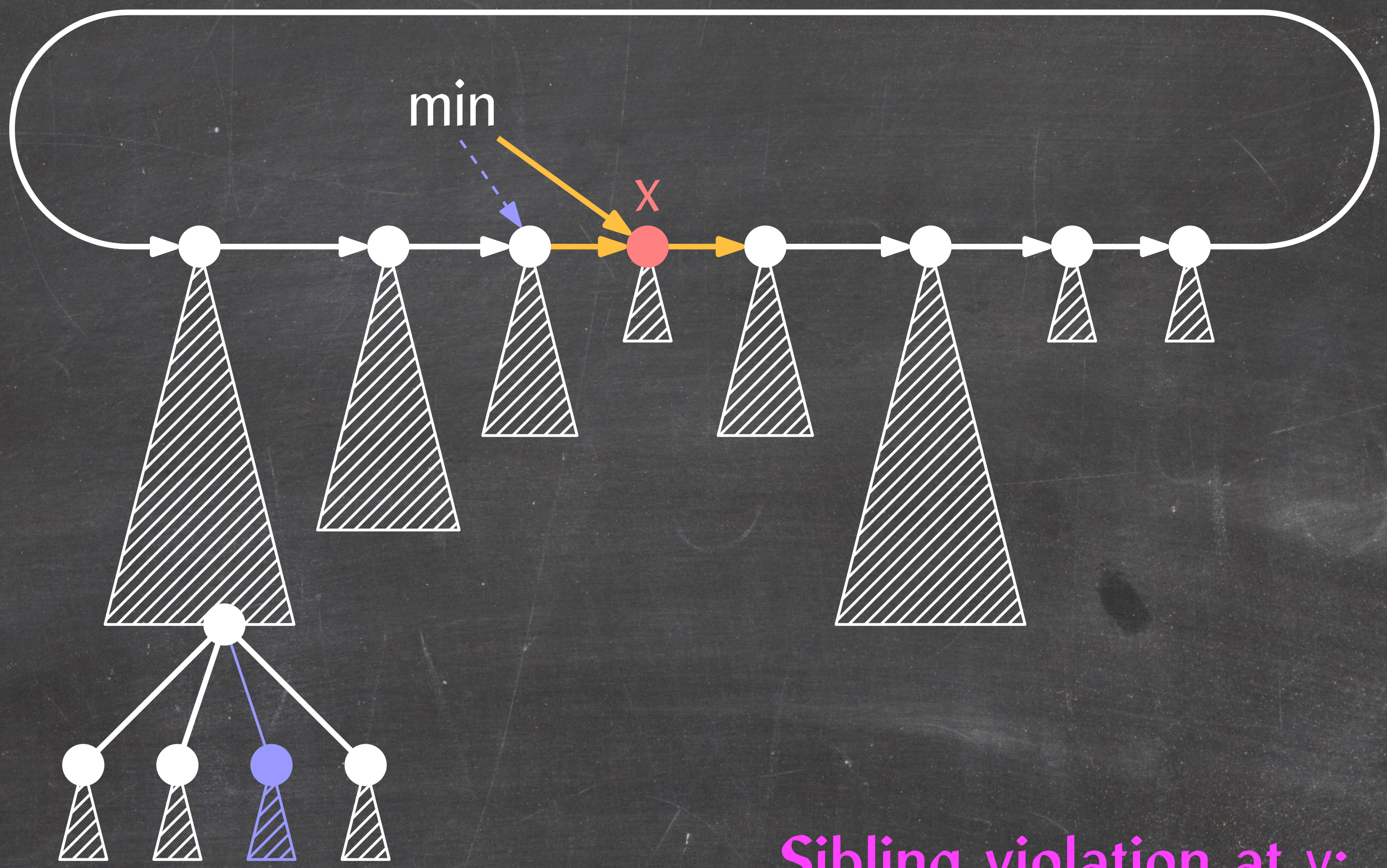

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- Update x's priority
- Make x a root
- Fix parent/sibling violations

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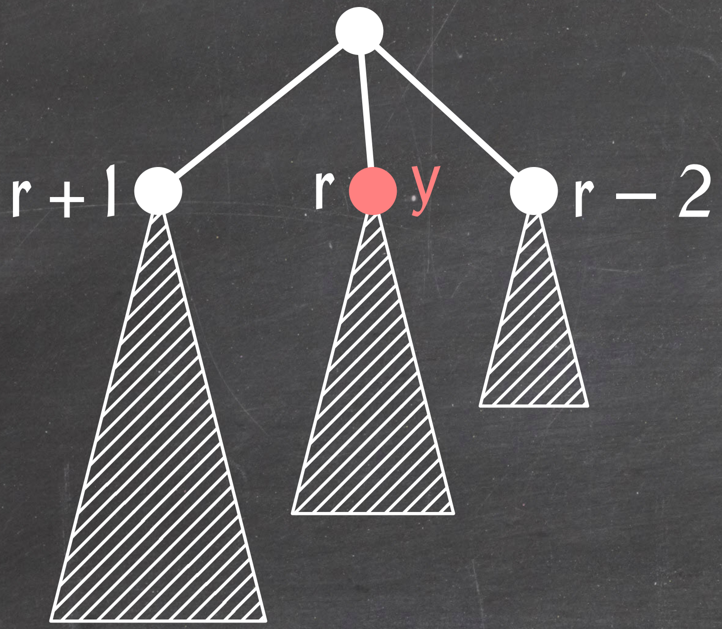

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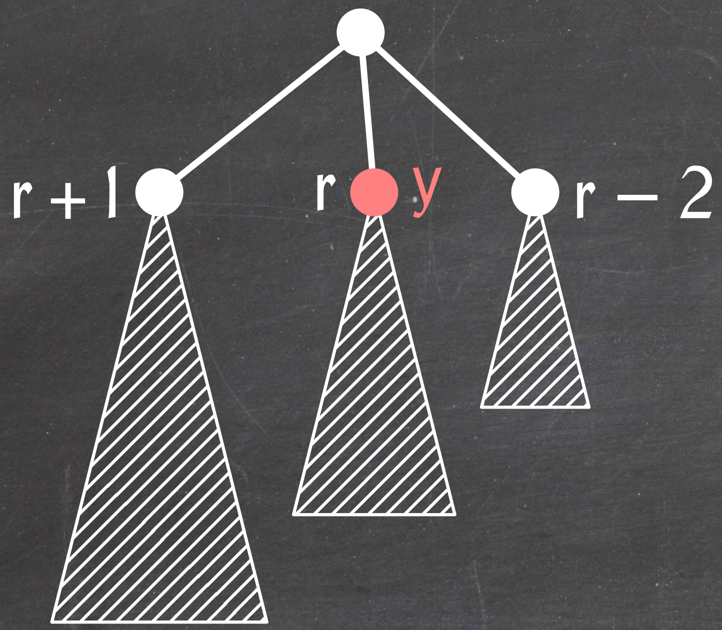

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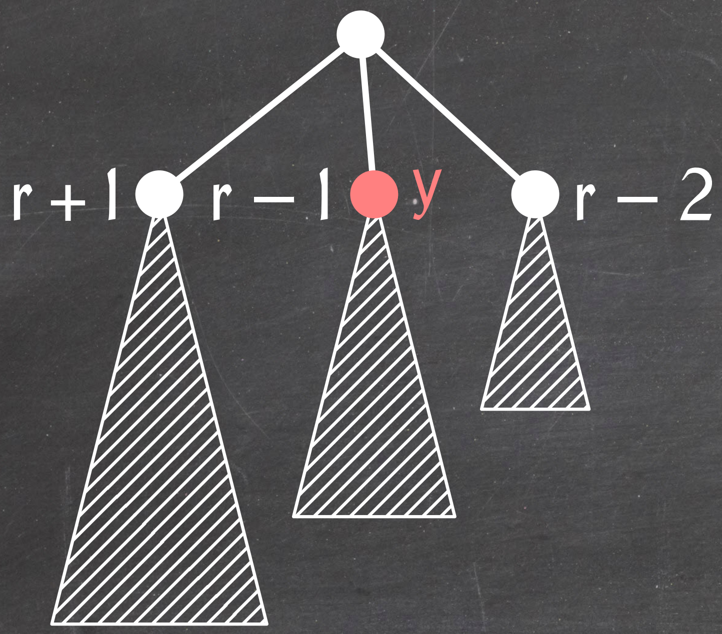

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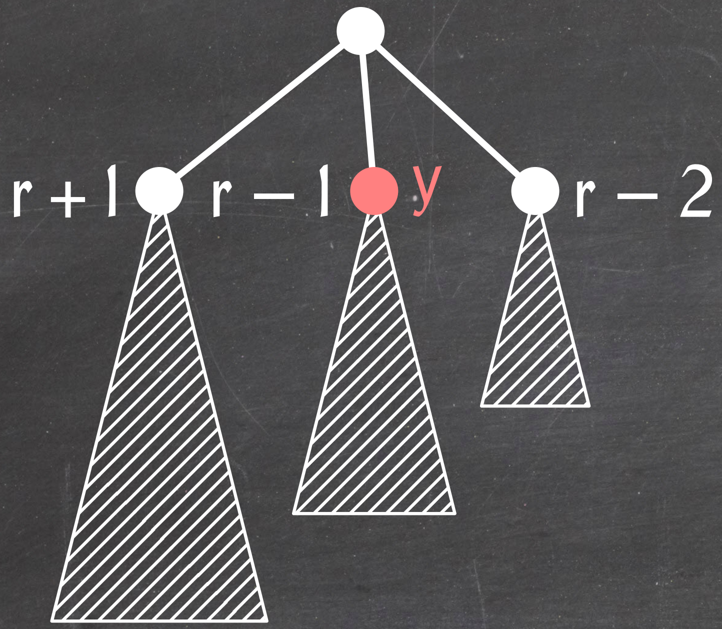

If y is thin, then

- decrease its rank by one and
- fix violation at y.leftSibOrParent.

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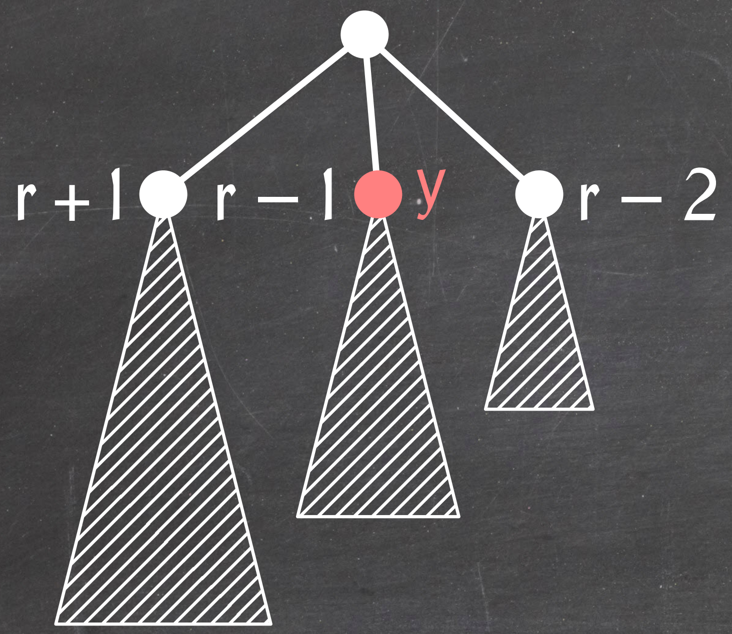

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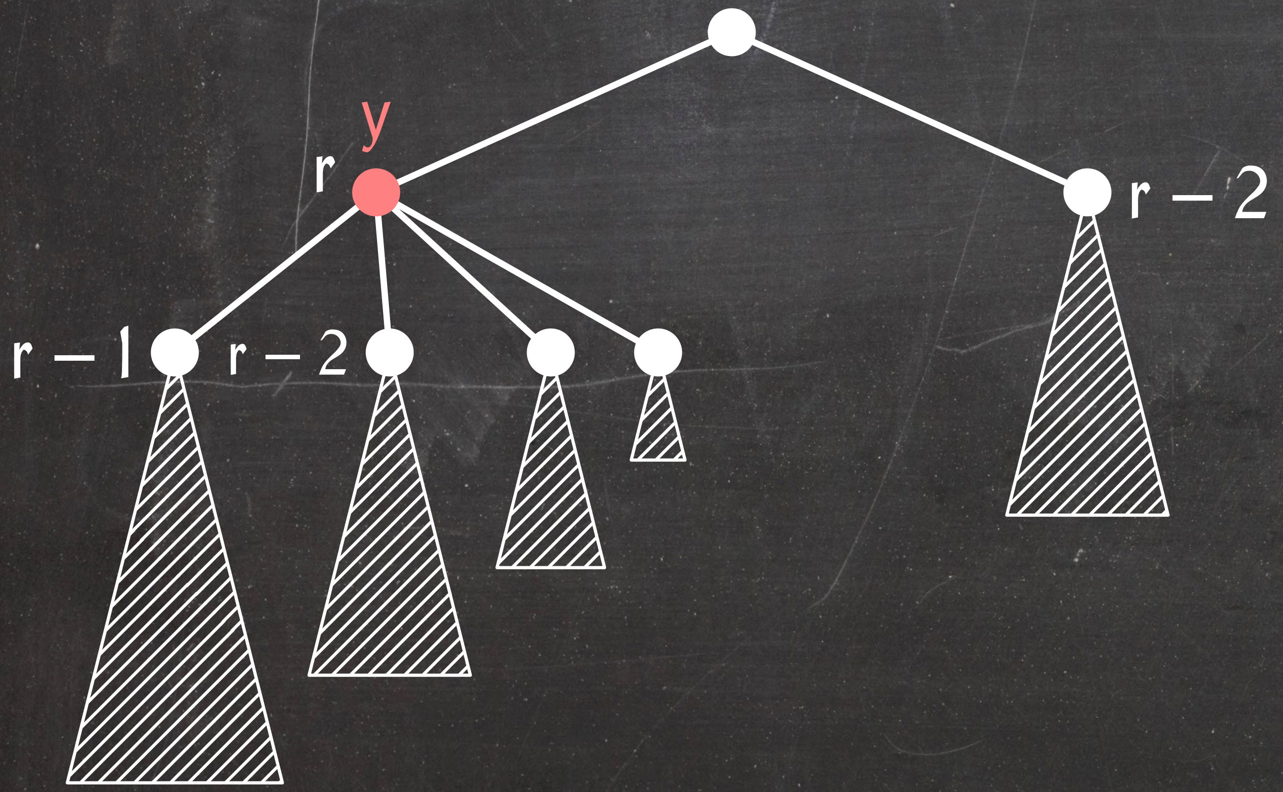

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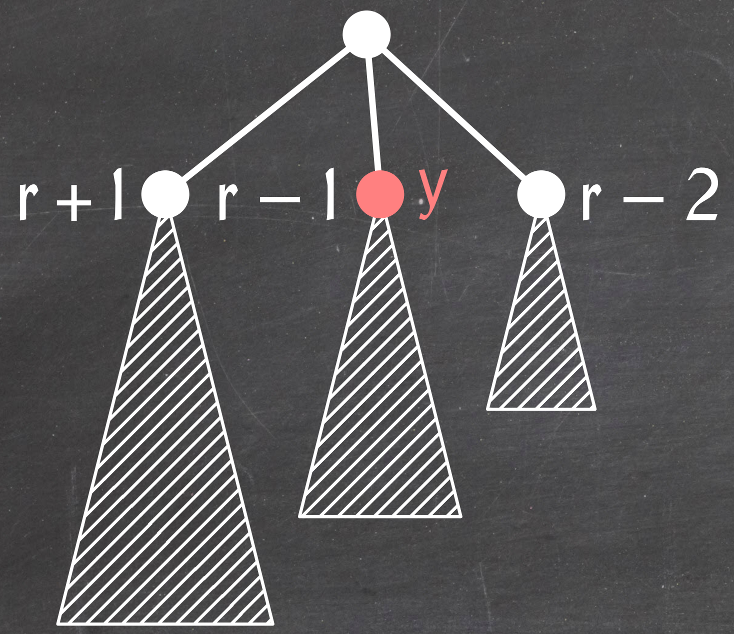

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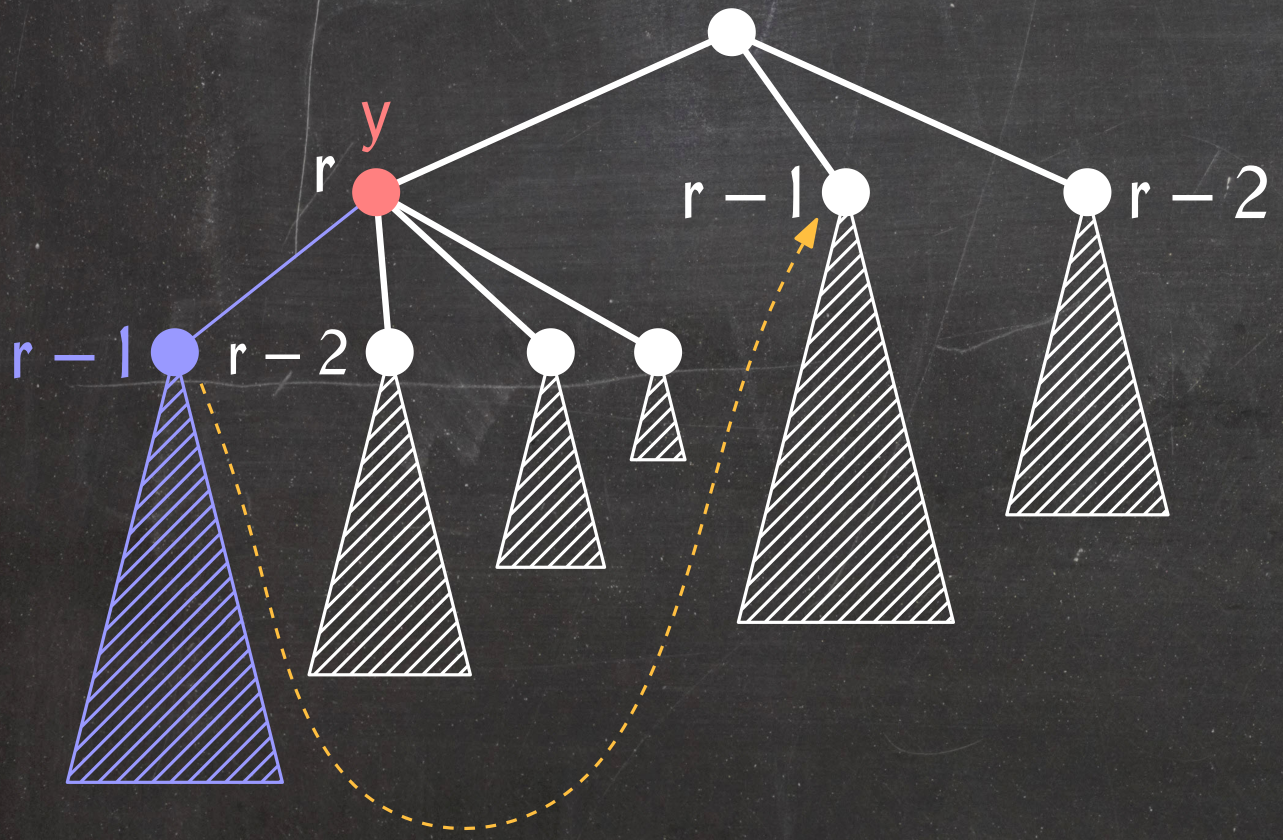

If y is thick, then make y.child y's right sibling.

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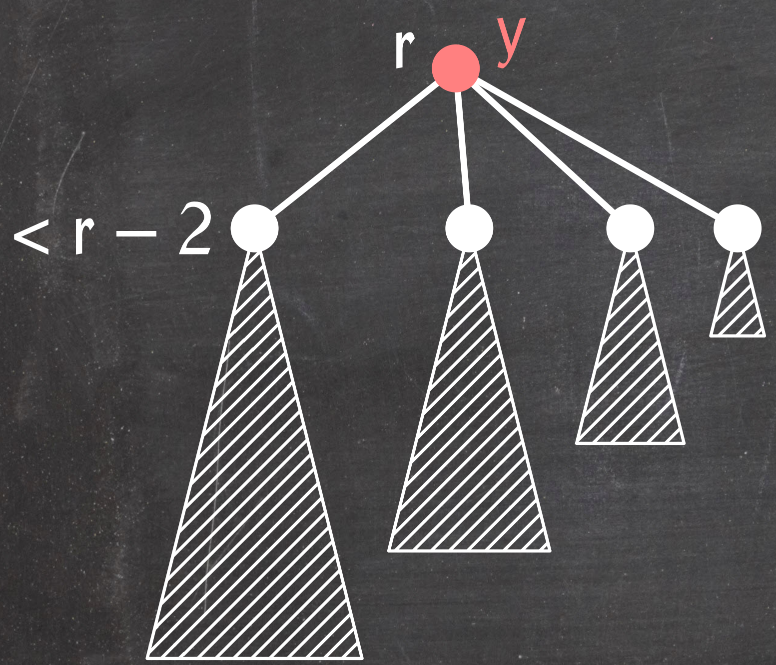

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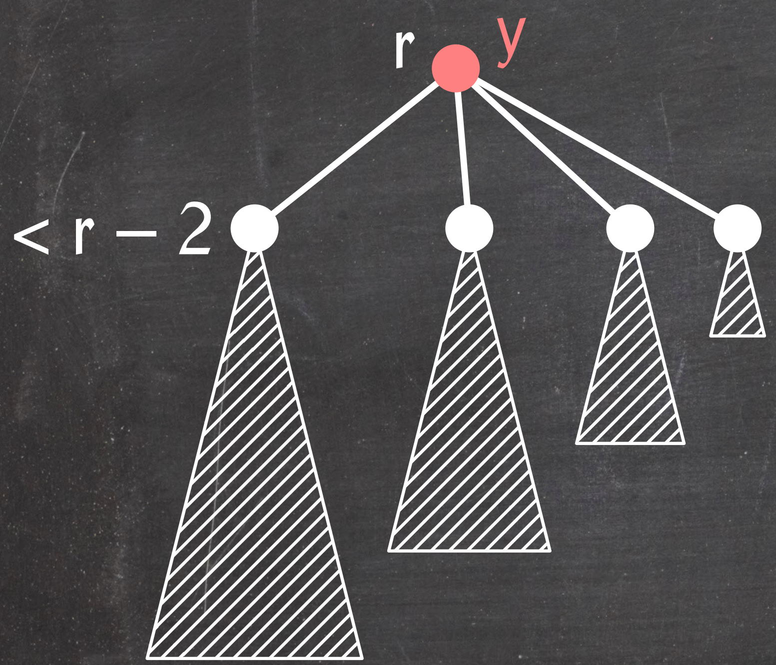

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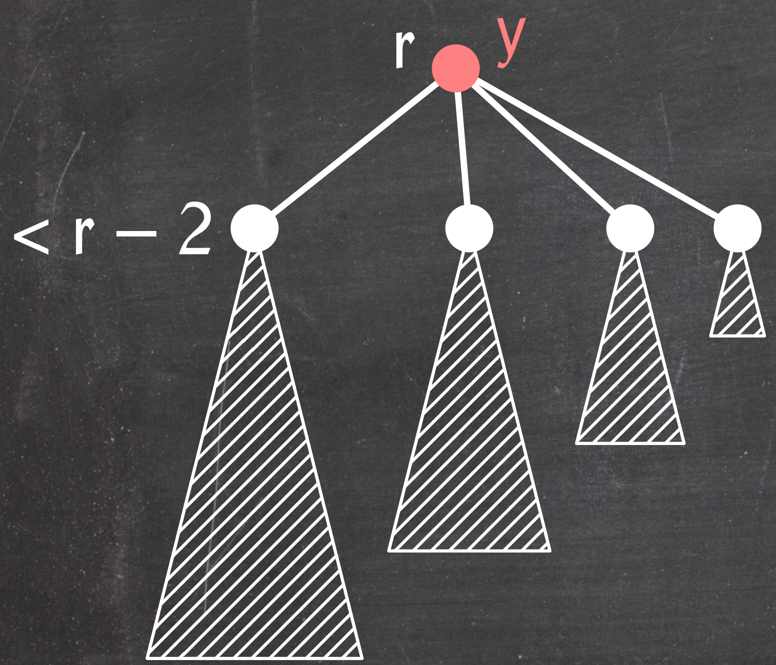

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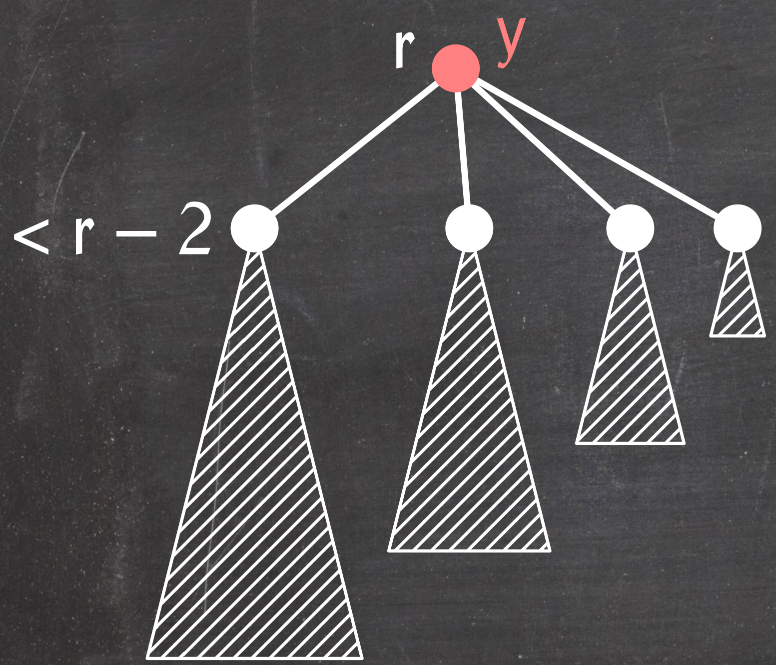

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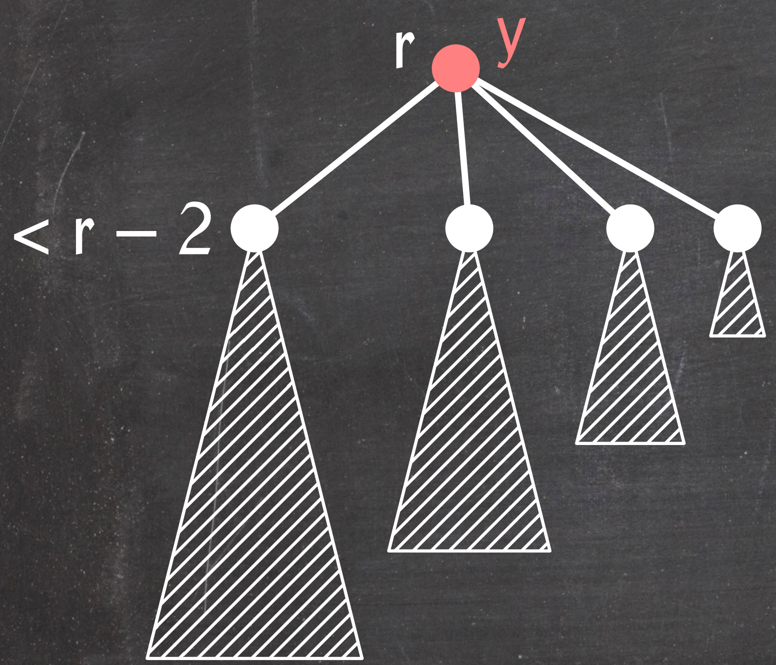

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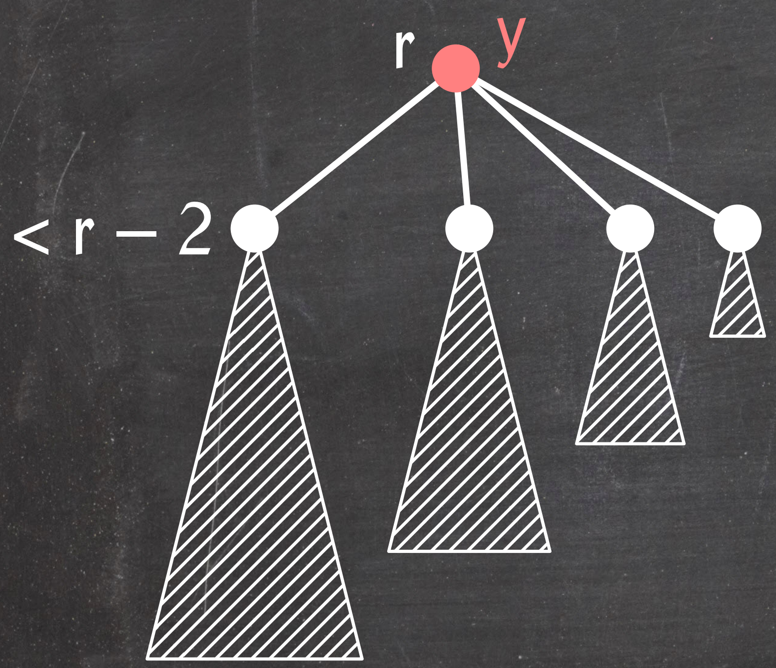

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These costs are completely fictitious but must satisfy an important condition to be useful:

$$\sum_{i=1}^{m} c_i \le \sum_{i=1}^{m} \hat{c}_i$$

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**Intuition:**

- The potential captures parts of the data structure that can make operations expensive.
- If operations that take long eliminate these "expensive" parts of the data structure, then there can't be many expensive operations without lots of operations that create these expensive parts.
- These operations can "pay" for the cost of the expensive operations.

# Amortized Analysis: Stack with MultiPop Operation

**Operations:**

| | |
|---|---|
| S.push(x) | Push element x on the stack |
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The only operation we want to support is Increment.

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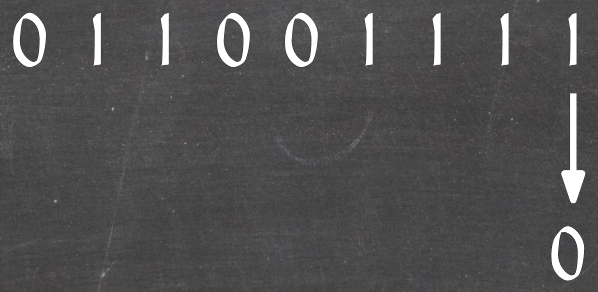

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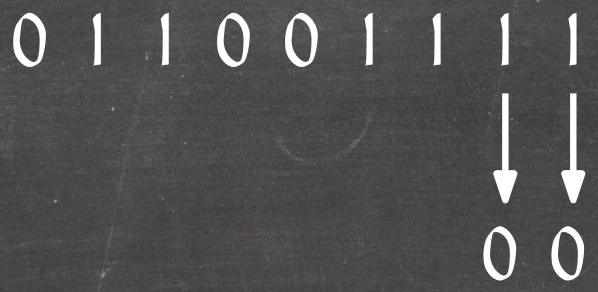

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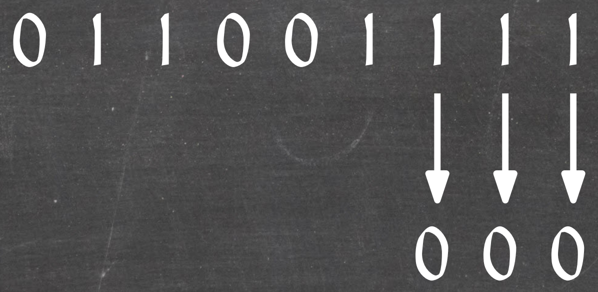

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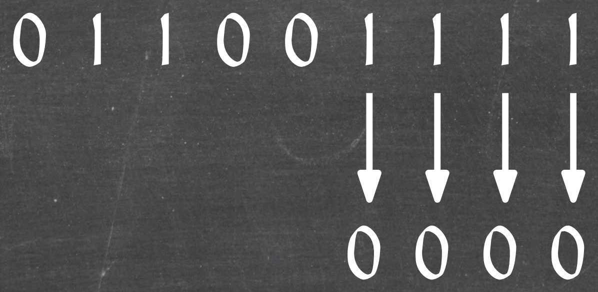

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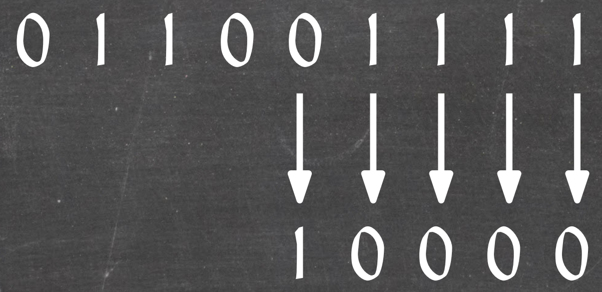

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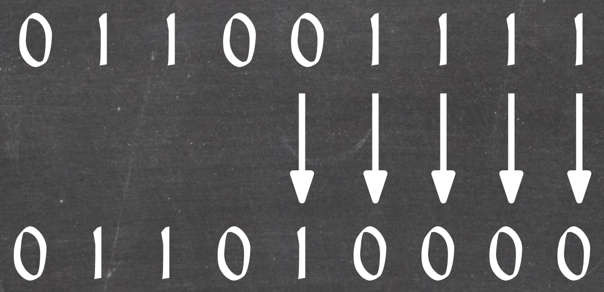

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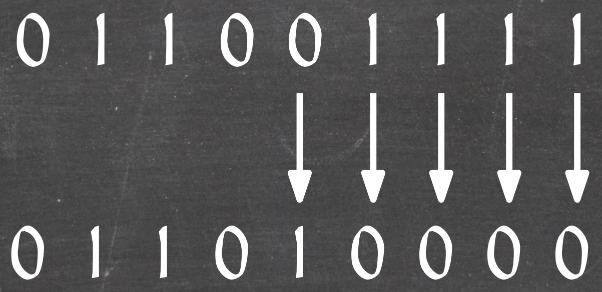

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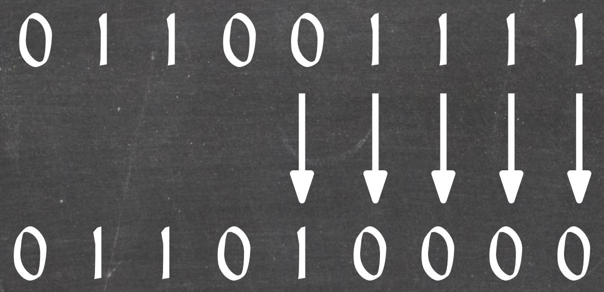

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$\Phi$ = #1s in the current counter value

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⇒ Thin nodes should be "more expensive" than roots.

$$\Phi = 2 \cdot \text{number of thin nodes} + \text{number of roots}$$

# Amortized Cost of Insert, FindMin, and Delete

**Insert:**

- $c \in O(1)$
- $\Delta\Phi = +1$:
  - $\Delta$(number of roots) $= +1$
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$\Rightarrow \hat{c} \in O(1)$

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**Delete:**

- We show that $\hat{c}$(DecreaseKey) $\in O(1)$.
- We show that $\hat{c}$(DeleteMin) $\in O(\lg n)$.

$\Rightarrow \hat{c} \in O(\lg n)$

# Amortized Cost of DeleteMin

**Actual cost:** O(lg n + number of roots + number of children of Q.min)

- O(lg n) for initializing R
- O(1) per addition to R
- O(1) per link operation
- O(lg n) to collect final list of roots from R
- Number of additions to R = number of roots and children of Q.min
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- $\Delta$(number of thin nodes) $\leq 0$
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$\Rightarrow$ $\Delta\Phi \leq 2 \lg n -$ number of roots

**Amortized cost:**

$\hat{c} = c + \Delta\Phi =$ O(lg n + number of roots) + 2 lg n − number of roots $\in$ O(lg n).

# Amortized Cost of DecreaseKey

Make affected element x a root (if it isn't already a root):

- $c \in O(1)$
- $\Delta$(number of roots) $\leq 1$
- $\Delta$(number of thin nodes) $\leq 1$:
  - x's parent becomes thin if it was thick and x is the leftmost child.

$\Rightarrow \Delta\Phi \leq 3$

$\Rightarrow \hat{c} \in O(1)$

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The remaining cost is the result of fixing violations.

We prove that

- Fixing the last violation has constant amortized cost,
- Fixing all other violations has amortized cost 0!

$\Rightarrow$ The amortized cost of fixing all violations is in $O(1)$.

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- $\Delta$(number of roots) $\leq 1$
- $\Delta$(number of thin nodes) $\leq 1$:
  - x's parent becomes thin if it was thick and x is the leftmost child.

$\Rightarrow \Delta\Phi \leq 3$

$\Rightarrow \hat{c} \in O(1)$

The remaining cost is the result of fixing violations.

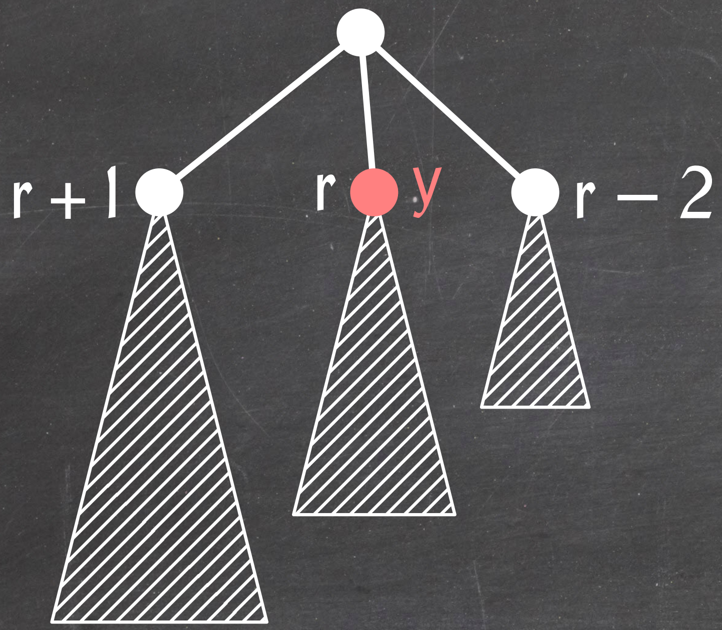
We prove that

- Fixing the last violation has constant amortized cost,
- Fixing all other violations has amortized cost 0!

$\Rightarrow$ The amortized cost of fixing all violations is in $O(1)$.

$\Rightarrow \hat{c}(\text{DecreaseKey}) \in O(1)$.

# Amortized Cost of Fixing Sibling Violations



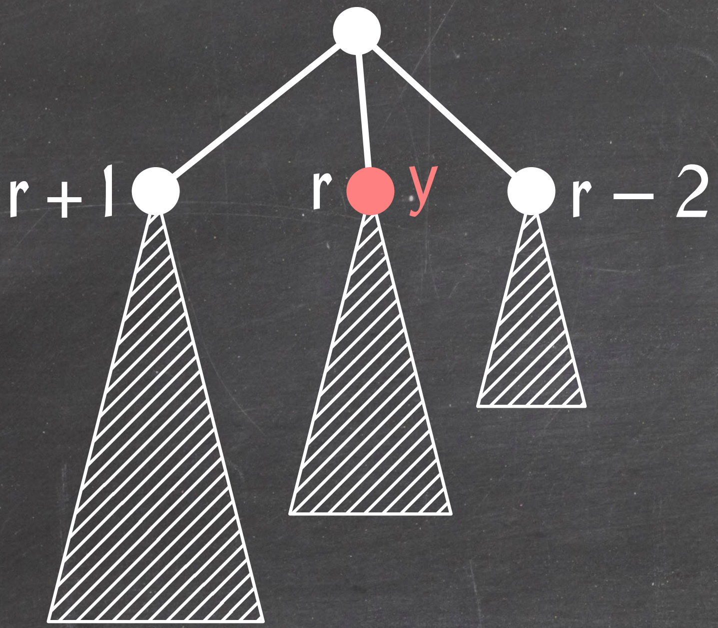
If y is thin,

- $c \in O(1)$
- $\Delta$(number of thin nodes) = $-1$
- $\Delta$(number of roots) = 0

$\Rightarrow \Delta\Phi = -2$

$\Rightarrow \hat{c} = 0$

# Amortized Cost of Fixing Sibling Violations



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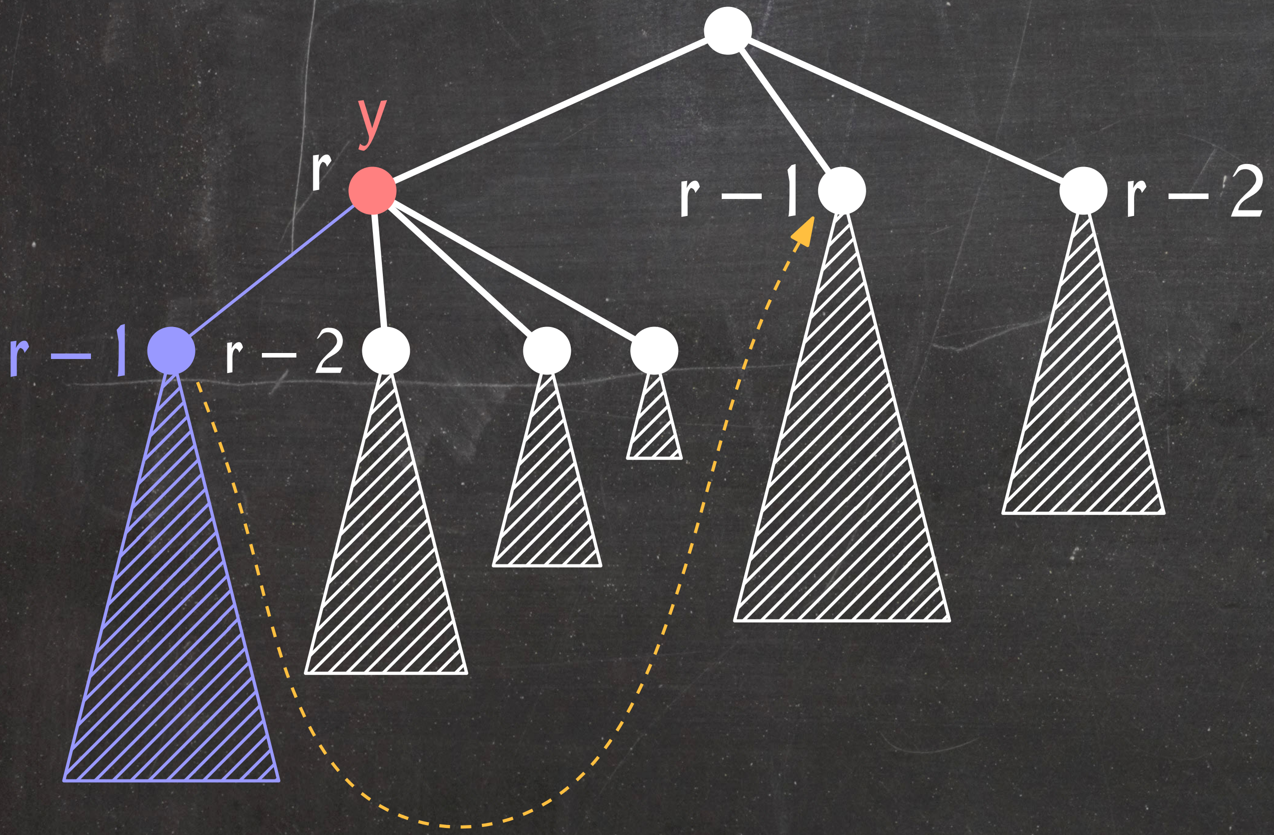


If y is thick,

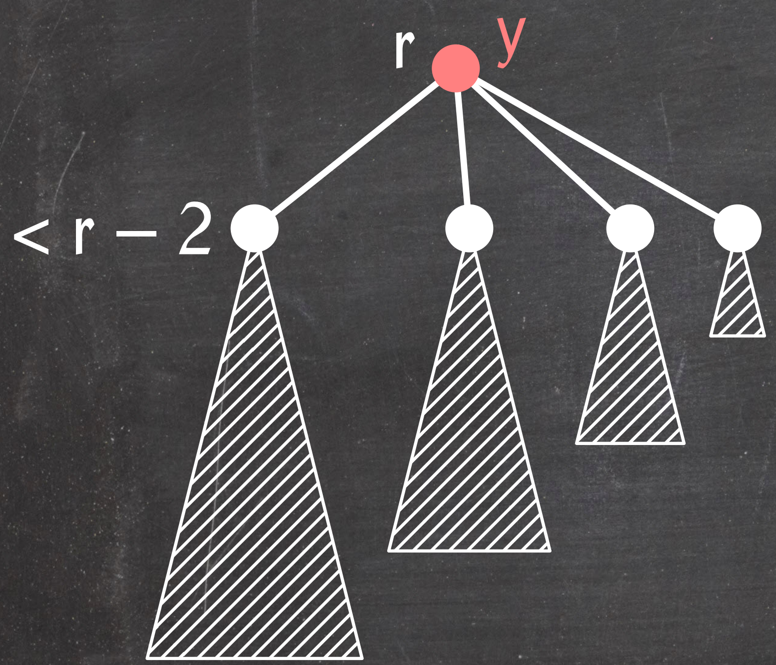
- $c \in O(1)$
- $\Delta$(number of thin nodes) = $+1$
- $\Delta$(number of roots) = 0

$\Rightarrow \Delta\Phi = +2$

$\Rightarrow \hat{c} \in O(1)$

After this, we're done!

# Amortized Cost of Fixing Parent Violations



If y is a root, then

- $c \in O(1)$
- $\Delta(\text{number of roots}) = 0$
- $\Delta(\text{number of thin nodes}) = -1$

$\Rightarrow \Delta\Phi = -2$

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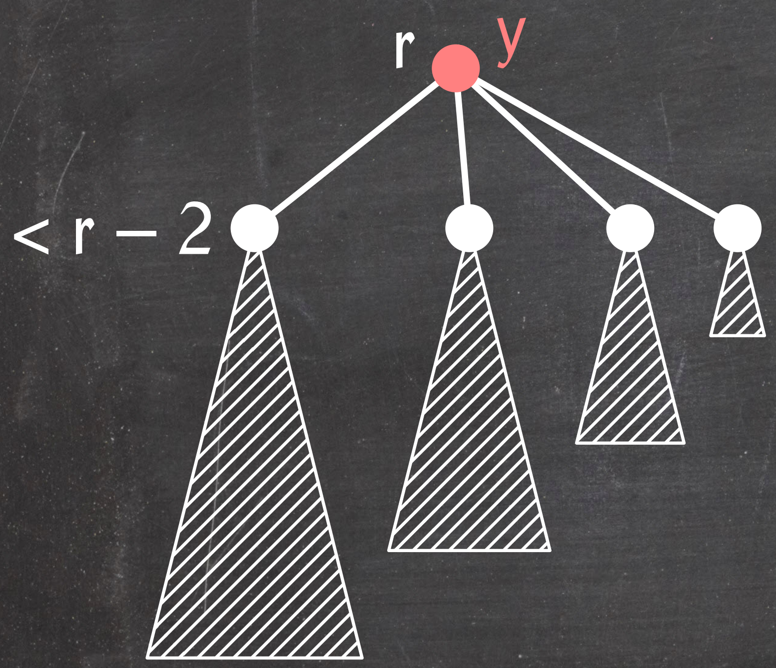

If y is a root, then

- $c \in O(1)$
- Δ(number of roots) = 0
- Δ(number of thin nodes) = −1

$\Rightarrow \Delta\Phi = -2$

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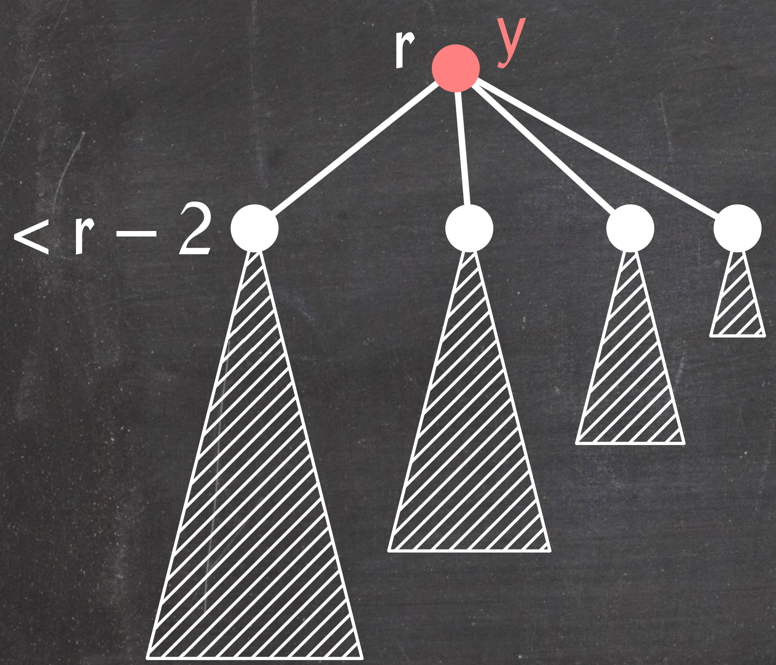
If y is not a root and is not the leftmost child of its parent, then

- $c \in O(1)$
- Δ(number of roots) = +1
- Δ(number of thin nodes) = −1

$\Rightarrow \Delta\Phi = -1$

$\Rightarrow \hat{c} = 0$

# Amortized Cost of Fixing Parent Violations



If y is not a root and is the leftmost child of its parent, and its parent is thin, then

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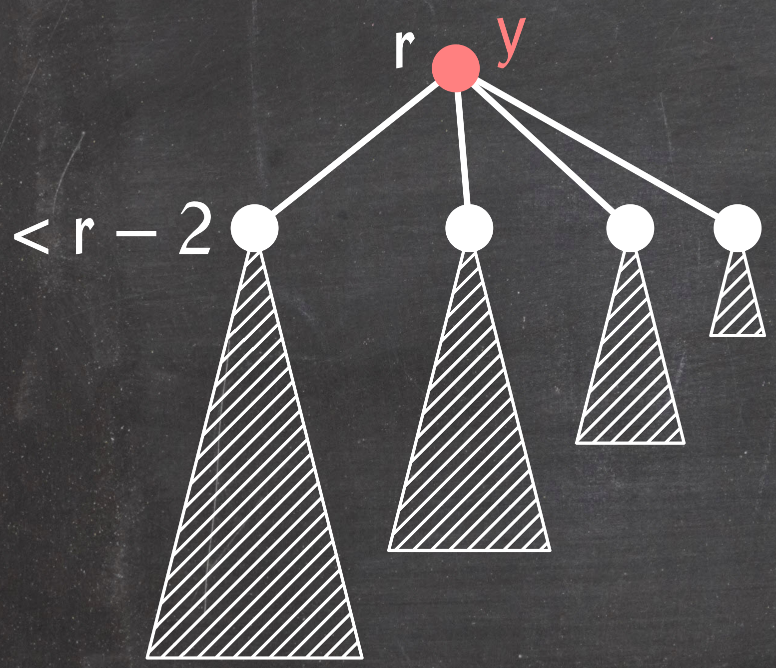

If y is not a root and is the leftmost child of its parent, and its parent is thin, then

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If y is not a root and is the leftmost child of its parent, and its parent is thick, then

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- $\Delta$(number of roots) = +1
- $\Delta$(number of thin nodes) = 0

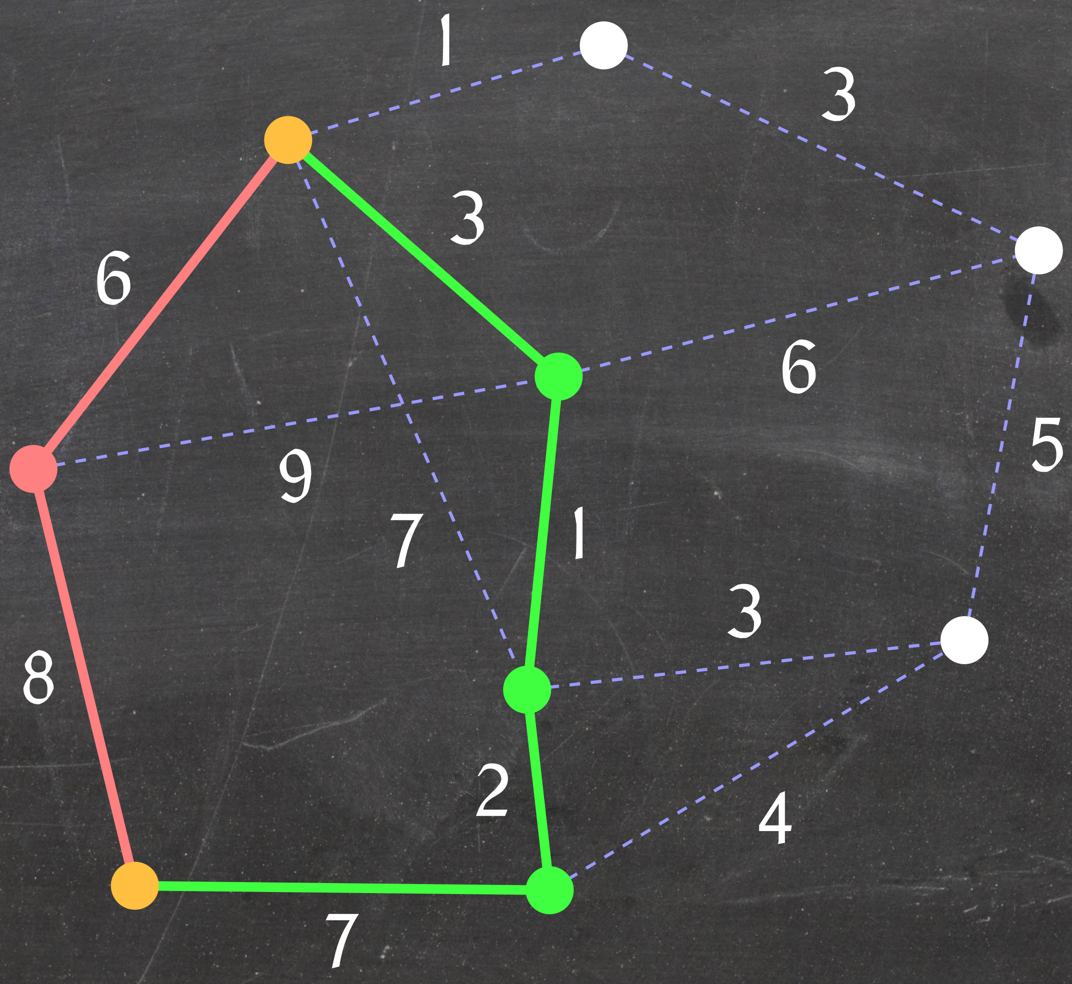
$\Rightarrow$ $\Delta\Phi = +1$

$\Rightarrow$ $\hat{c} \in O(1)$

After this, we're done!

# Shortest Path

Given a graph $G = (V, E)$ and an assignment of weights (costs) to the edges of G, a shortest path from u to v is a path from u to v with minimum total edge weight among all paths from u to v.



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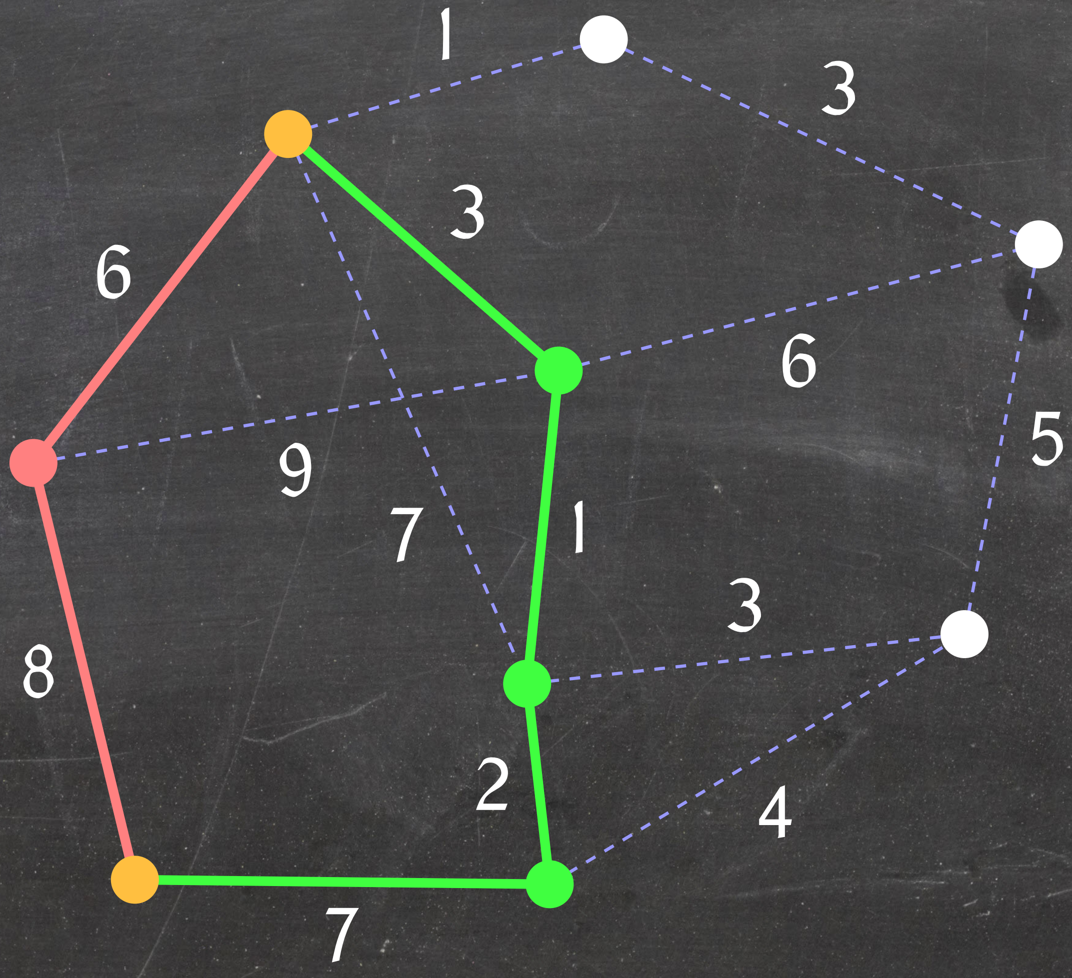

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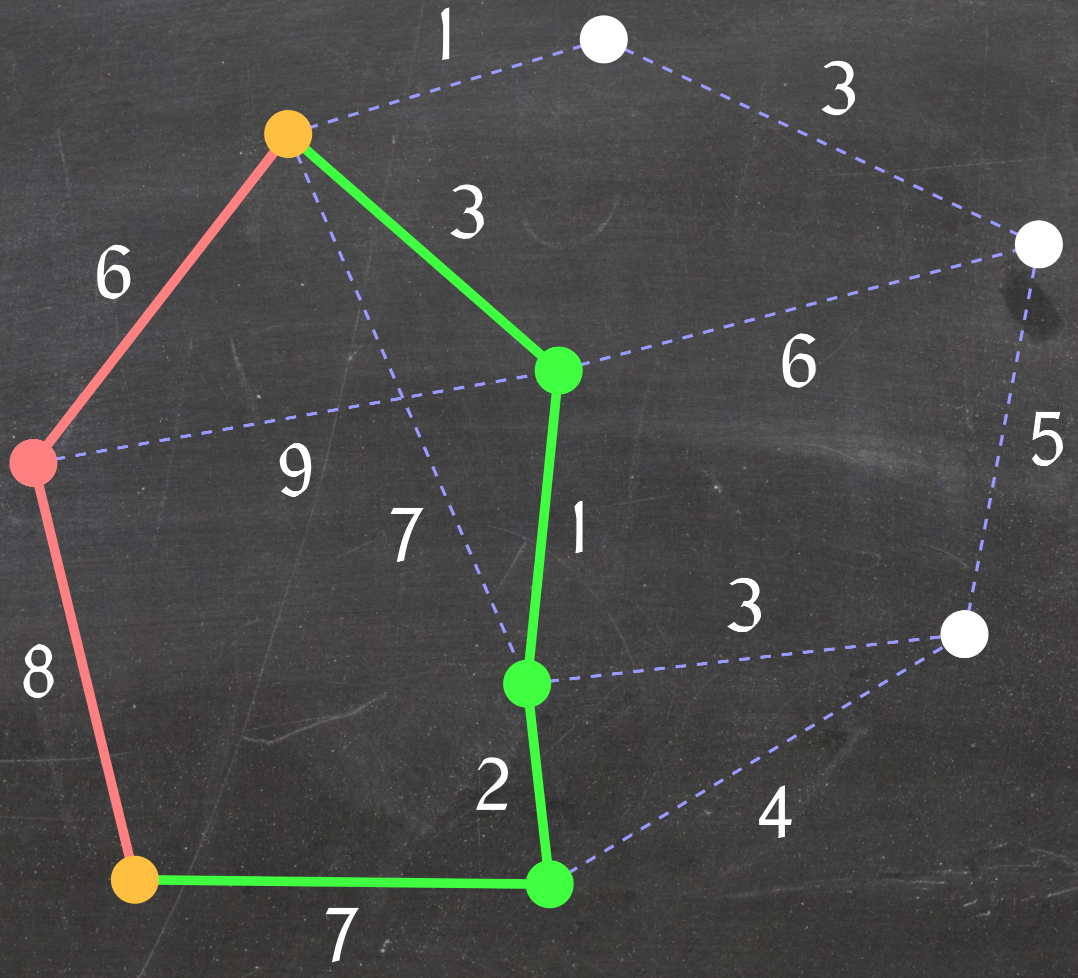

This is well-defined only if there is no negative cycle (cycle with negative total edge weight) that has a vertex on a path from u to v.

# Optimal Substructure of Shortest Paths

For a path P and two vertices u and w in P, let P[u, w] be the subpath of P from u to w.

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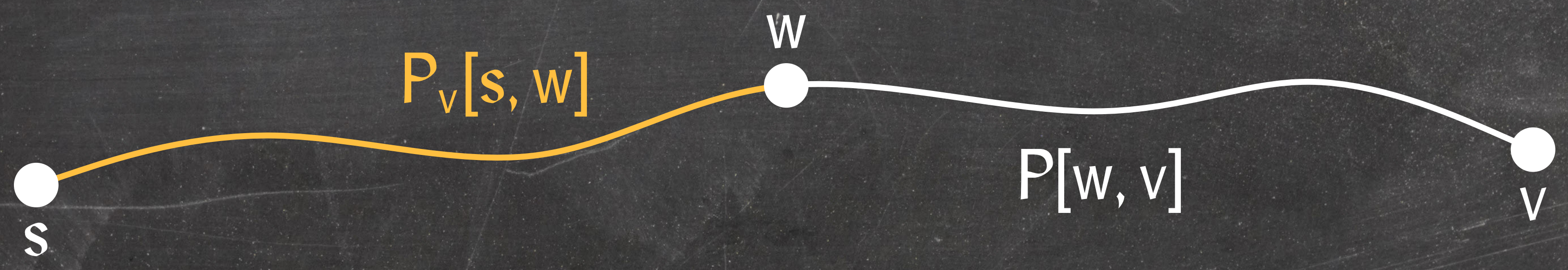

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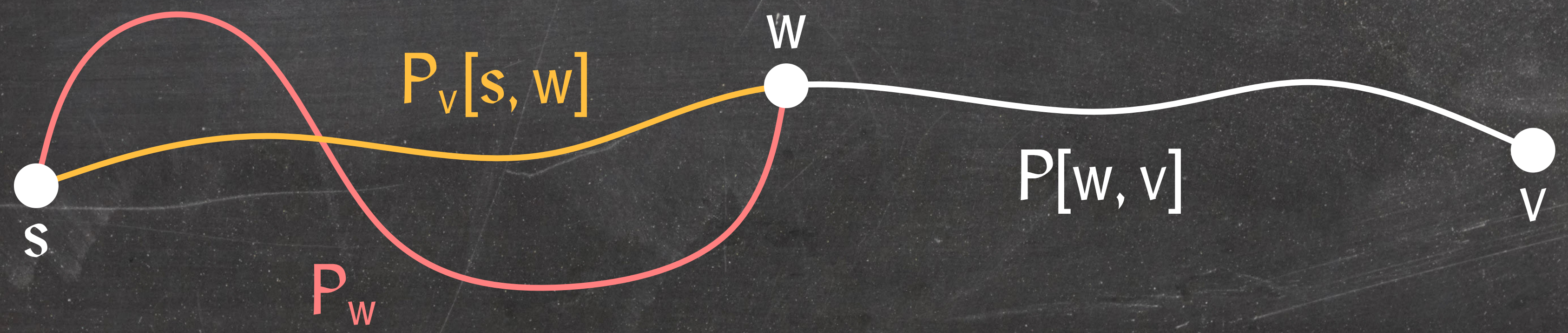
Assume there exists a path $P_w$ from s to w with $w(P_w) < w(P_v[s, w])$.



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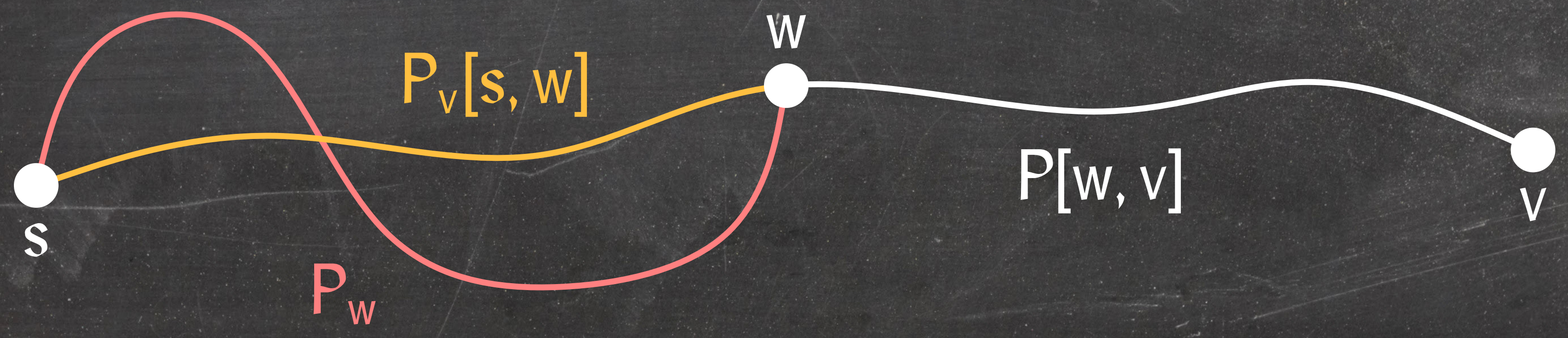

Then $w(P_w \circ P_v[w, v]) < w(P_v[s, w] \circ P_v[w, v]) = w(P_v)$, a contradiction because $P_v$ is a shortest path from s to v.

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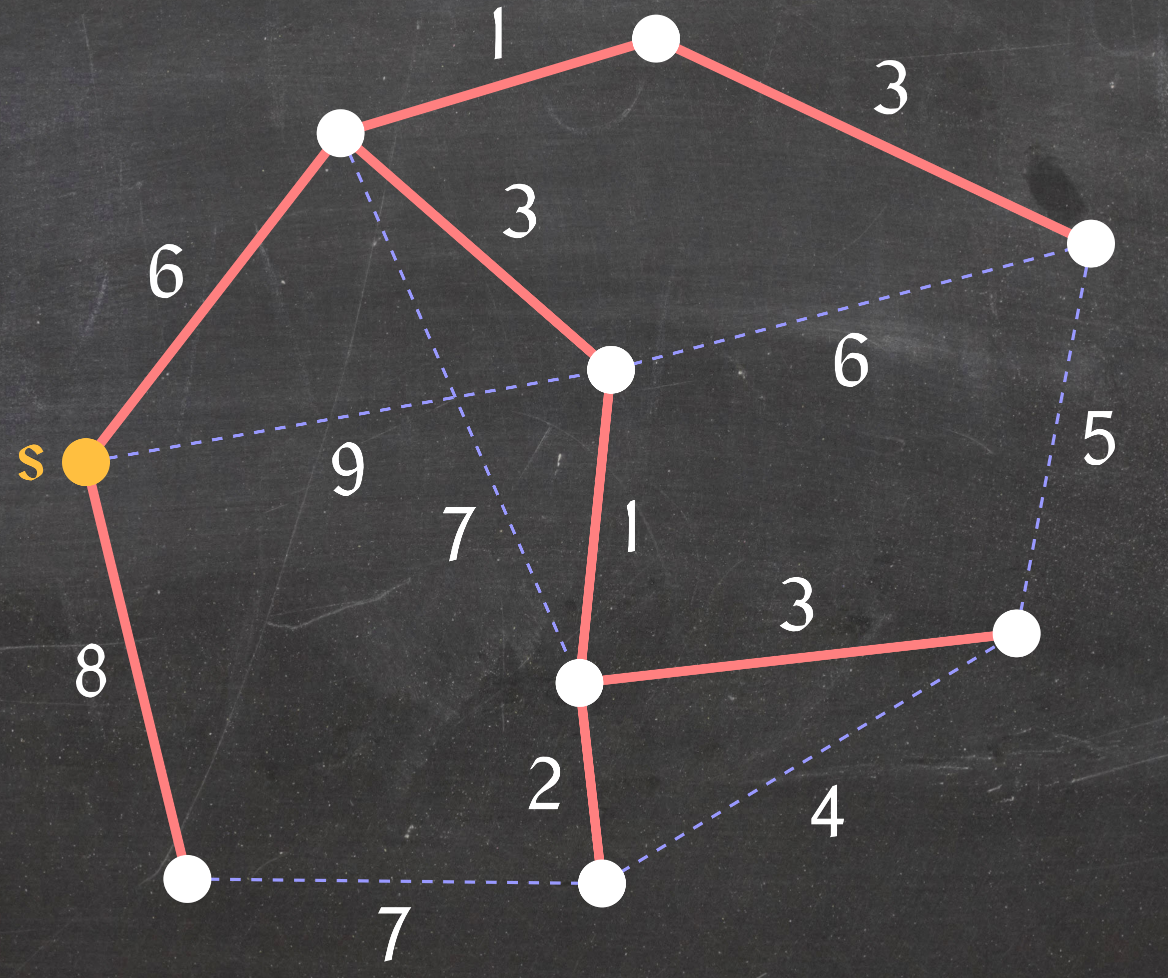
For a vertex $s \in G$, let $R(s)$ be the set of vertices reachable from $s$: for every vertex $v \in R(s)$, there exists a path from $s$ to $v$.

**Lemma:** For every node $s \in G$, there exists a collection of paths $S = \{P_v \mid v \in R(s)\}$ such that $P_v$ is a shortest path from $s$ to $v$ and $\bigcup_{v \in R(s)} P_v$ is a tree.



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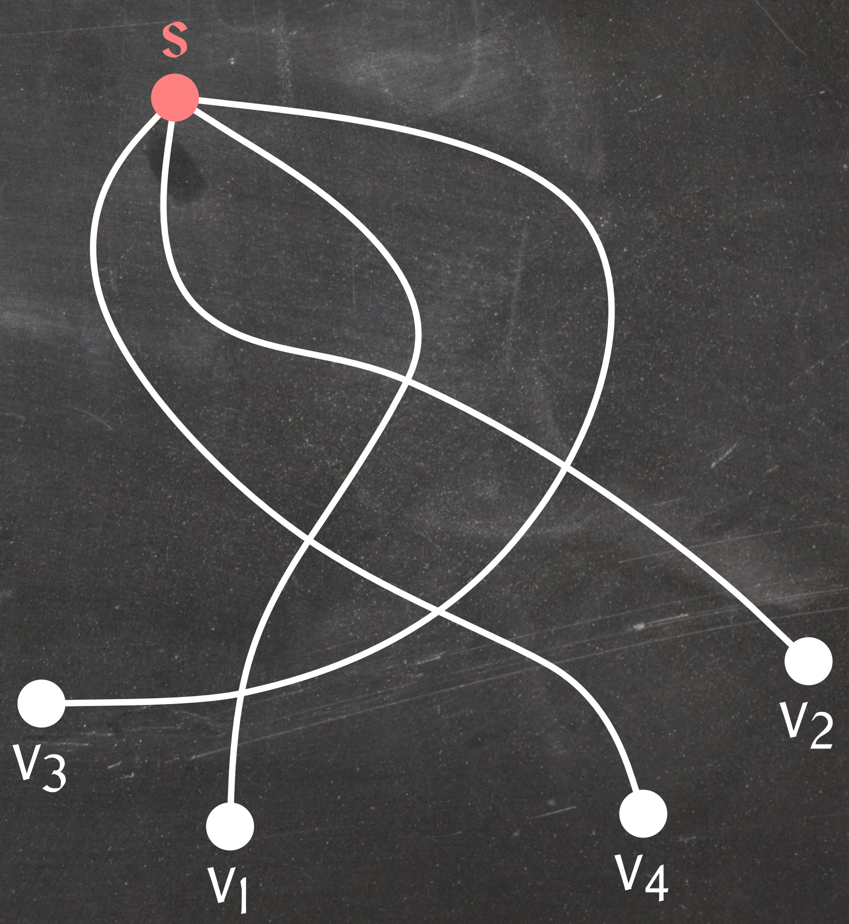
Let $R(s) = \{v_1, v_2, \ldots, v_t\}$ and let $\{P'_{v_1}, P'_{v_2}, \ldots, P'_{v_t}\}$ be a collection of shortest paths from s to these vertices.

We define a sequence of trees $\langle T_1, T_2, \ldots, T_t \rangle$ and shortest paths $\langle P_{v_1}, P_{v_2}, \ldots, P_{v_t} \rangle$ as follows:



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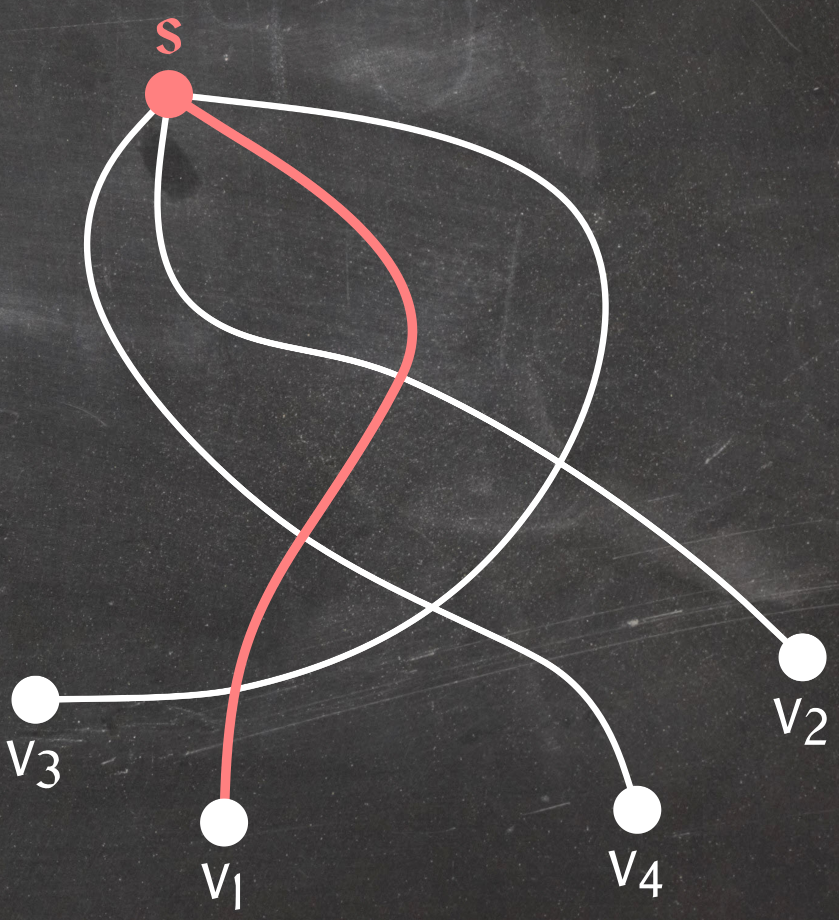

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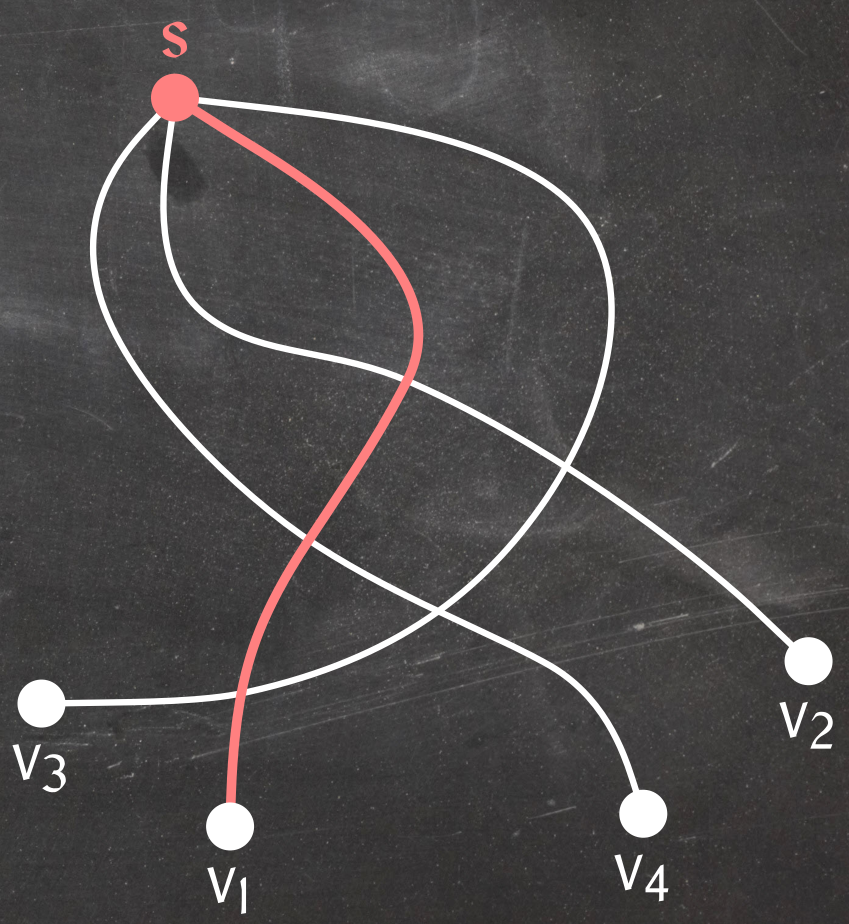

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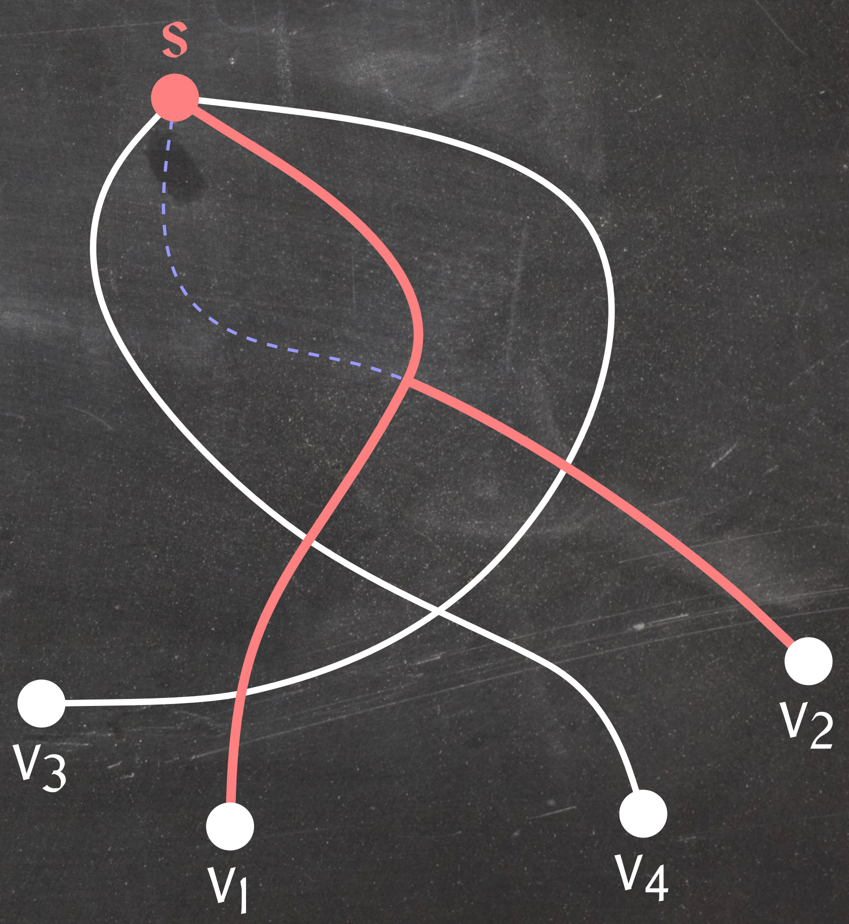

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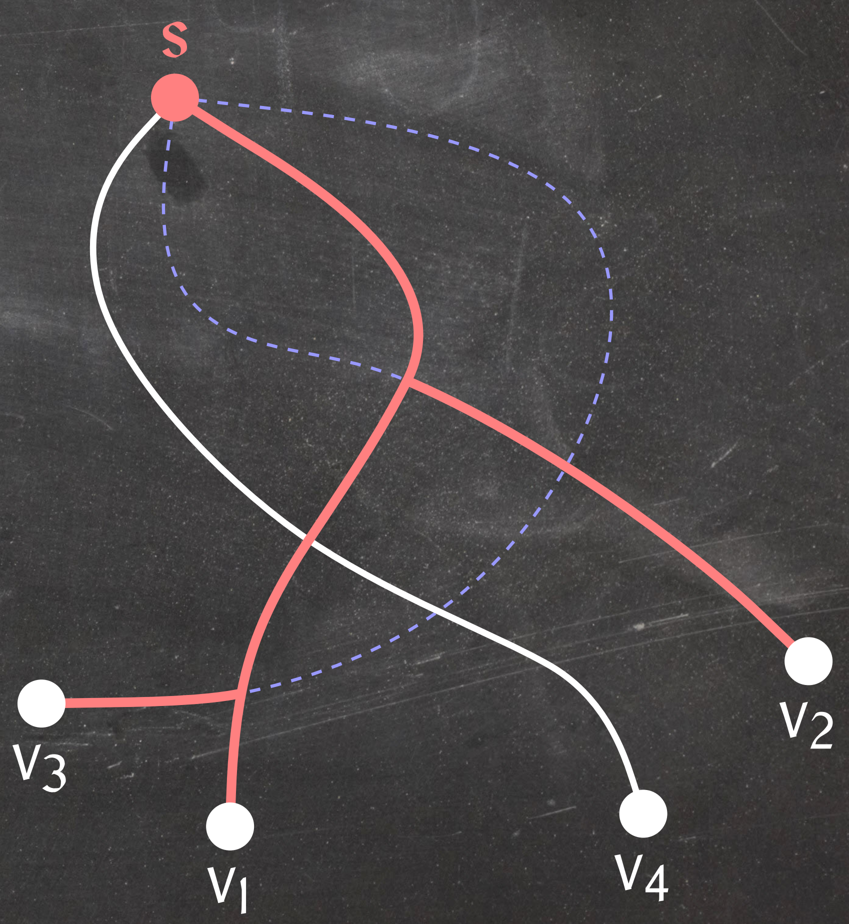

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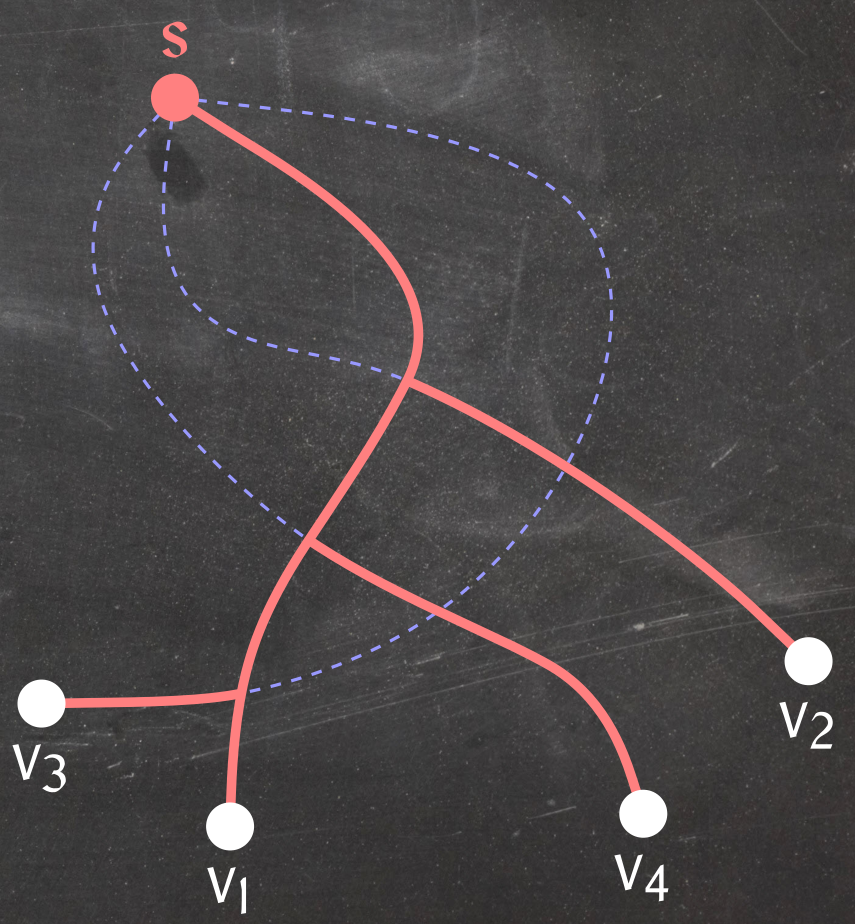

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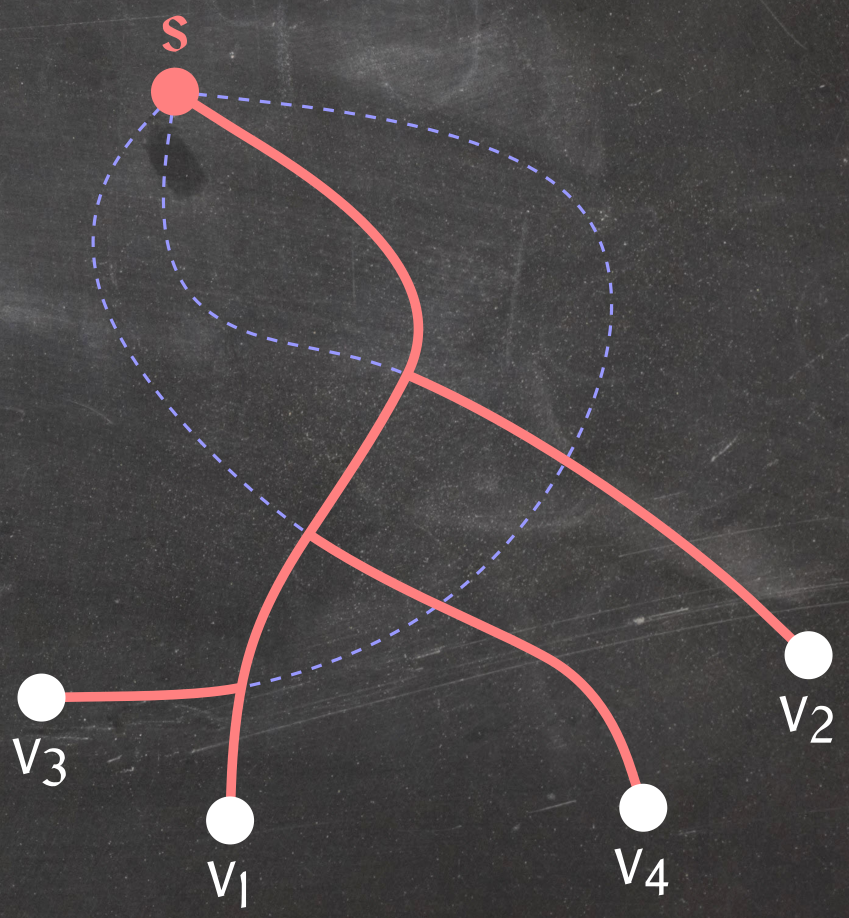

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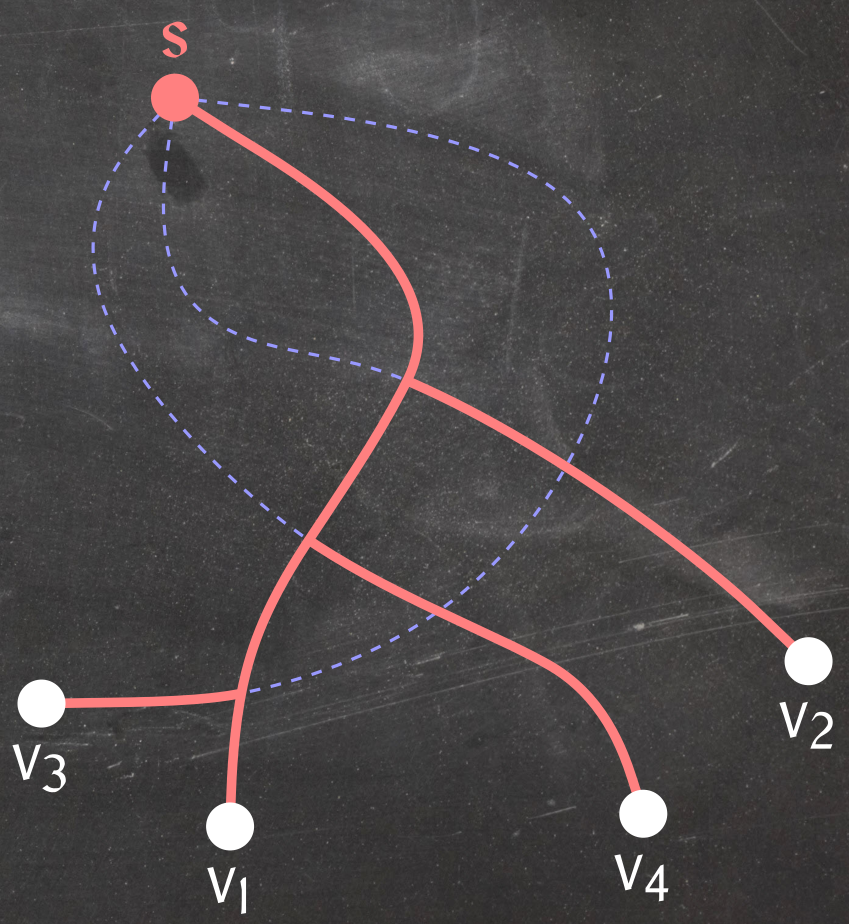
$T_t$ is a tree:

- $T_1$ is a tree.
- $T_i$ is obtained by adding a path to $T_{i-1}$ that shares only one vertex with $T_{i-1}$.
- To create a cycle, the added path would have to share two vertices with $T_{i-1}$.



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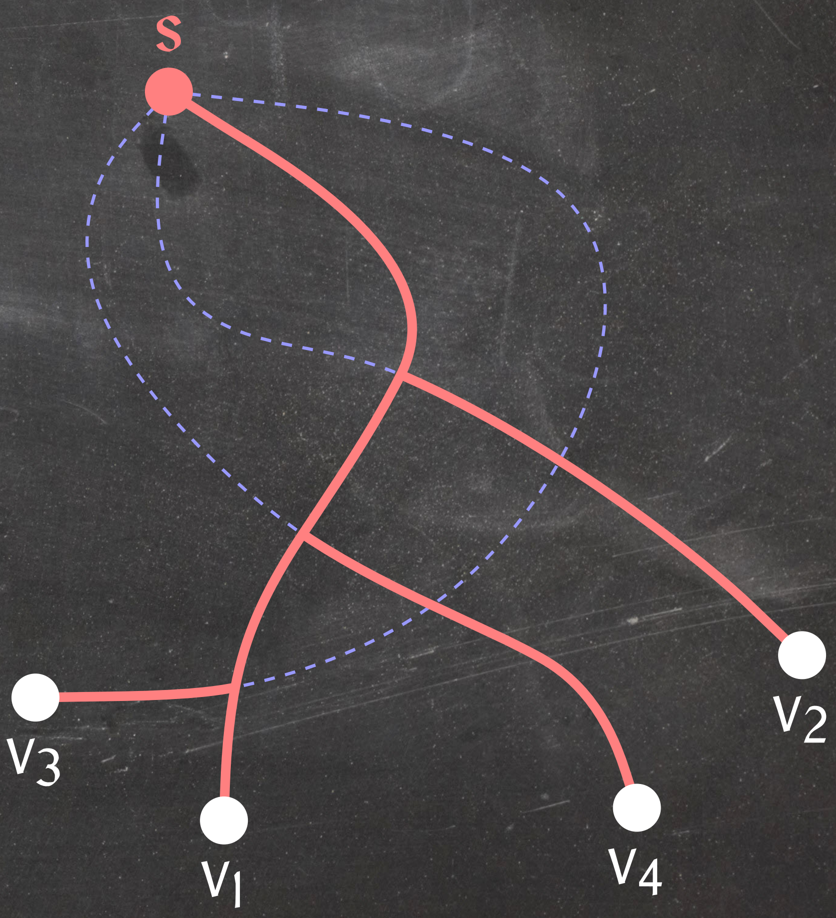

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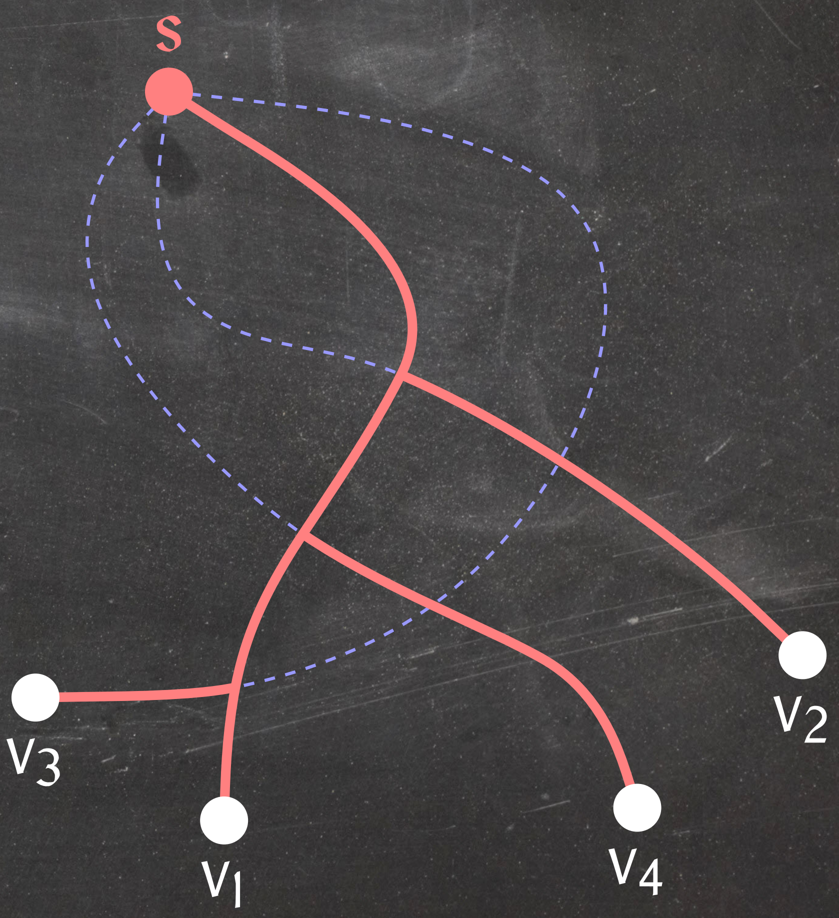
Prove by induction on $i$ that $T_i[s, v]$ is a shortest path from $s$ to $v$, for all $v \in T_i$.



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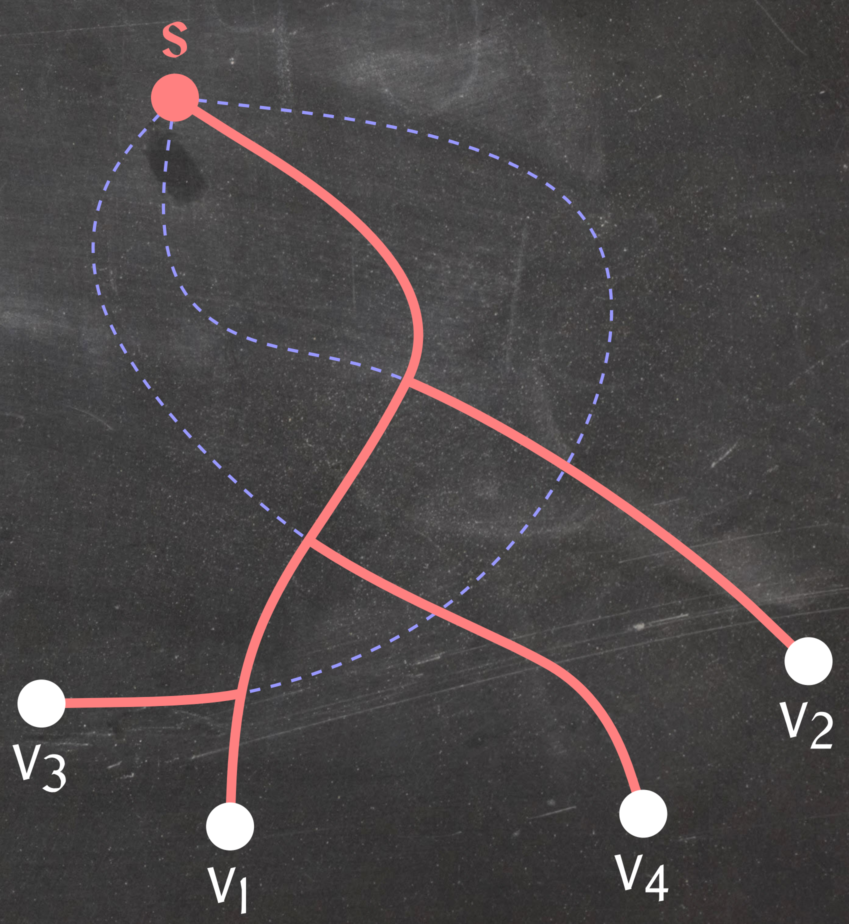

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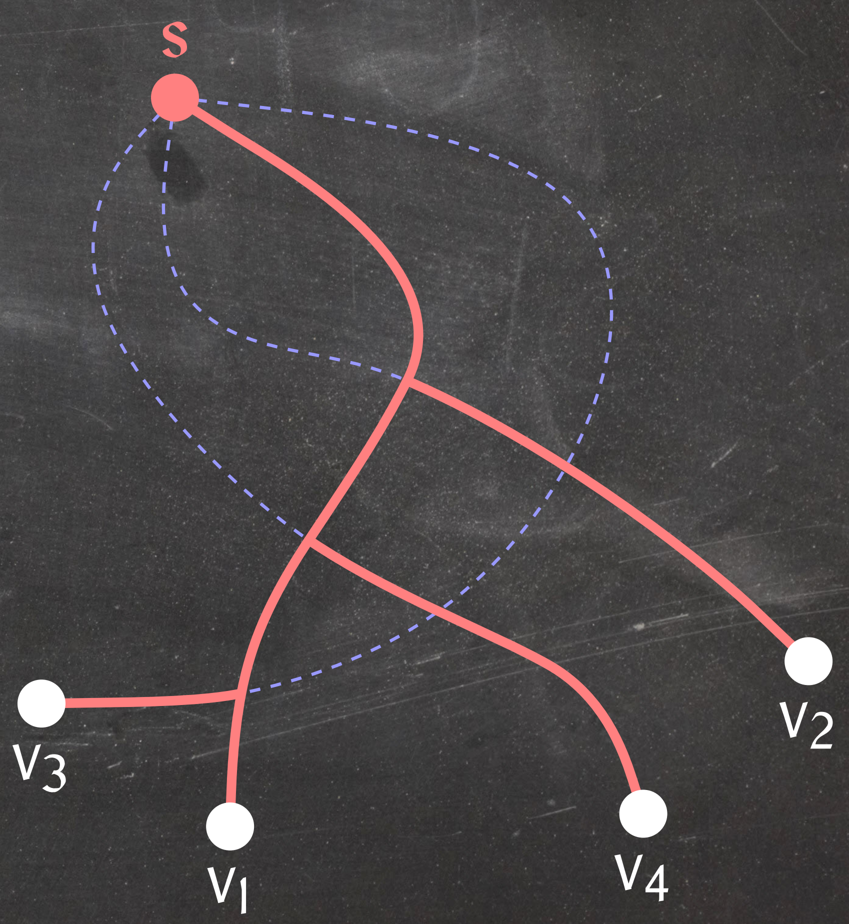

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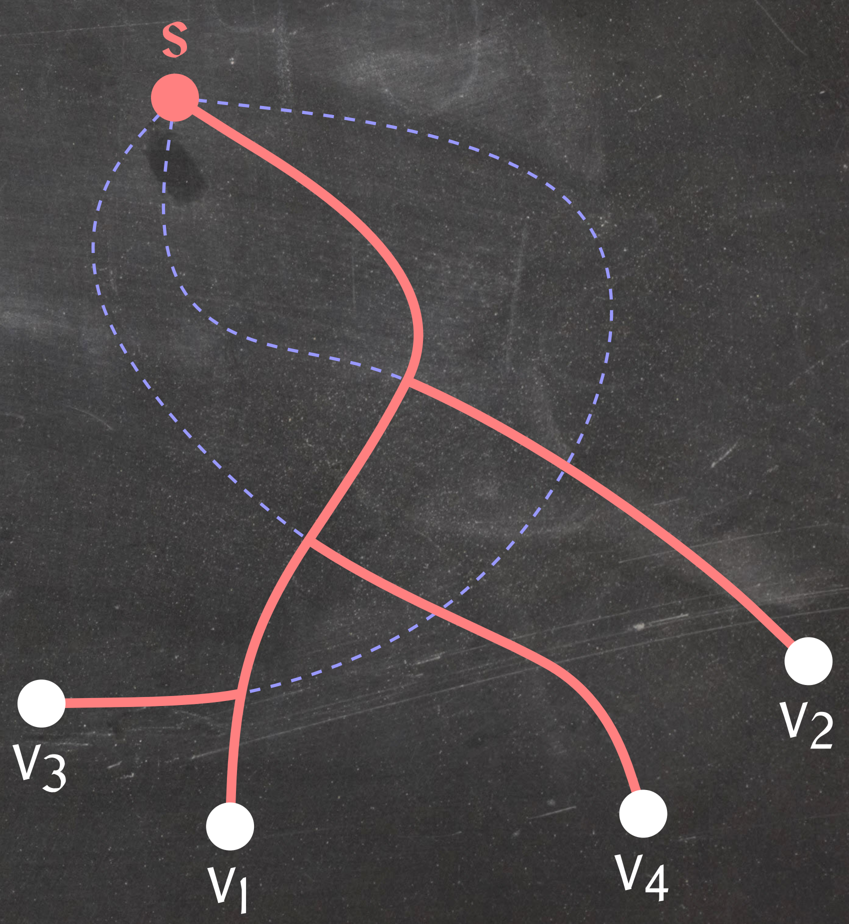
For $i = 1$, $T_1 = P_{v_1} = P'_{v_1}$ is a shortest path from $s$ to $v_1$. By optimal substructure, $T_1[s, v] = P'_{v_1}[s, v]$ is a shortest path from $s$ to $v$ for all $v \in T_1$.

For $i > 1$, $T_{i-1}[s, v]$ is a shortest path from $s$ to $v$ for all $v \in T_{i-1}$, by the inductive hypothesis.

Thus, $w(T_{i-1}[s, w]) \leq w(P'_{v_i}[s, w])$ and therefore $w(P_{v_i}) = w(T_{i-1}[s, w]) + w(P'_{v_i}[w, v_i]) \leq w(P'_{v_i})$.



# Shortest Path Tree

For a vertex $s \in G$, let $R(s)$ be the set of vertices reachable from s: for every vertex $v \in R(s)$, there exists a path from s to v.

**Lemma:** For every node $s \in G$, there exists a collection of paths $\mathcal{S} = \{P_v \mid v \in R(s)\}$ such that $P_v$ is a shortest path from s to v and $\bigcup_{v \in R(s)} P_v$ is a tree.

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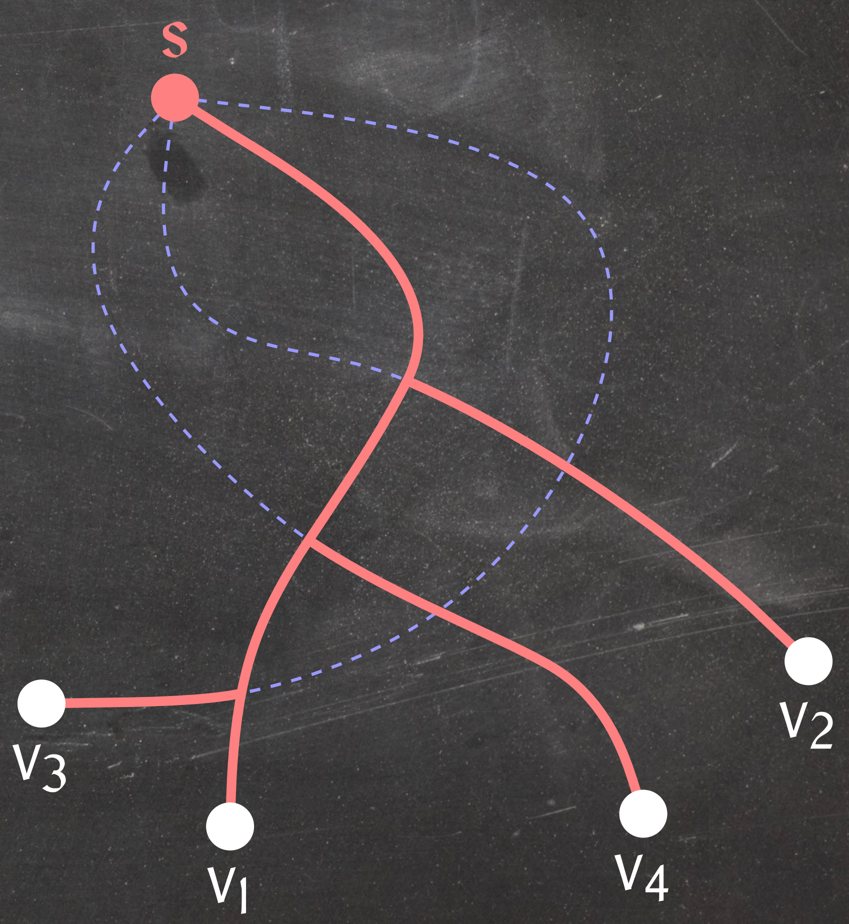

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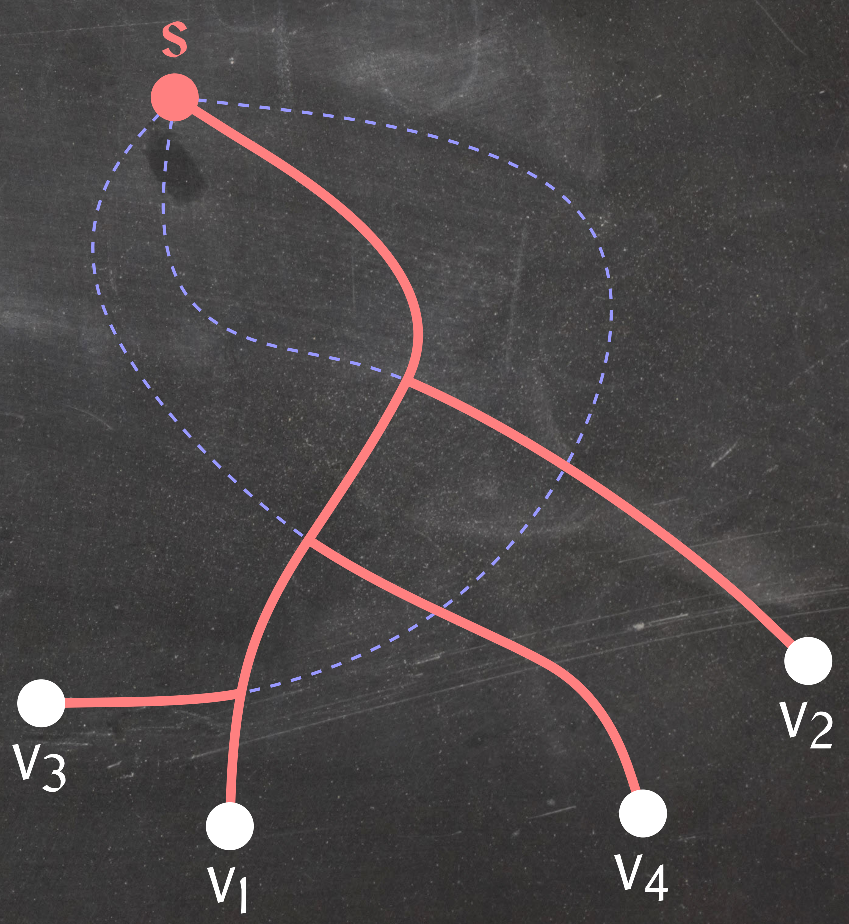

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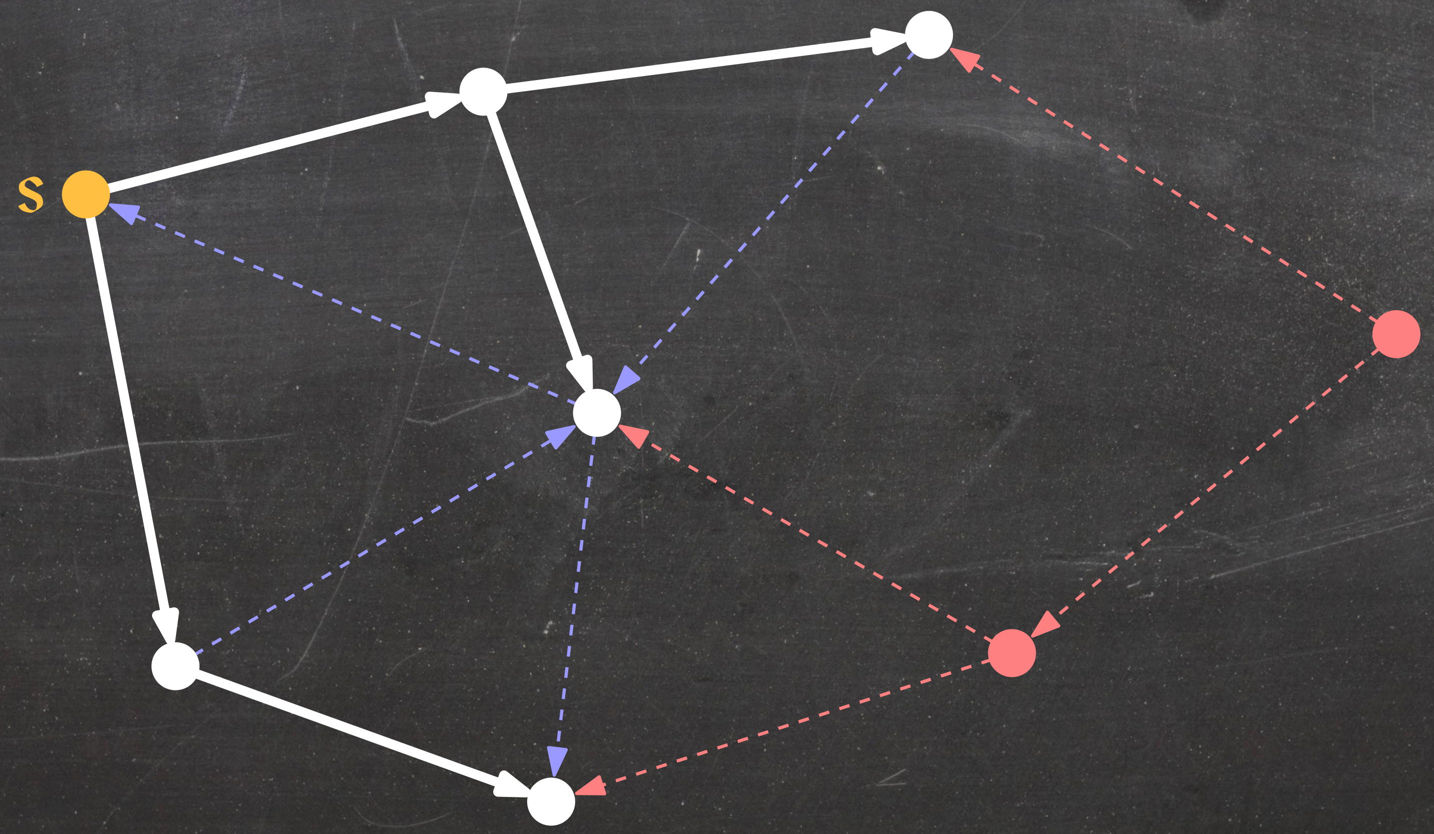
An out-tree of s is a spanning tree T of G[R(s)] = (R(s), E[R(s)]), where E[R(s)] = {(v, w) ∈ E | v, w ∈ R(s)}, such that there exists a path from s to v in T, for all v ∈ R(s).



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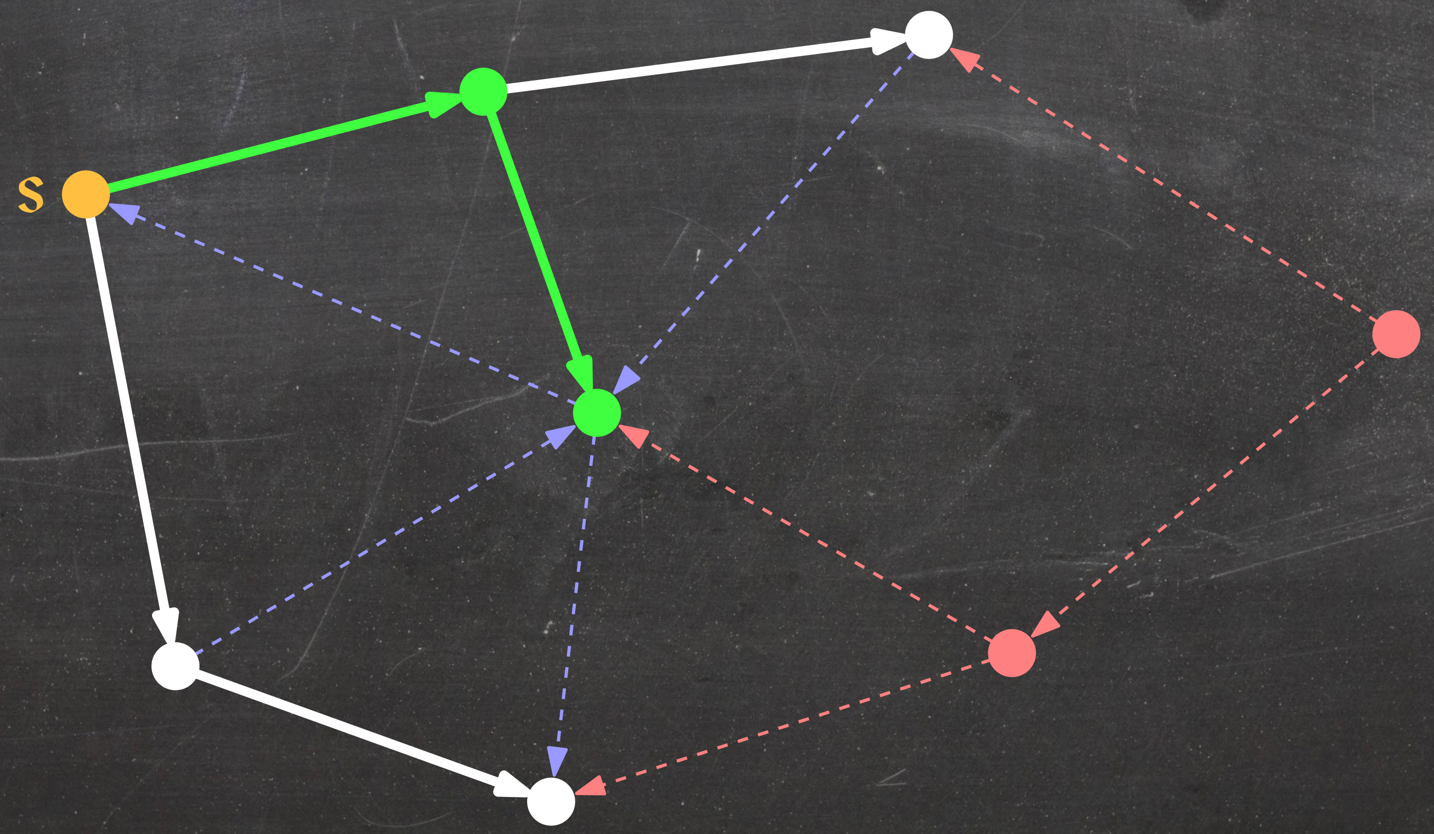

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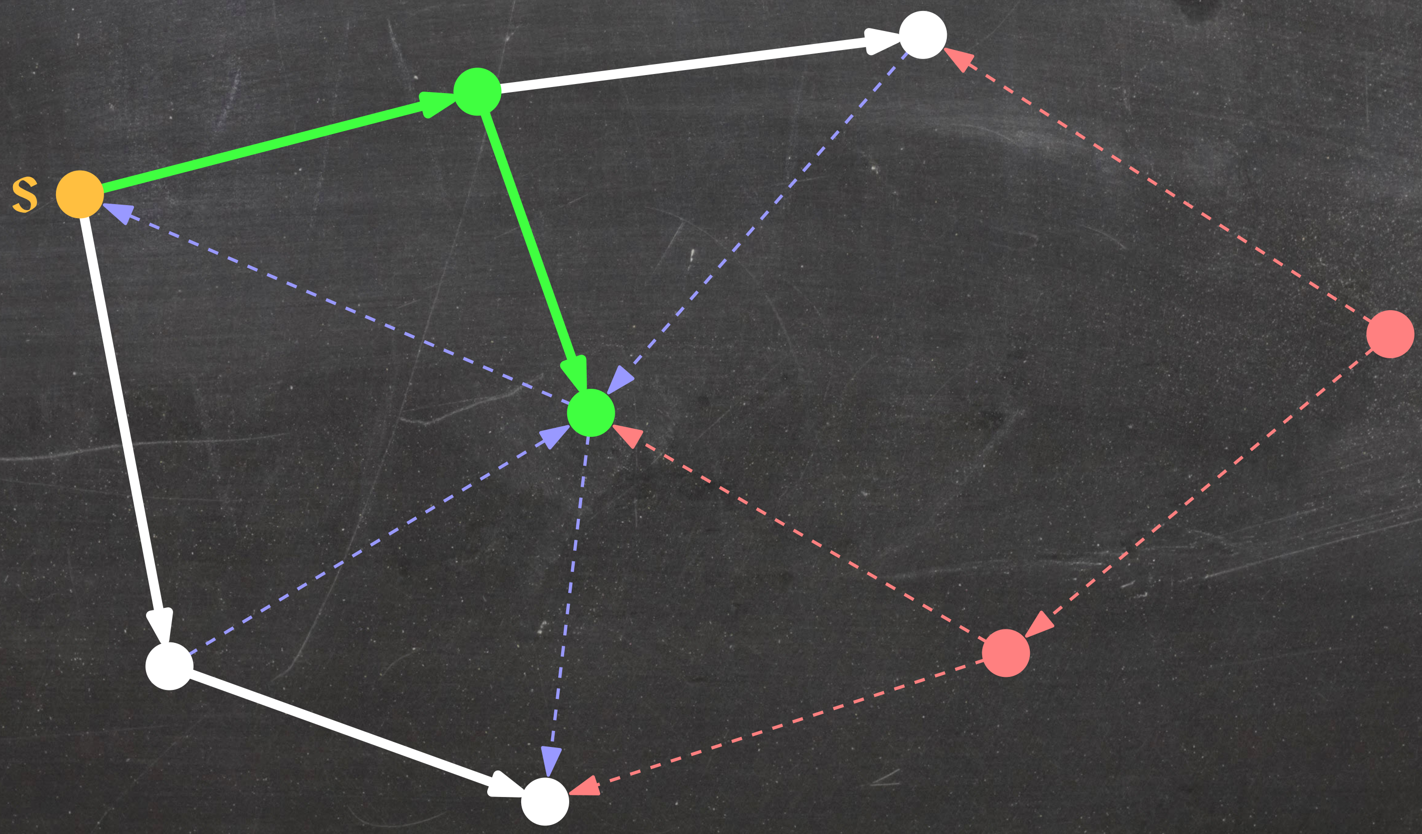

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**Dijkstra(G, s)**

```
T = $(\{s\}, \emptyset)$
while some vertex in T has an out-neighbour not in T
    do choose an edge $(u, v)$ such that
          • $u \in T$,
          • $v \notin T$, and
          • $d_T(u) + w(u, v)$ is minimized.
       add v and $(u, v)$ to T
return T
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Dijkstra(G, s)

T = $(V, \emptyset)$
mark every vertex of G as unexplored
set $d(v) = +\infty$ and $e(v)$ = nil for every vertex $v \in G$
mark s as explored and set $d(v) = 0$
Q = an empty priority queue
for every edge (s, v) incident to s
    do Q.insert(v, w(s, v))
        $d(v) = w(s, v)$
        $e(v) = (s, v)$
while not Q.isEmpty()
    do u = Q.deleteMin()
        mark u as explored
        add e(u) to T
        for every edge (u, v) incident to u
            do if v is unexplored and ($v \notin Q$ or $d(u) + w(u, v) < d(v)$)
                then $d(v) = d(u) + w(u, v)$
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$\Rightarrow$ Dijkstra's algorithm takes $O(n \lg n + m)$ time.

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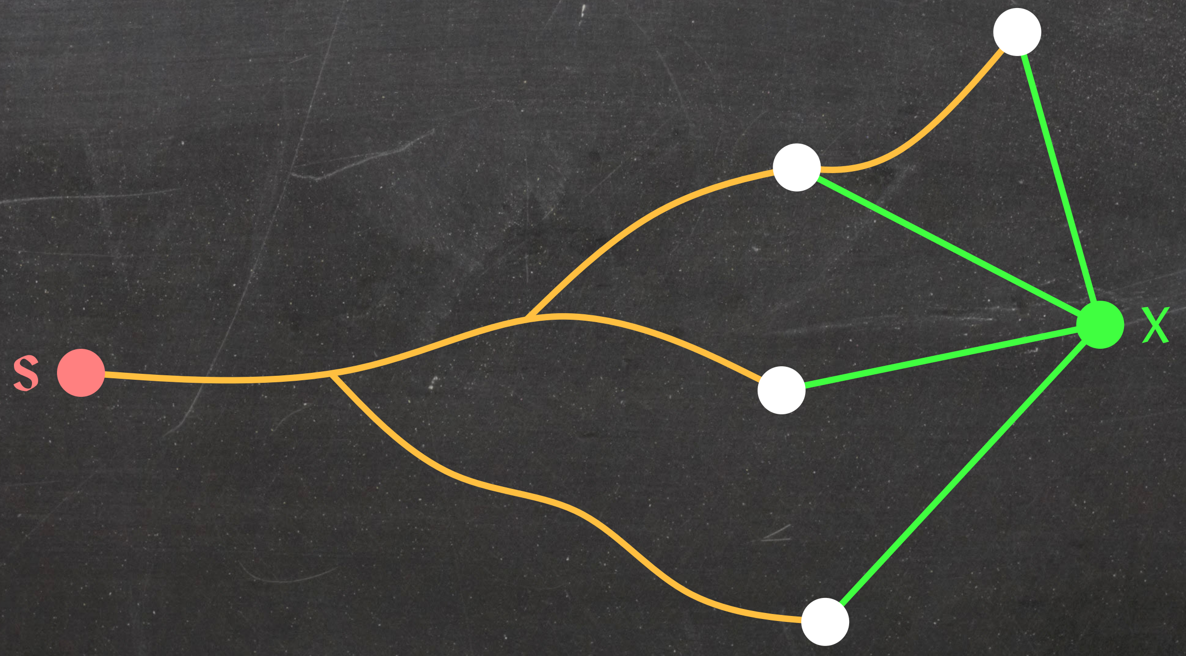
For every vertex $x \notin T$, we have

$$d(x) = \min_{\substack{(u,x)\in E\\ u\in T}} d(u) + w(u, x) = \min_{\substack{(u,x)\in E\\ u\in T}} \mathrm{dist}(s, u) + w(u, x).$$



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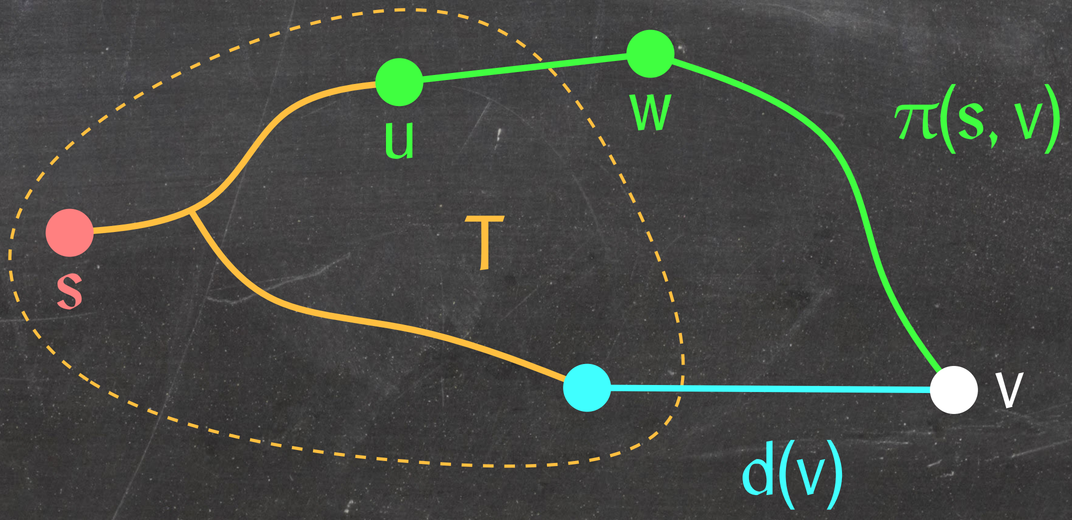

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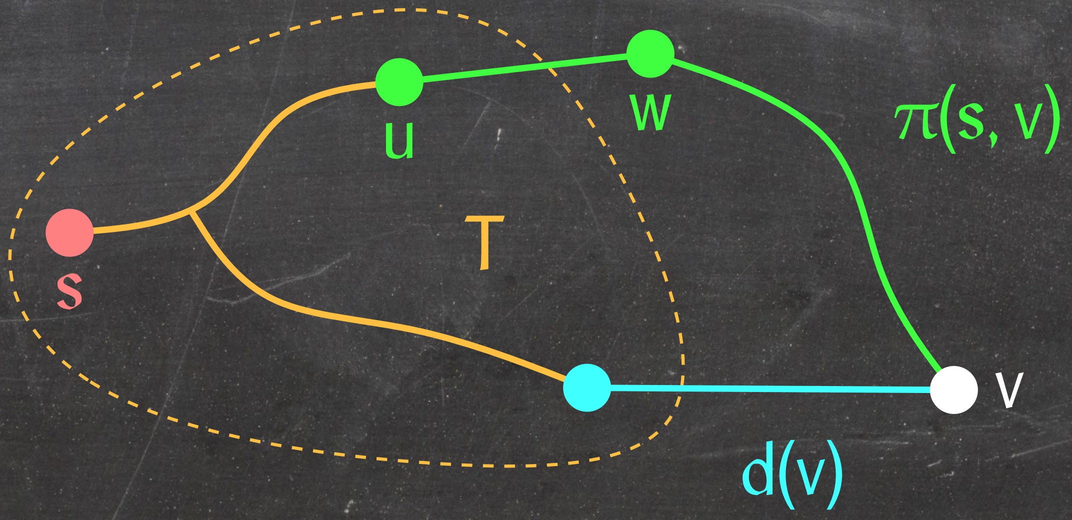

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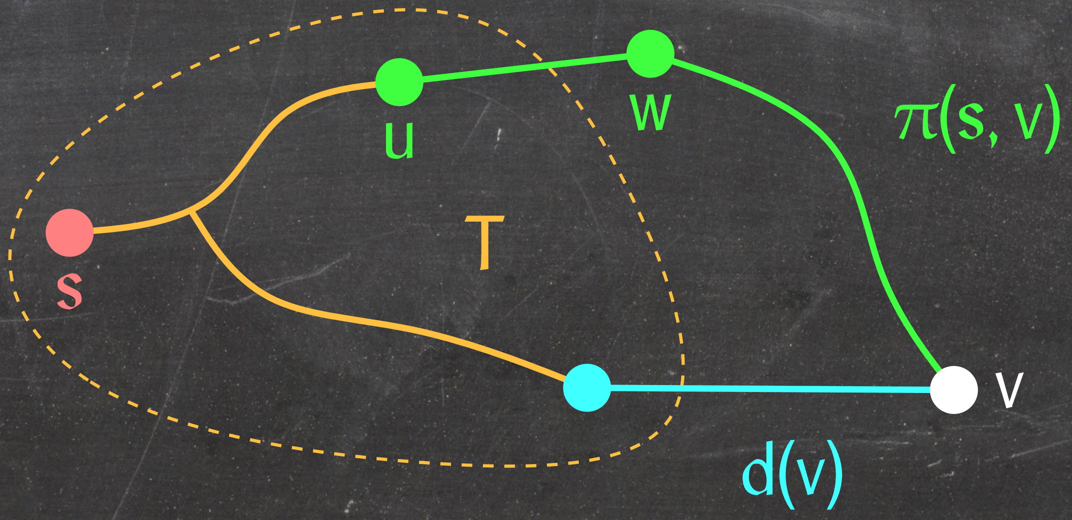

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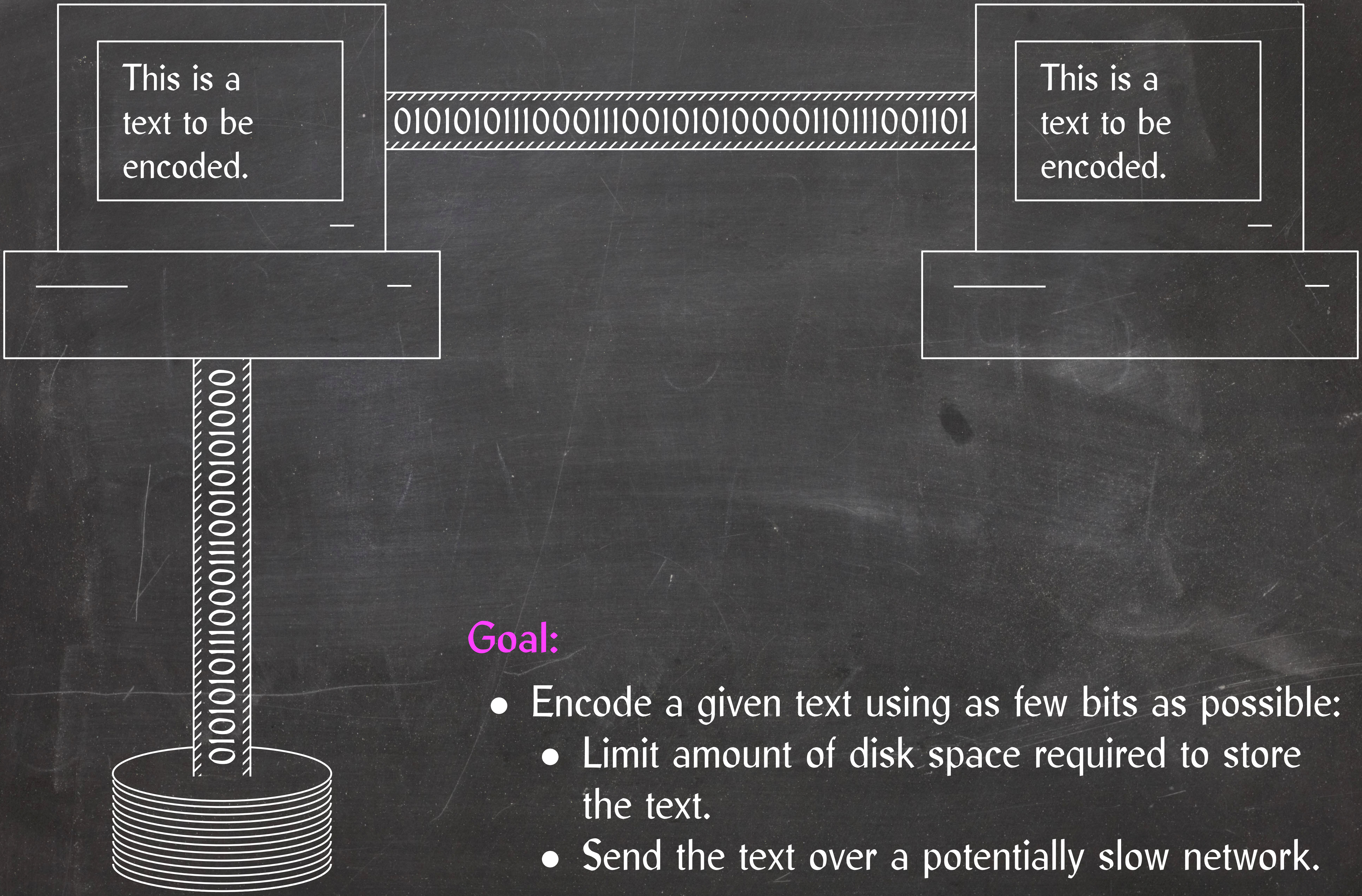
# Minimum Length Codes



**Goal:**

- Encode a given text using as few bits as possible:
  - Limit amount of disk space required to store the text.
  - Send the text over a potentially slow network.
  - . . .

# Codes That Can Be Decoded

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|---|---|---|---|---|---|---|---|
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For a text $T = \langle x_1, x_2, \ldots, x_n \rangle$, let $C(T) = C(x_1) \circ C(x_2) \circ \cdots \circ C(x_n)$ be the bit string obtained by concatenating the encodings of its characters. We call $C(T)$ the encoding of $T$.

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For a text $T = \langle x_1, x_2, \ldots, x_n \rangle$, let $C(T) = C(x_1) \circ C(x_2) \circ \cdots \circ C(x_n)$ be the bit string obtained by concatenating the encodings of its characters. We call $C(T)$ the encoding of $T$.

"prefix-free"

| | e | f | i | p | r | x | - |
|---|---|---|---|---|---|---|---|
| $C_1$ | 000 | 001 | 010 | 011 | 100 | 101 | 110 |

$C_1$(prefix-free) = 011 100 000 001 010 101 110 001 100 000 000 (33 bits)

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Non-prefix-free codes cannot always be decoded uniquely!

# Codes That Can Be Decoded

**Lemma:** If $C(\cdot)$ is a prefix-free code and $T \neq T'$, then $C(T) \neq C(T')$.

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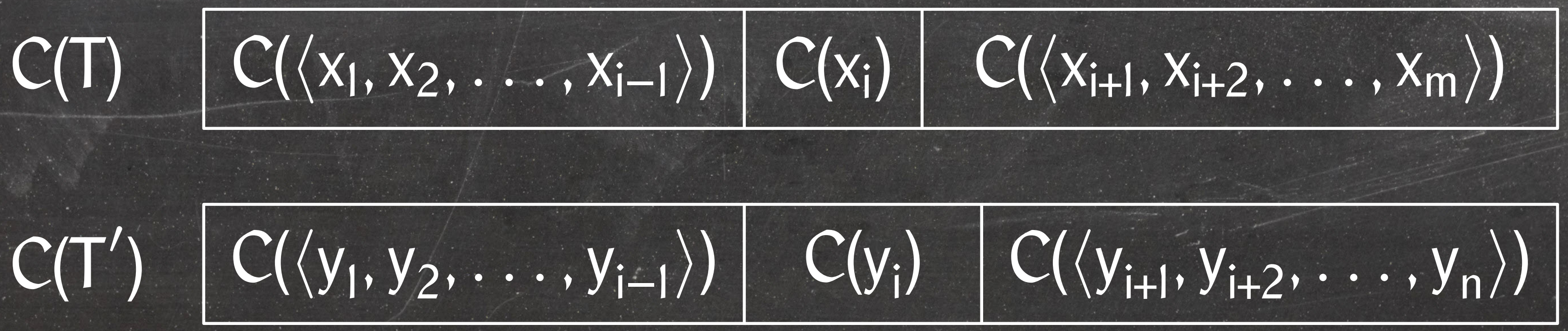
Let $i$ be the minimum index such that $x_i \neq y_i$.

$\Rightarrow$ $C(\langle x_1, x_2, \ldots, x_{i-1} \rangle) = C(\langle y_1, y_2, \ldots, y_{i-1} \rangle)$ and
$C(\langle x_i, x_{i+1}, \ldots, x_m \rangle) = C(\langle y_i, y_{i+1}, \ldots, y_n \rangle)$.



| | | | |
|---|---|---|---|
| $C(T)$ | $C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)$ | $C(x_i)$ | $C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)$ |
| $C(T')$ | $C(\langle y_1, y_2, \ldots, y_{i-1} \rangle)$ | $C(y_i)$ | $C(\langle y_{i+1}, y_{i+2}, \ldots, y_n \rangle)$ |

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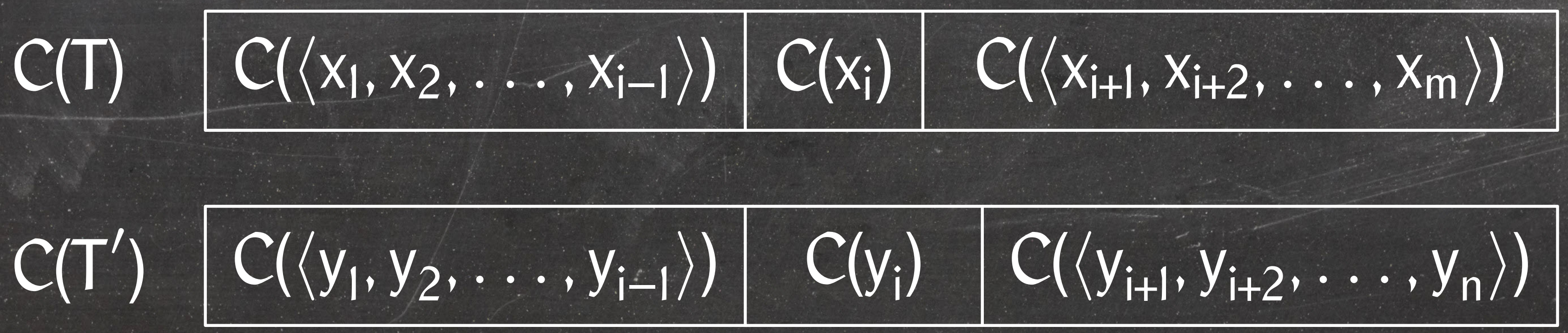
$$C(\langle x_i, x_{i+1}, \ldots, x_m \rangle) = C(\langle y_i, y_{i+1}, \ldots, y_n \rangle).$$

Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.



| | | | |
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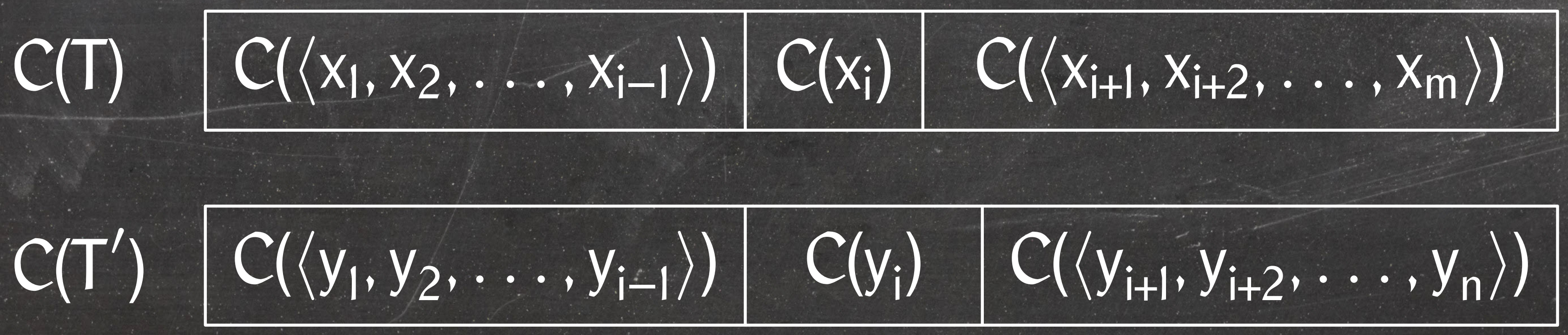
Assume w.l.o.g. that $|C(x_i)| \leq |C(y_i)|$.

Since both $C(x_i)$ and $C(y_i)$ are prefixes of $C(\langle x_i, x_{i+1}, \ldots, x_m \rangle)$, $C(x_i)$ must be a prefix of $C(y_i)$, a contradiction.



$C(T)$ $\boxed{C(\langle x_1, x_2, \ldots, x_{i-1} \rangle)}\boxed{C(x_i)}\boxed{C(\langle x_{i+1}, x_{i+2}, \ldots, x_m \rangle)}$

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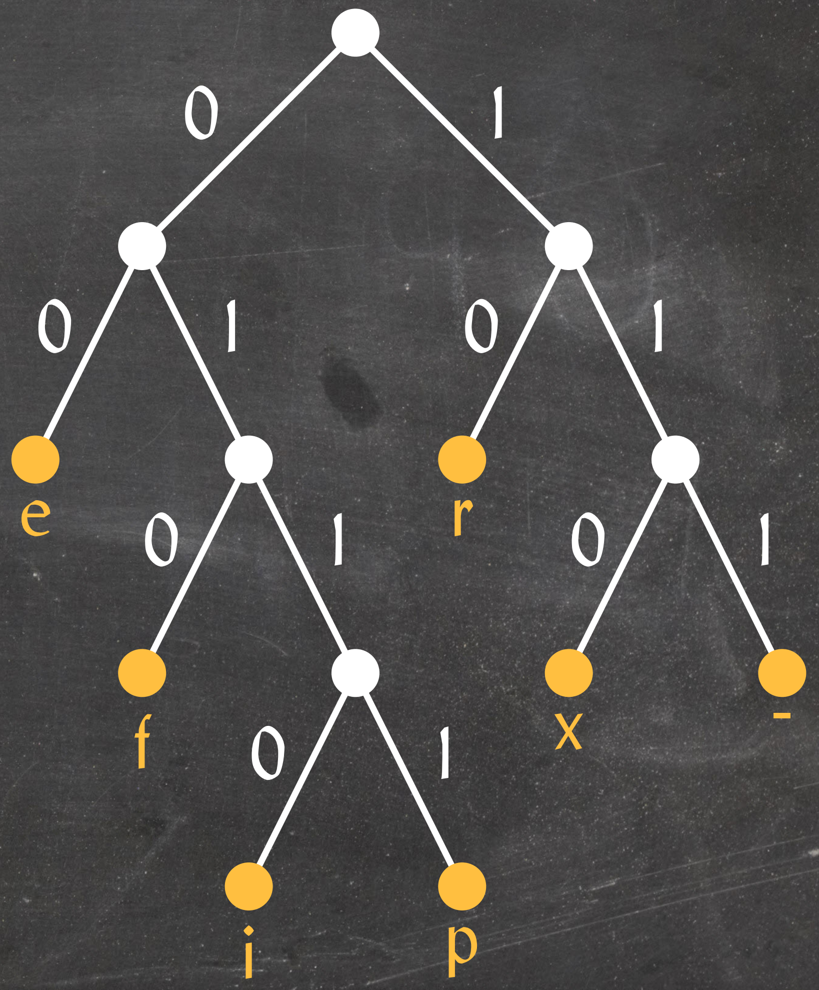
# Prefix Codes and Binary Trees

**Observation:** Every prefix-free code $C(\cdot)$ can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

|   | e | f | i | p | r | x | - |
|---|---|---|---|---|---|---|---|
| C | 00 | 010 | 0110 | 0111 | 10 | 110 | 111 |



# Prefix Codes and Binary Trees

**Observation:** Every prefix-free code C(·) can be represented as a binary tree $\mathcal{T}_C$ whose leaves correspond to the letters in the alphabet.

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|---|---|---|---|---|---|---|---|
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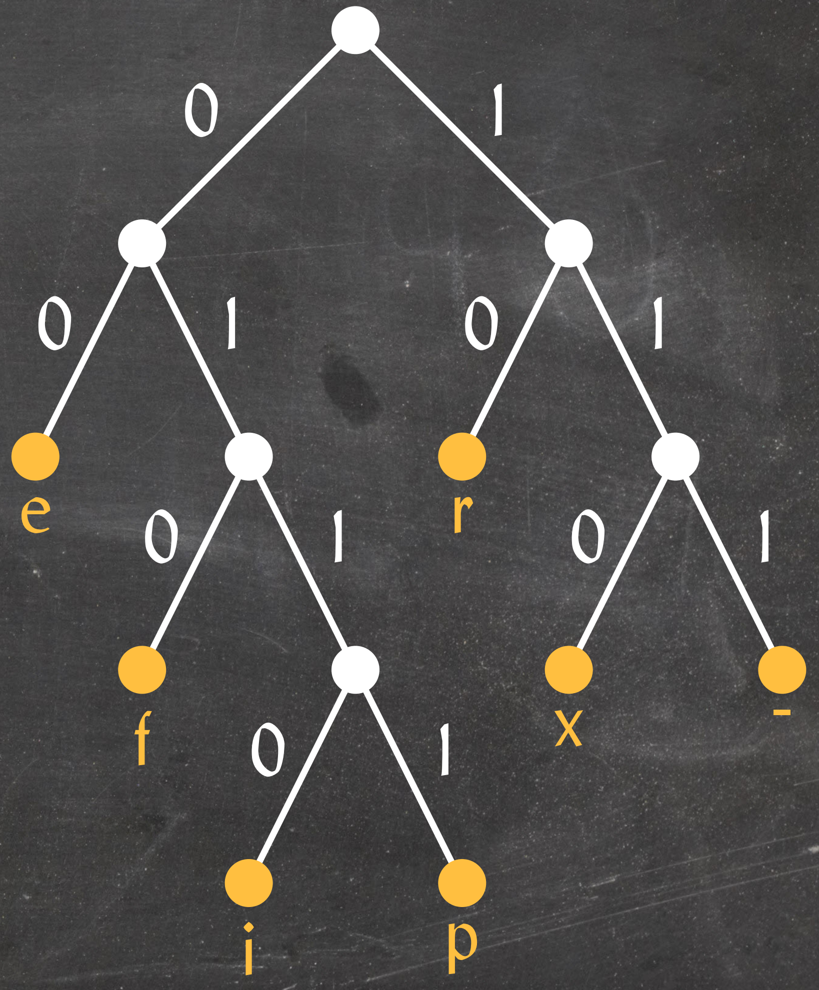

The depth of character x in $\mathcal{T}_C$ is the number of bits |C(x)| used to encode x using C(·).

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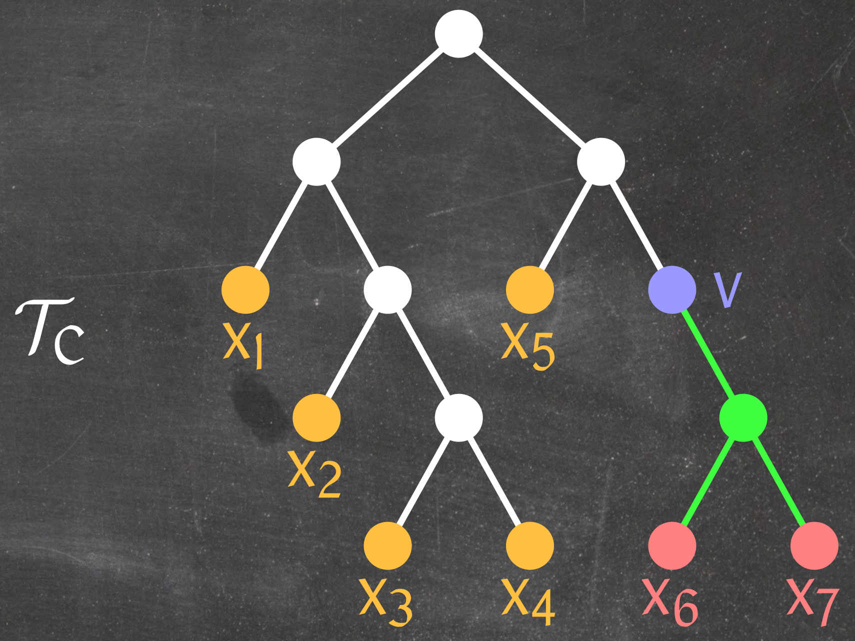

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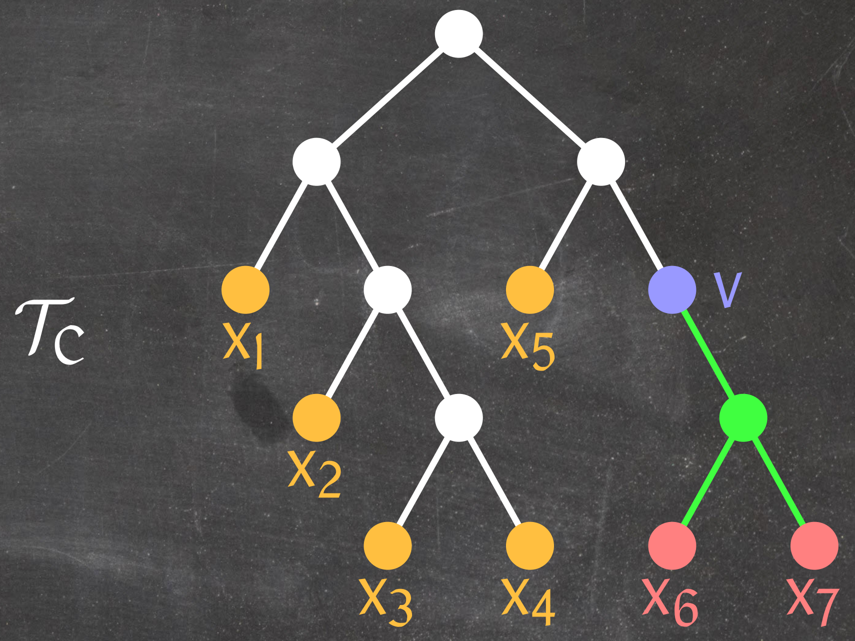
Choose C(·) so that $\mathcal{T}_C$ has as few internal nodes with only one child as possible among all optimal prefix-free codes for T.



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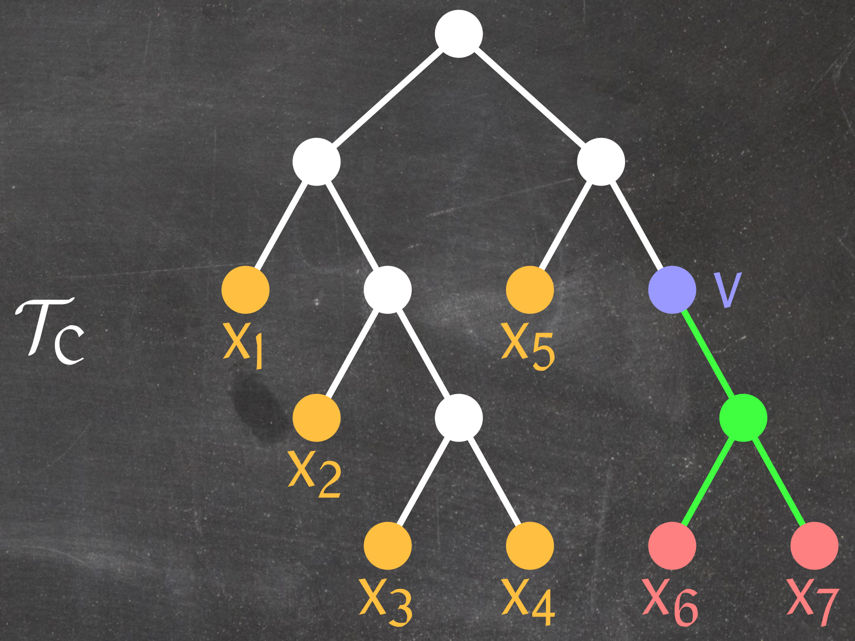
If $\mathcal{T}_C$ has no internal node with only one child, the lemma holds.



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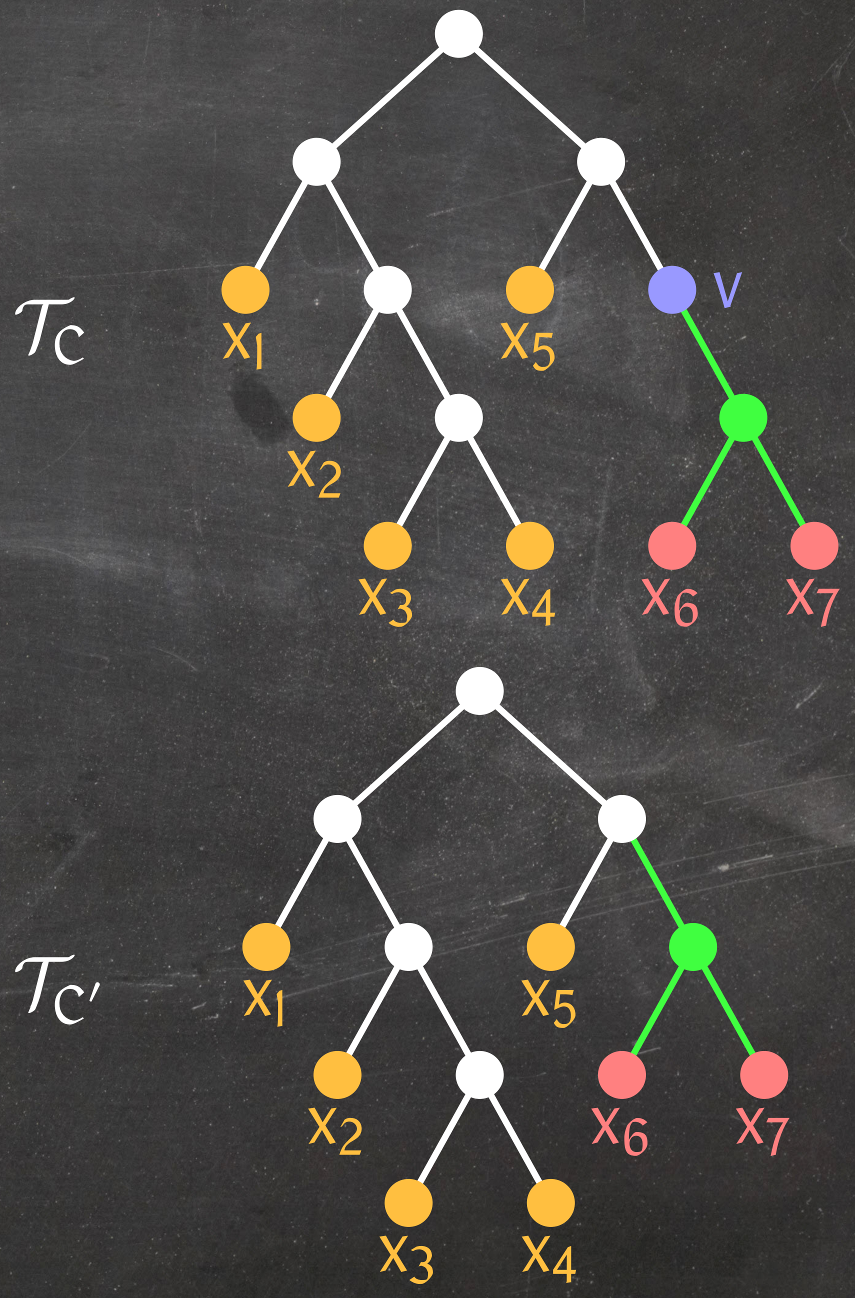
Otherwise, choose an internal node v with only one child w and contract the edge (v, w).



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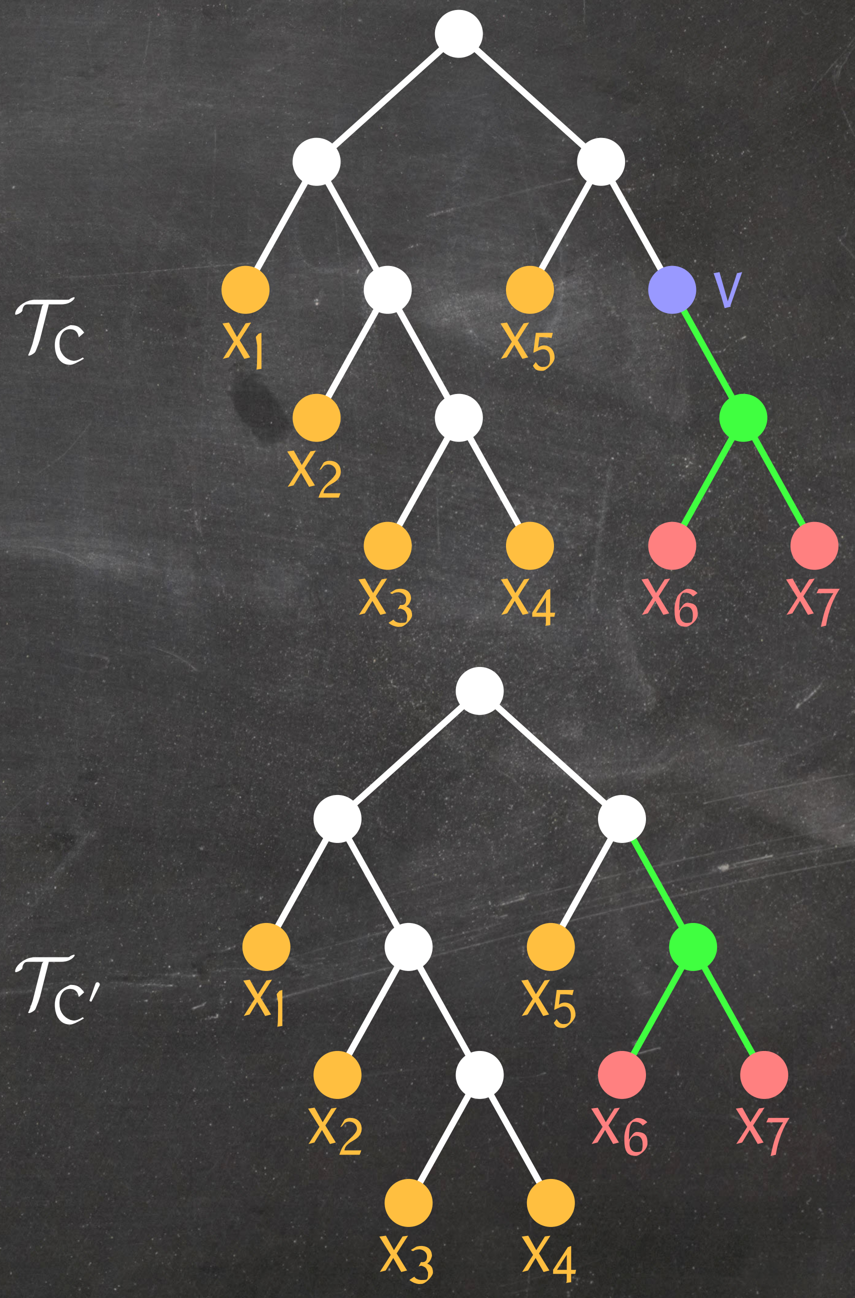
The resulting tree $\mathcal{T}_{C'}$ has one less internal node with only one child and represents a prefix-free code $C'(\cdot)$ with the property that $|C'(x)| \leq |C(x)|$ for every character x.



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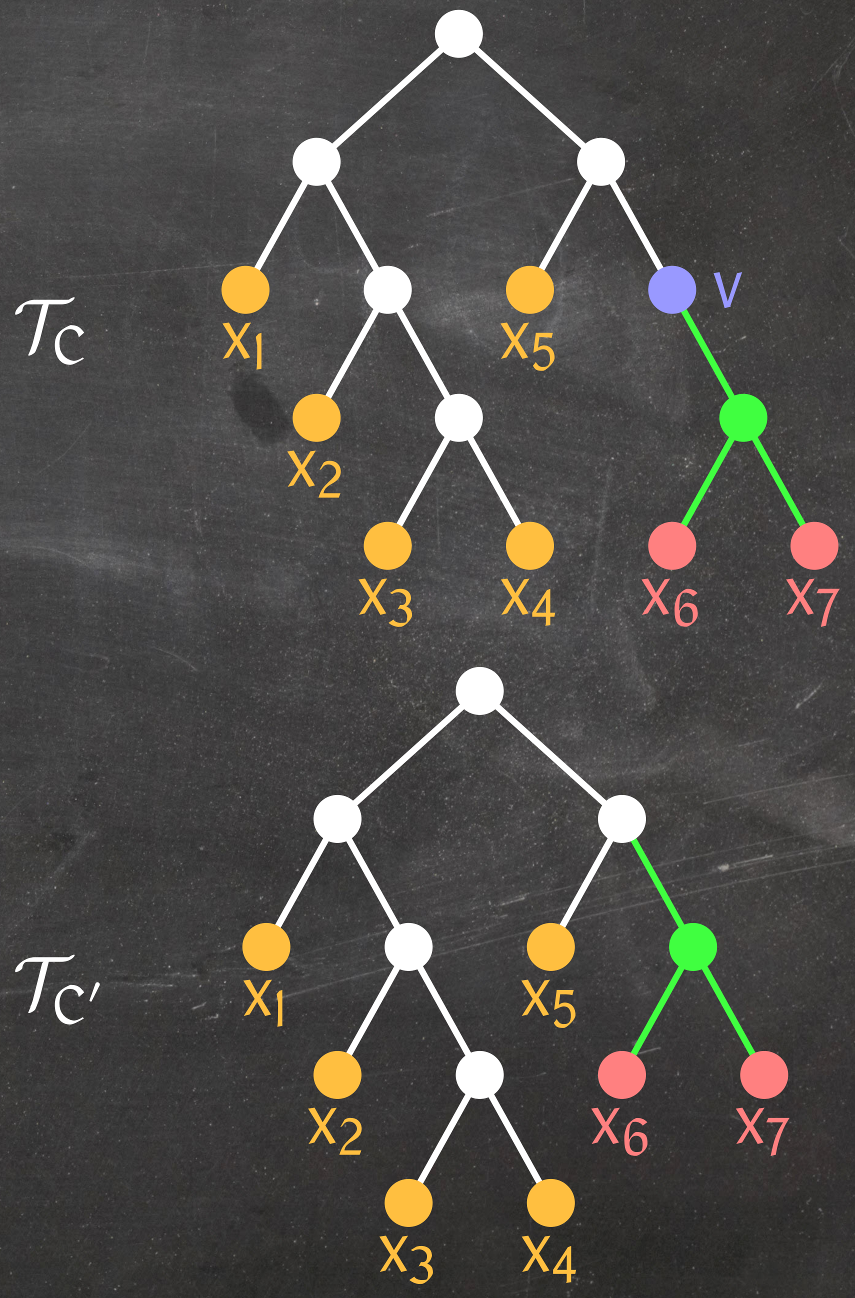
$\Rightarrow$ $|C'(T)| \le |C(T)|$, contradicting the choice of C.



# A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

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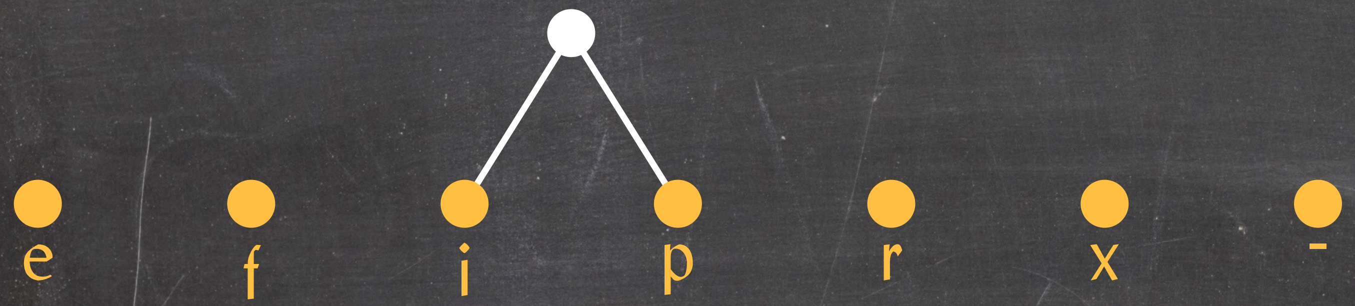

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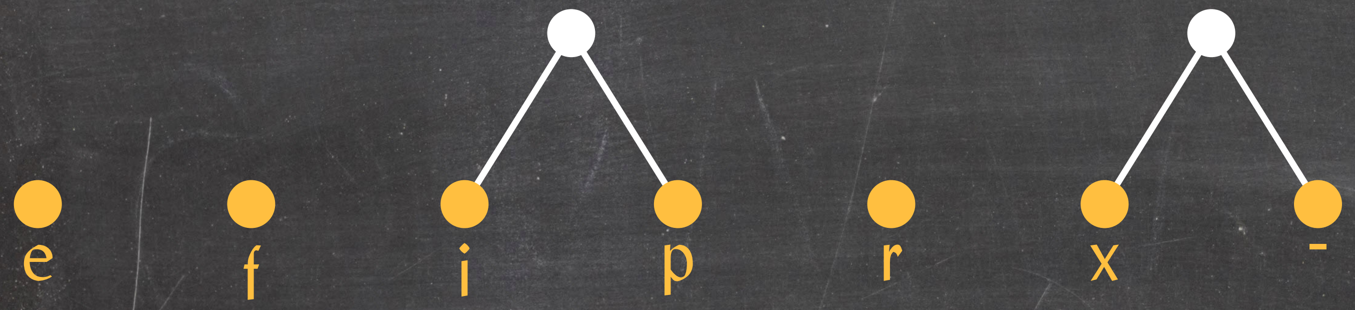

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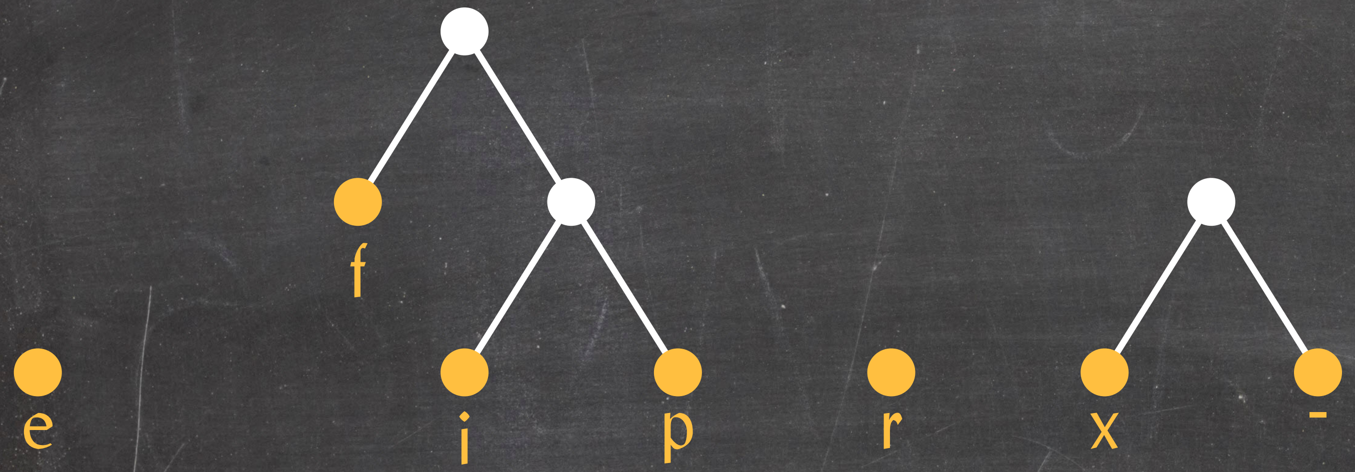

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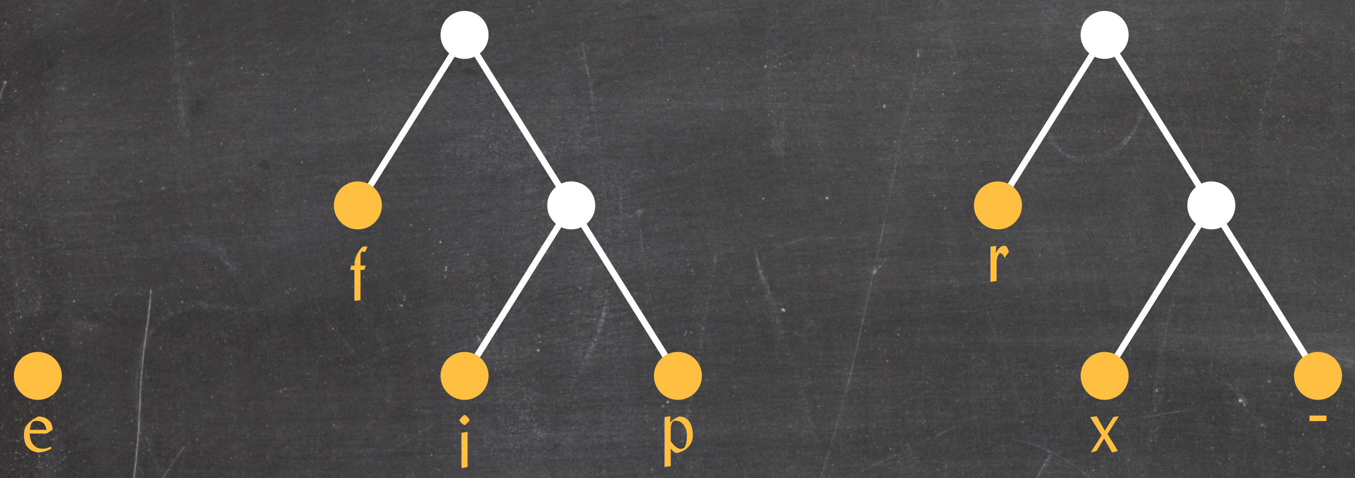

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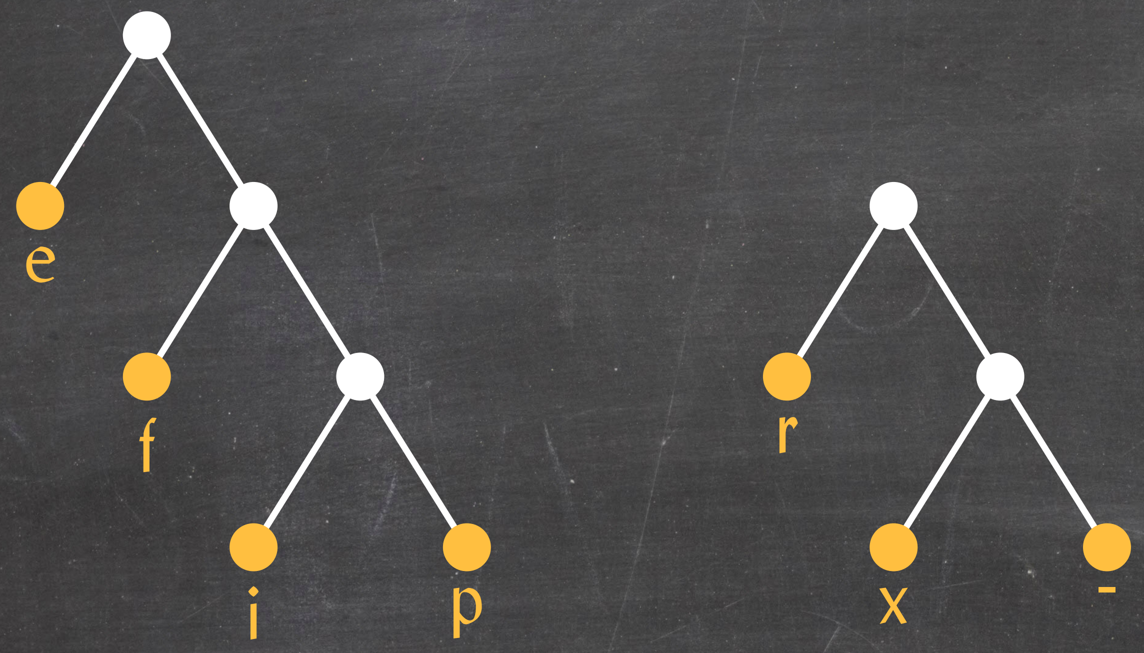

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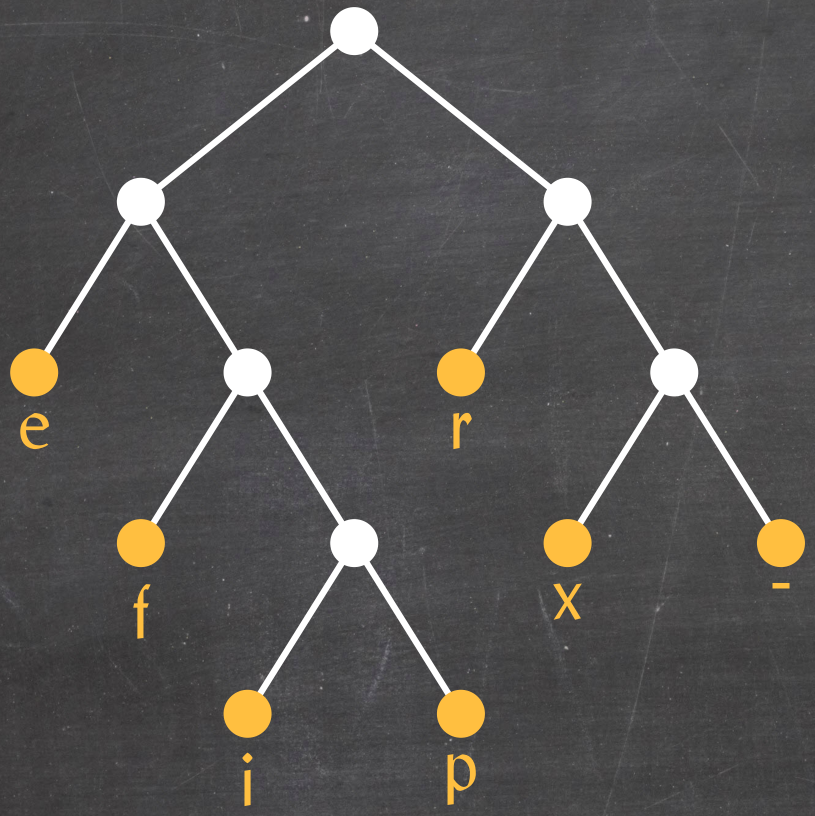

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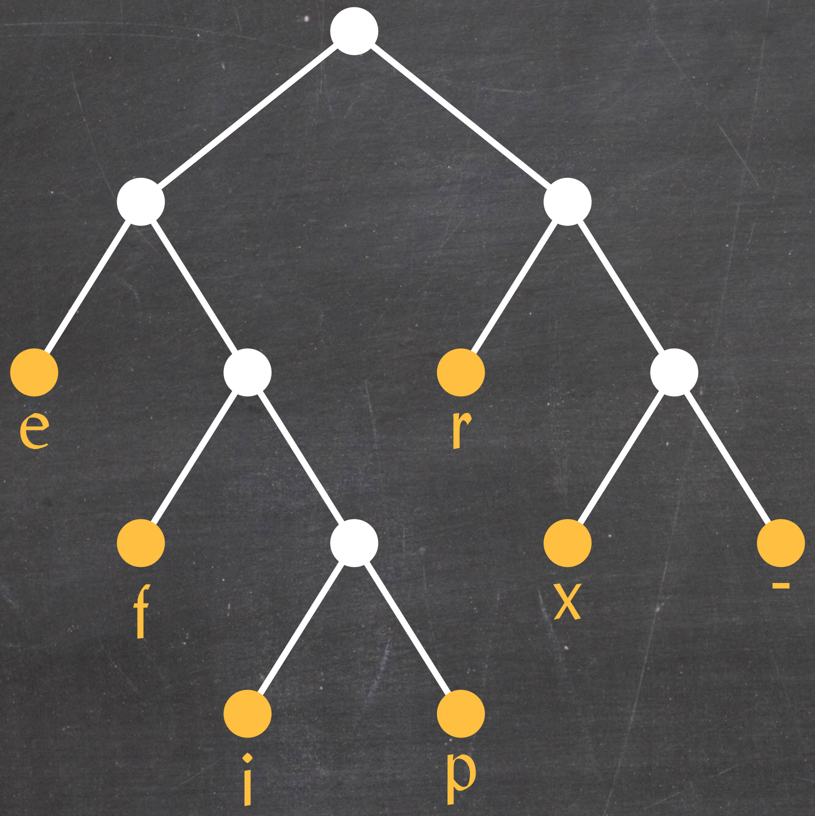

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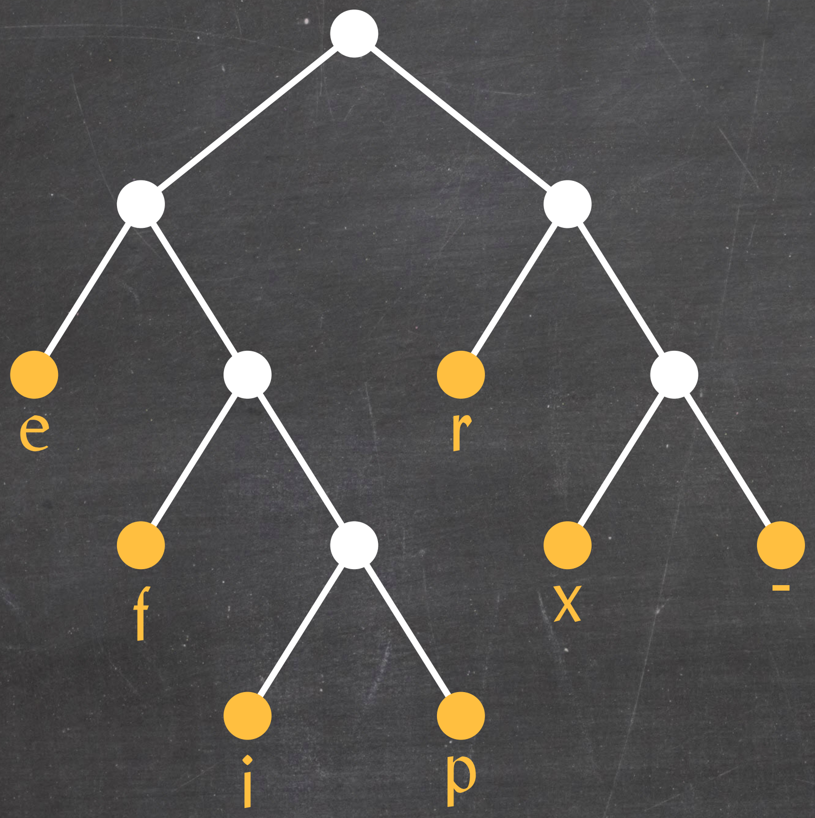

The length of the encoding of T is $|C(T)| = \sum_x f_T(x)|C(x)|$, where $f_T(x)$ is the frequency of x in T.

When making a node r a child of a new parent, we add 1 bit to the encoding C(x) of every descendant leaf x of r.

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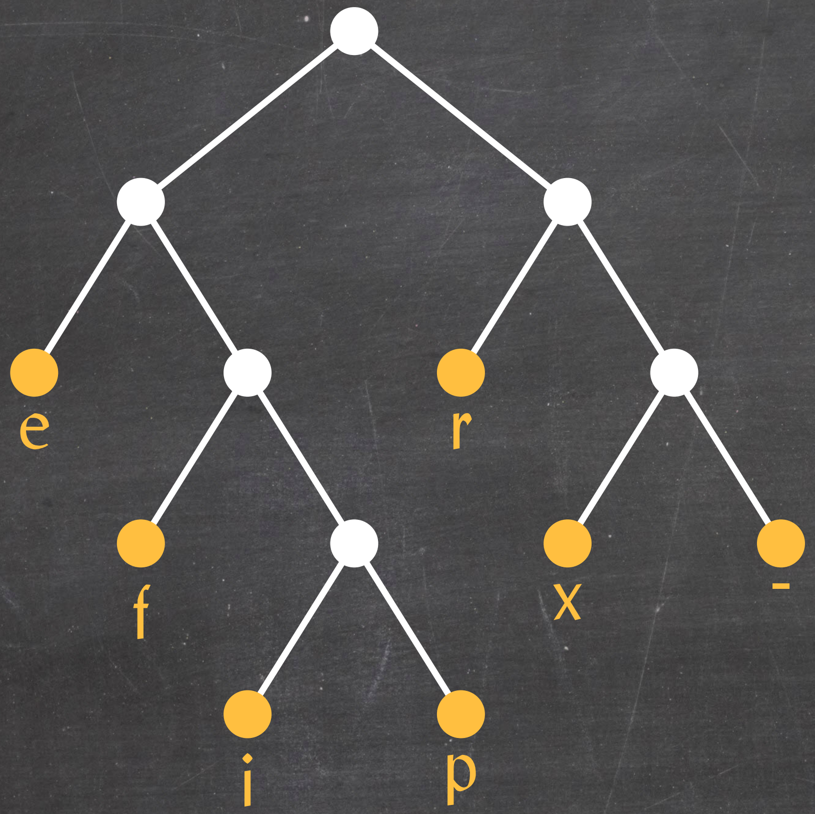

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⇒ By choosing the two roots with minimum total frequency of their descendent leaves, we minimize the increase in |C(T)|.

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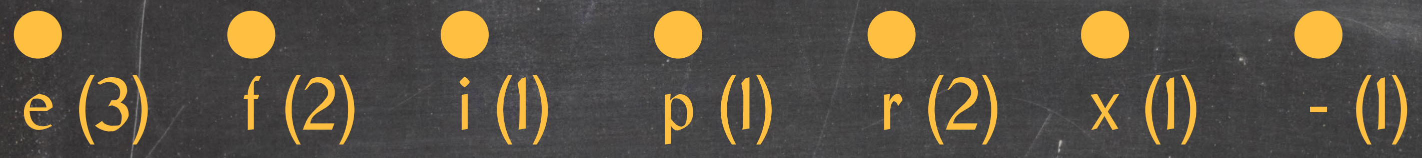
We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.

“prefix-free”

| x | e | f | i | p | r | x | - |
|---|---|---|---|---|---|---|---|
| $f_T(x)$ | 3 | 2 | 1 | 1 | 2 | 1 | 1 |



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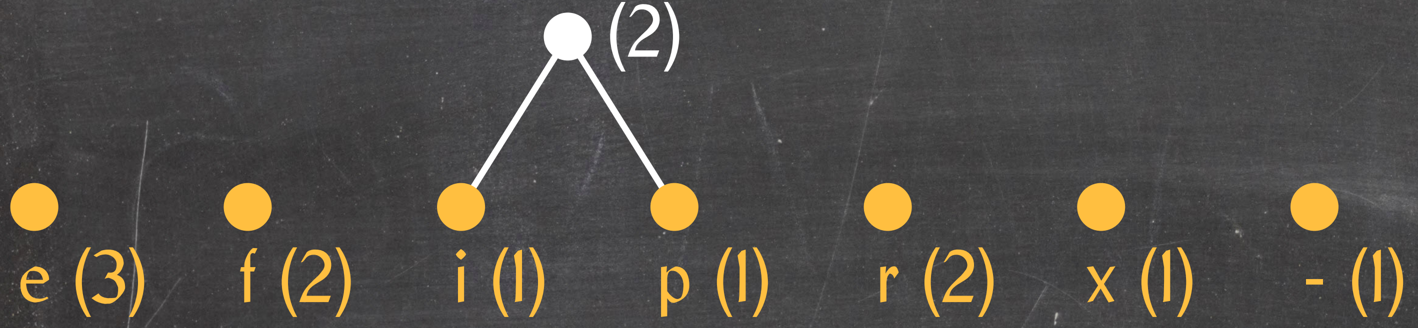

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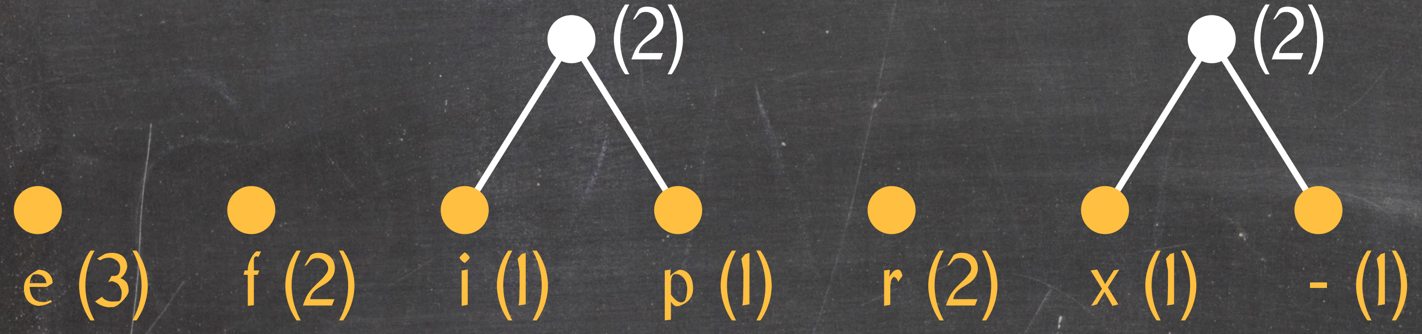
|---|---|---|---|---|---|---|---|
| $f_T(x)$ | 3 | 2 | 1 | 1 | 2 | 1 | 1 |



The length of the encoding of T is $|C(T)| = \sum_x f_T(x)|C(x)|$, where $f_T(x)$ is the frequency of x in T.

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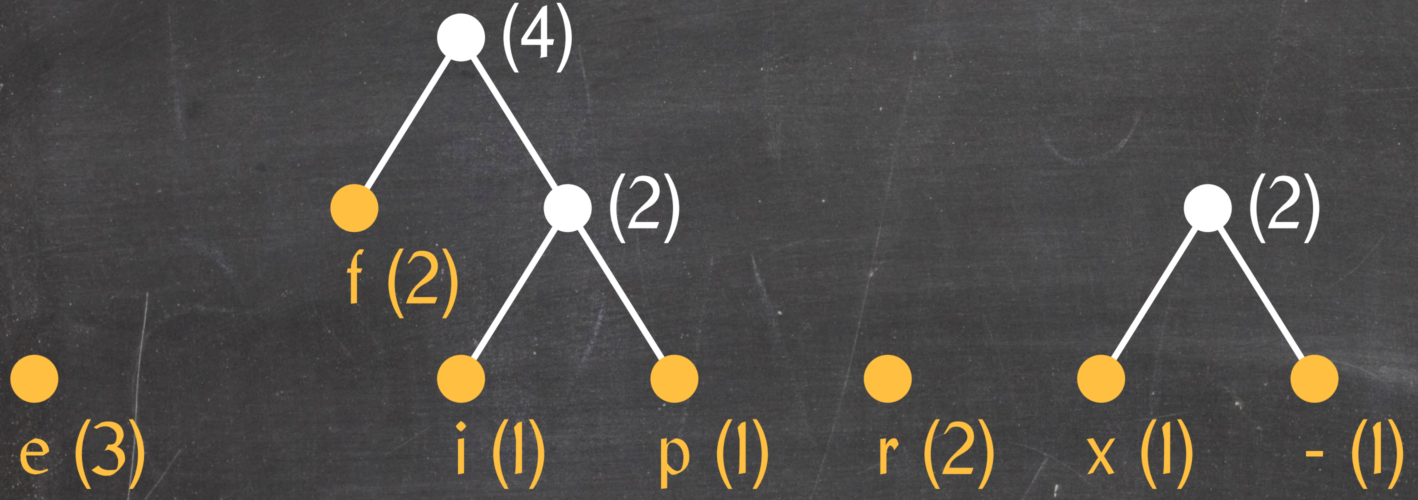
# A Greedy Choice for Optimal Prefix Codes

We can build binary trees by starting with each leaf in its own tree, joining two trees under a common parent, and repeating this until only one tree is left.



“prefix-free”

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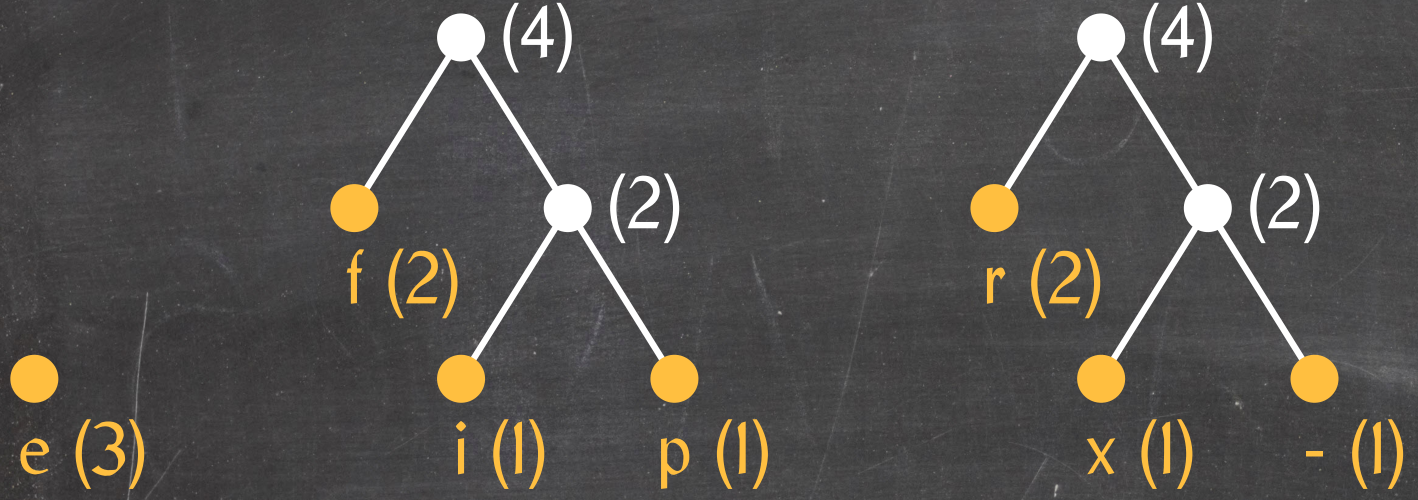

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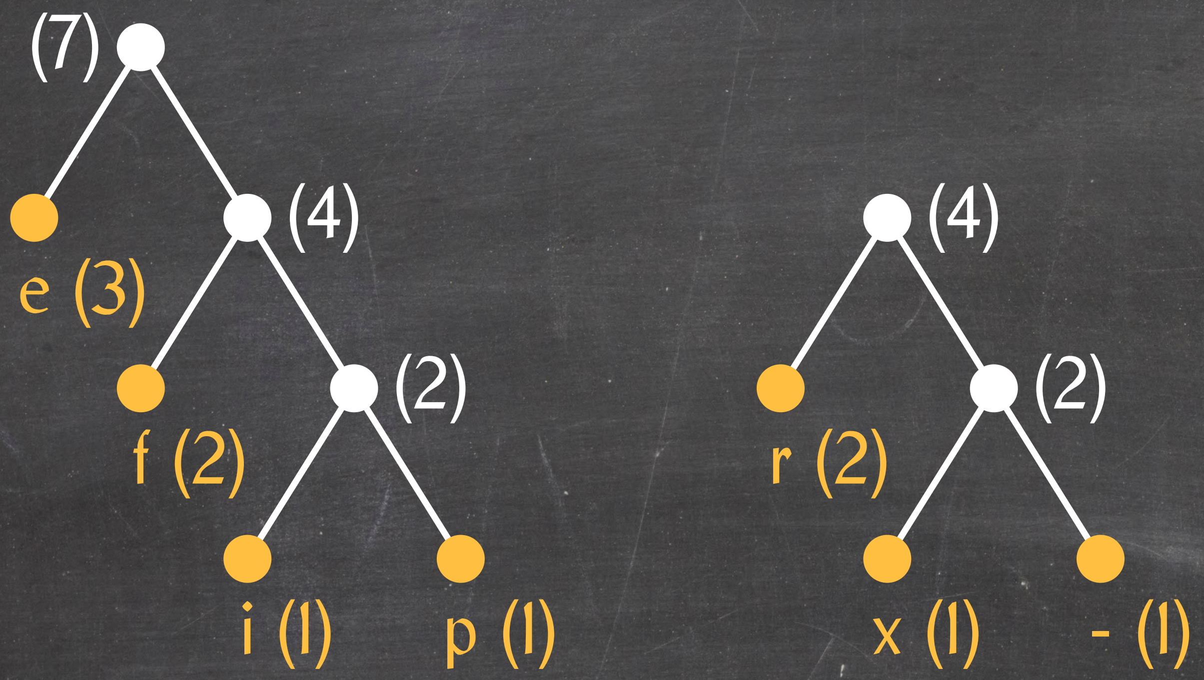

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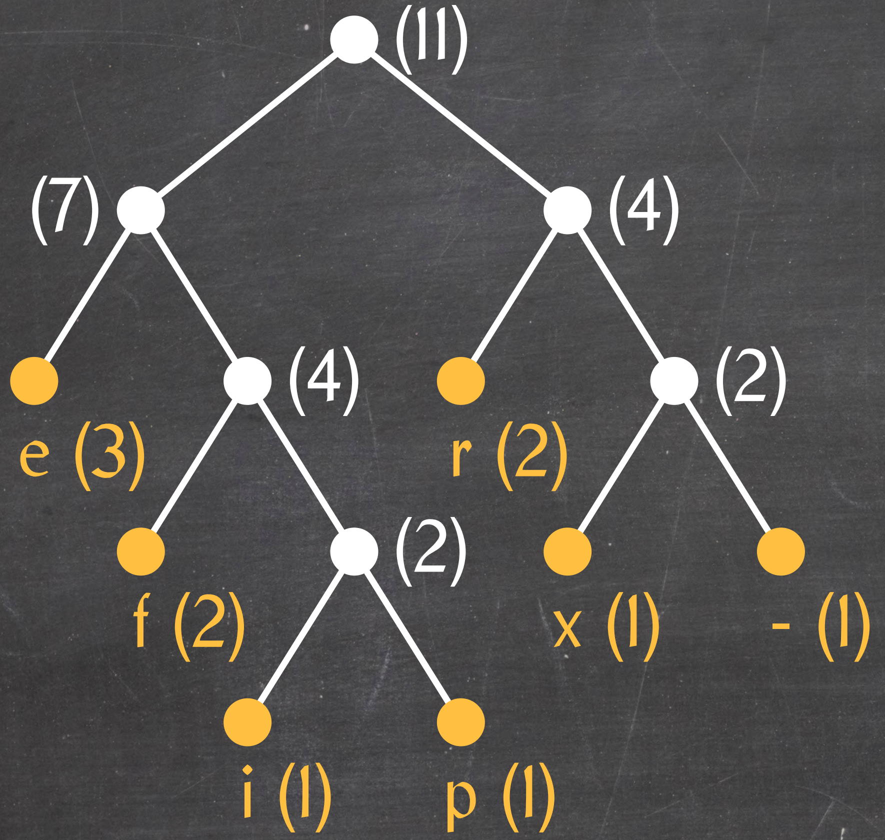

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# Huffman's Algorithm

**Huffman(T)**

```
determine the set A of characters that occur in T and their frequencies
Q = an empty priority queue
for every character $x \in A$
    do create a node v associated with x and define $f(v) = f(x)$
       Q.insert(v, f(v))
while $|Q| > 1$
    do v = Q.deleteMin()
       w = Q.deleteMin()
       u = a new node with frequency $f(u) = f(v) + f(w)$
       make v and w children of u
       Q.insert(u, f(u))
return Q.deleteMin()
```

**Lemma:** Huffman's algorithm runs in $O(m \lg n)$ time, where $m = |T|$ and $n$ is the size of the alphabet.

# Correctness of Huffman's Algorithm

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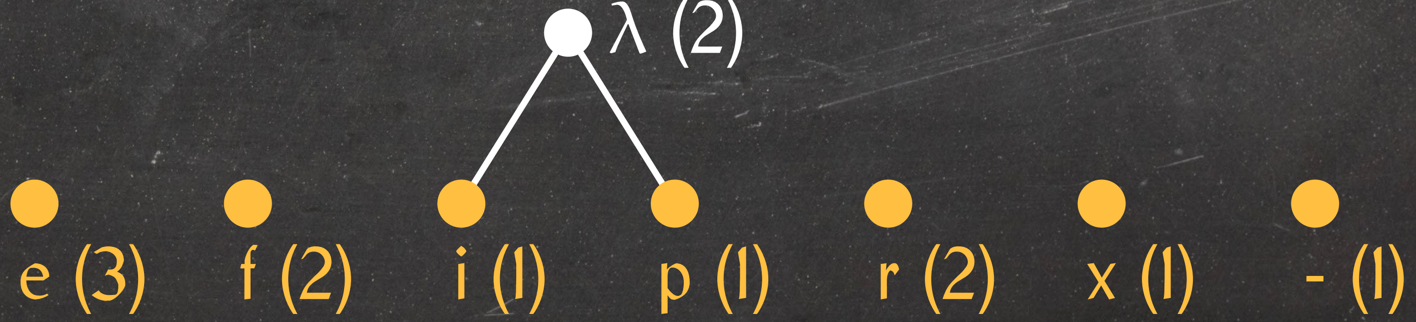

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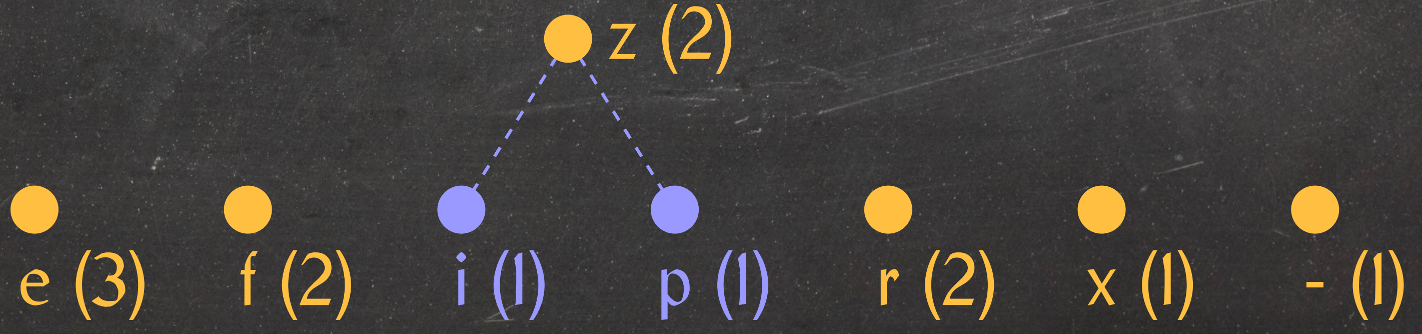
Replacing a and b with z in T produces a new text $T'$ over an alphabet of size $n - 1$ where z has frequency $f(z)$.

"prefix-free"

⇓

"zrefzx-free"



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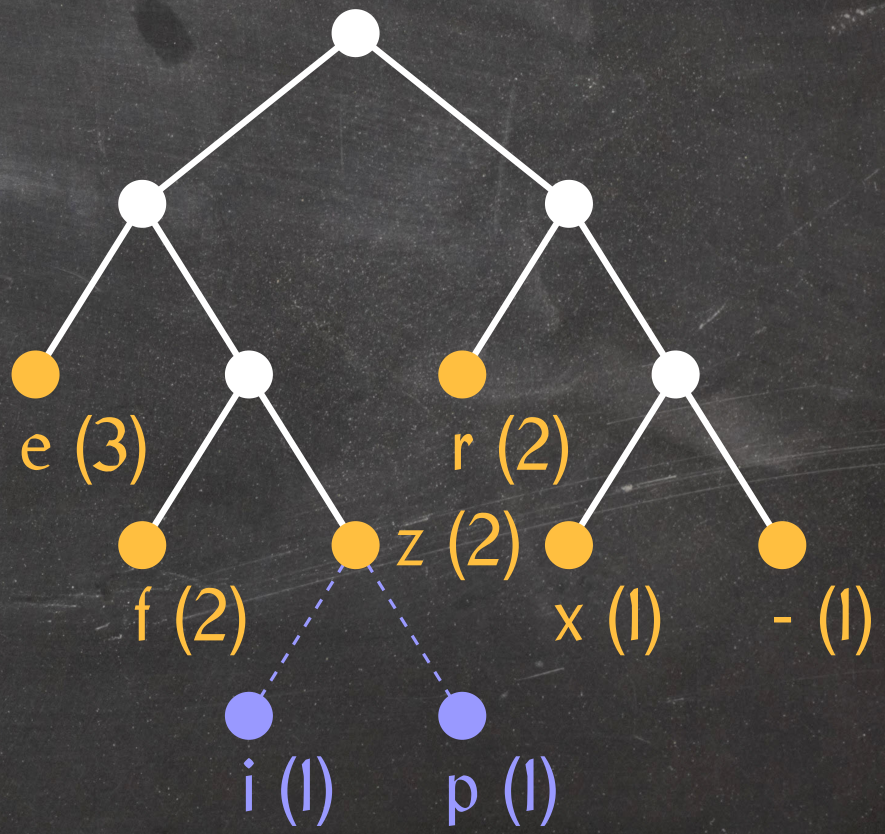

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By induction, it produces an optimal code $C'(\cdot)$ for $T'$.

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⇓

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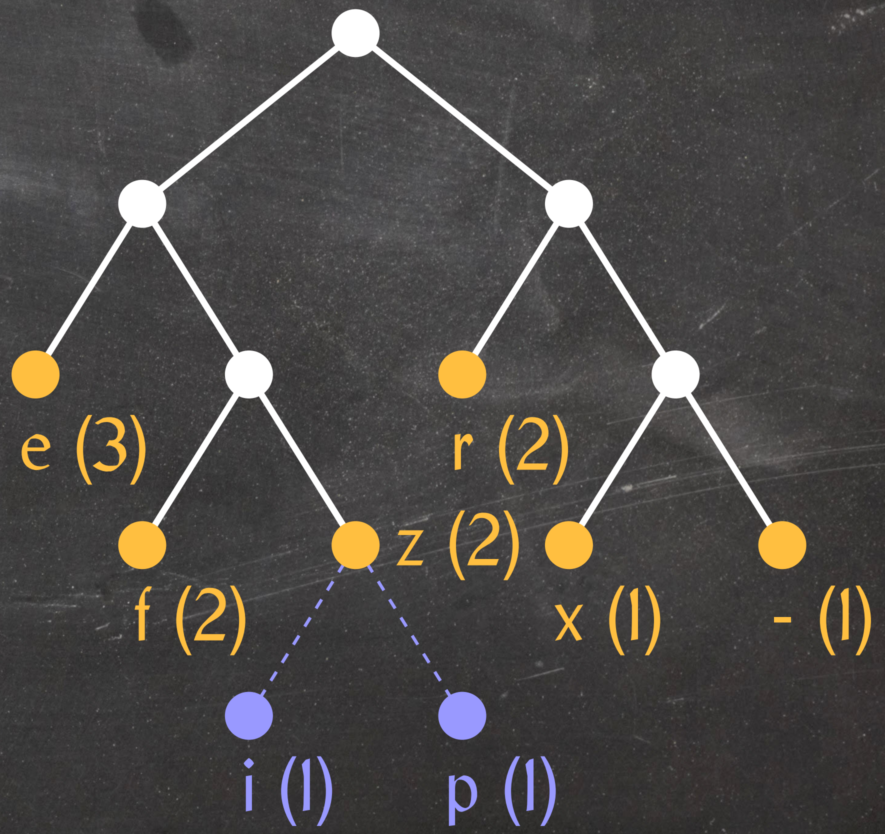

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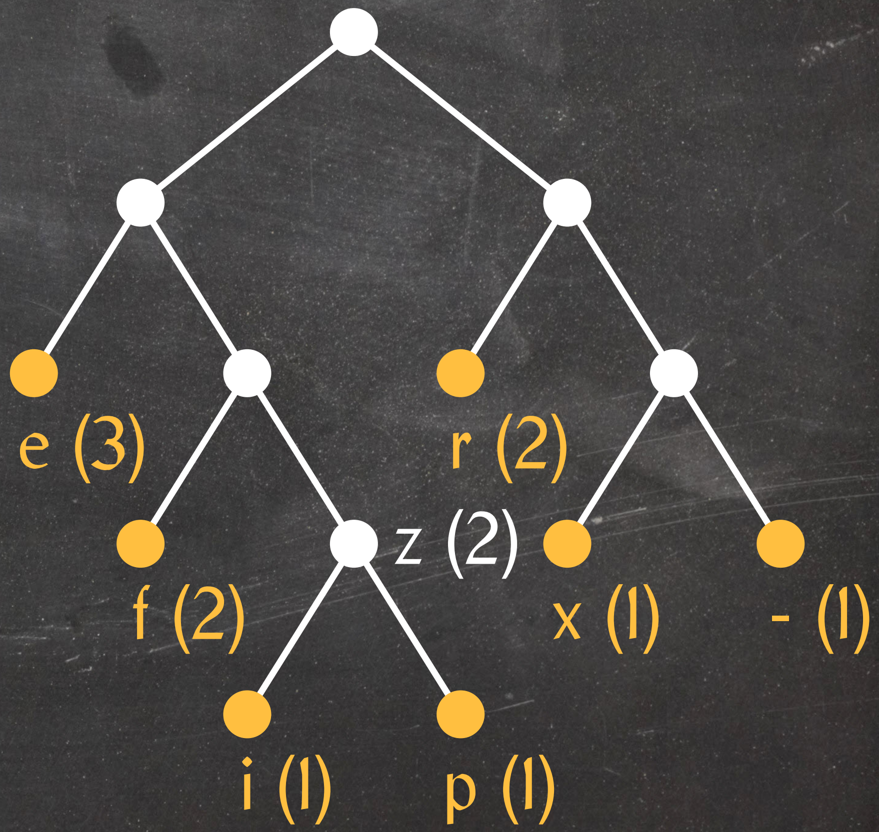
"prefix-free"



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$$C''(x) = \begin{cases} C^*(x) & x \neq z \\ \sigma & x = z \text{ and } C^*(a) = \sigma 0 \end{cases}$$

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⇓

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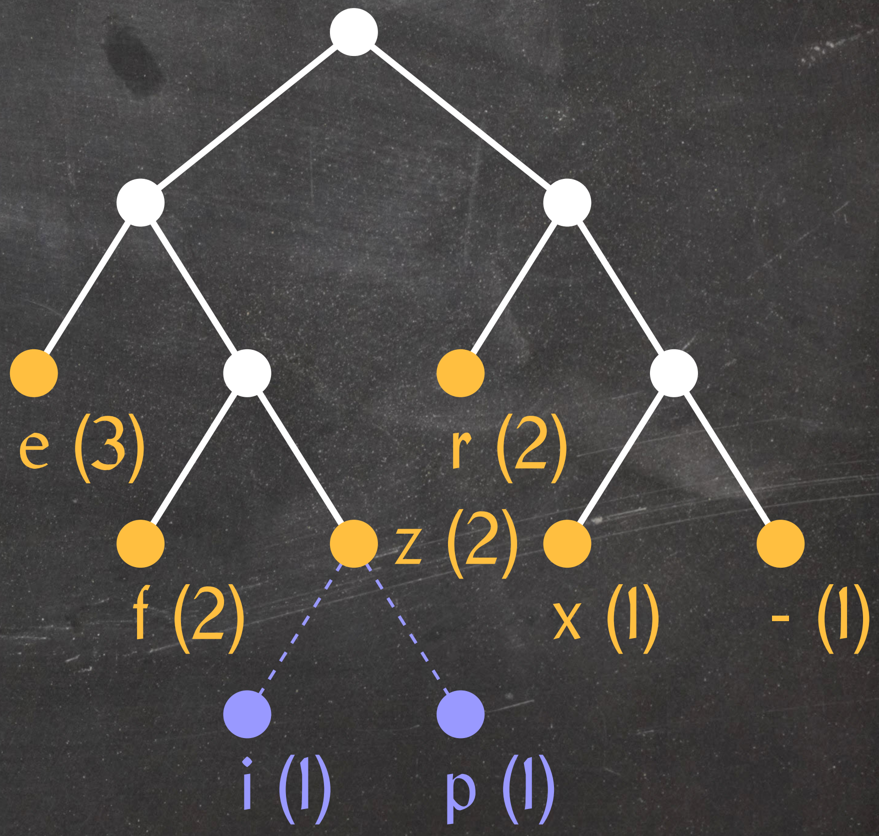

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⇓

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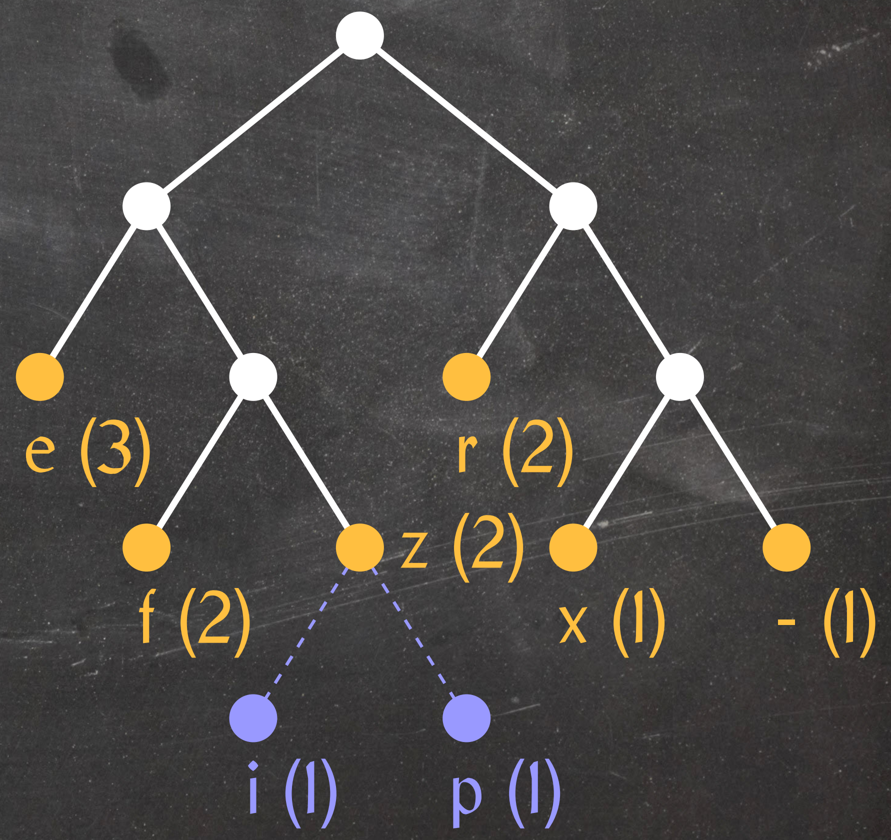

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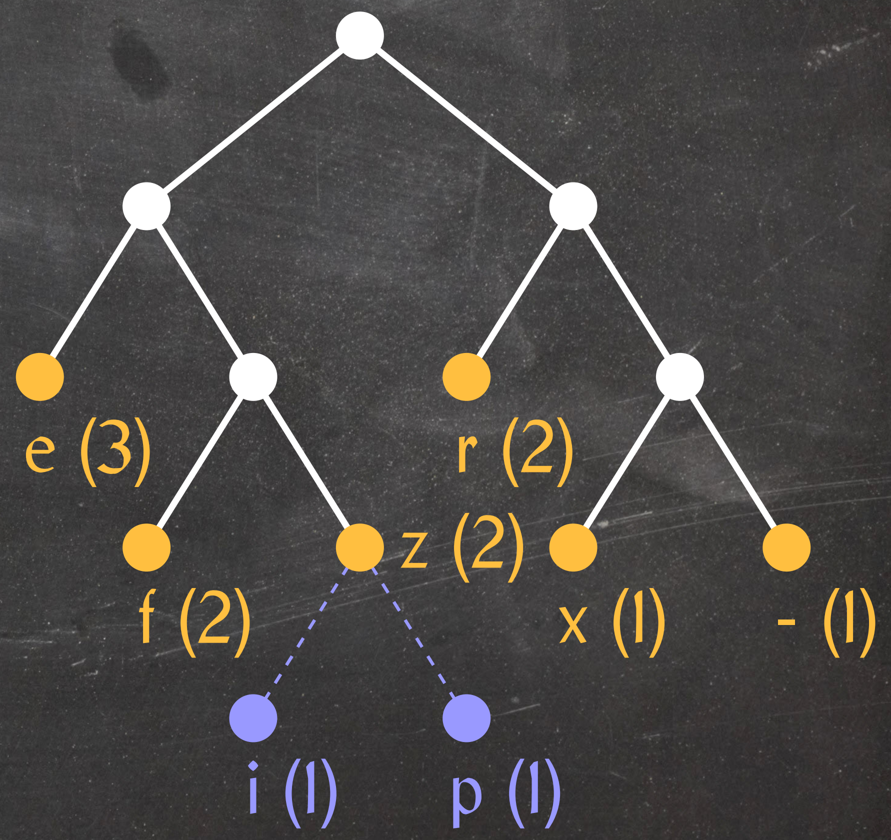

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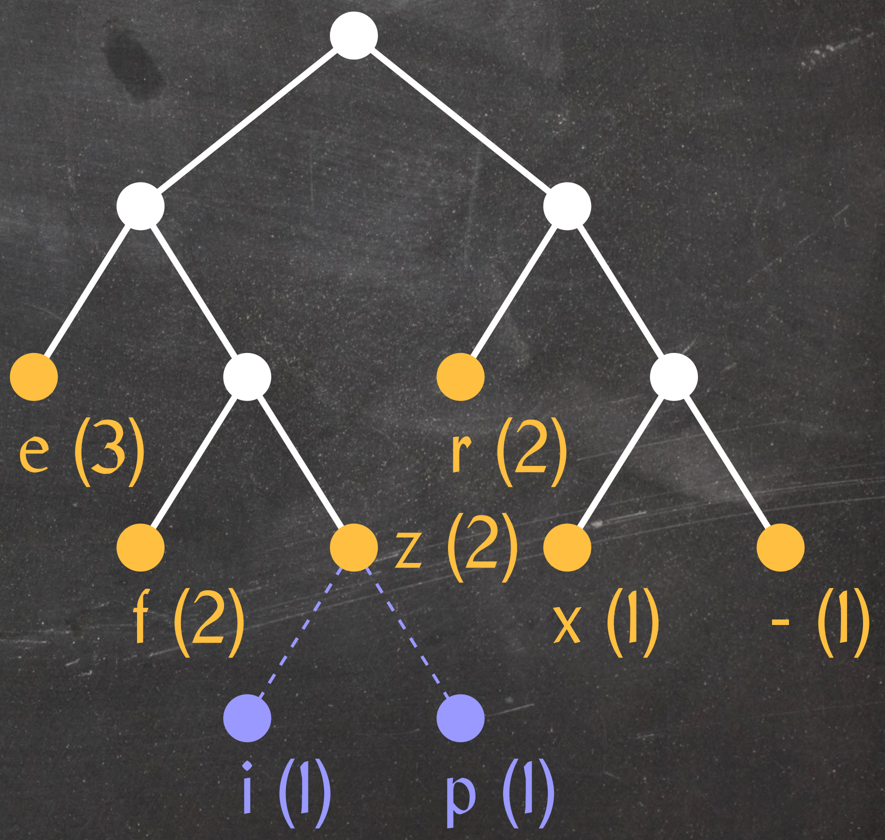
$\Rightarrow$ $|C''(T')| < |C'(T')|$, a contradiction because $C'(\cdot)$ is optimal for $T'$.

“prefix-free”

$\Downarrow$

“zrefzx-free”



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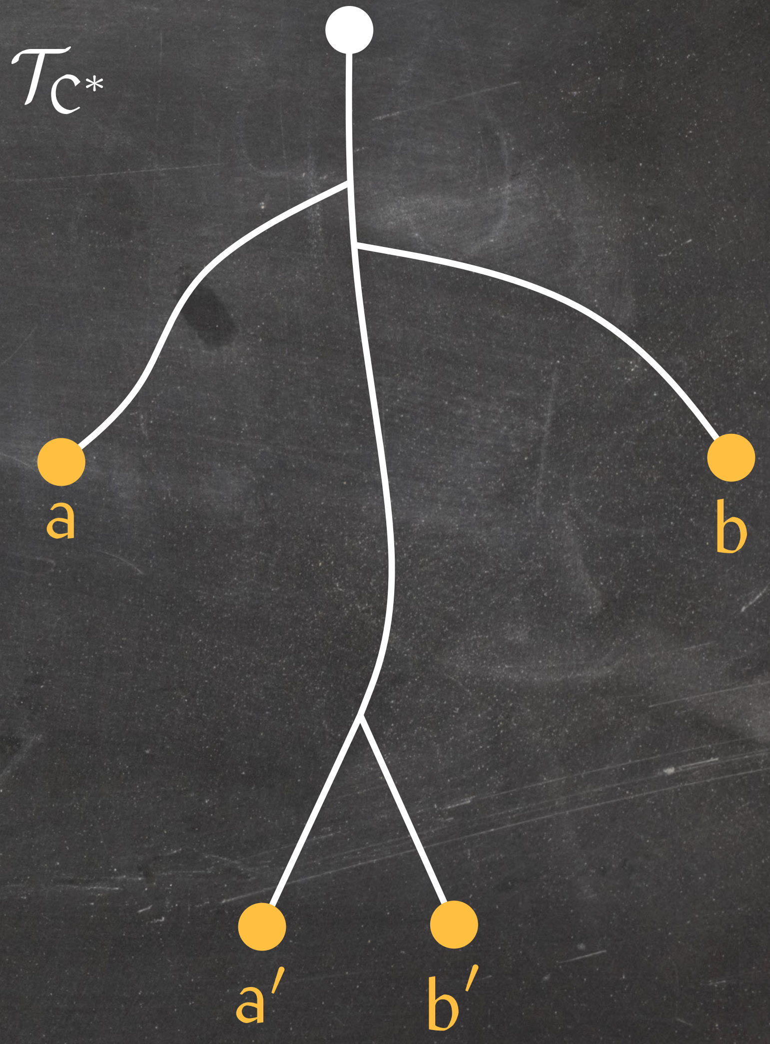

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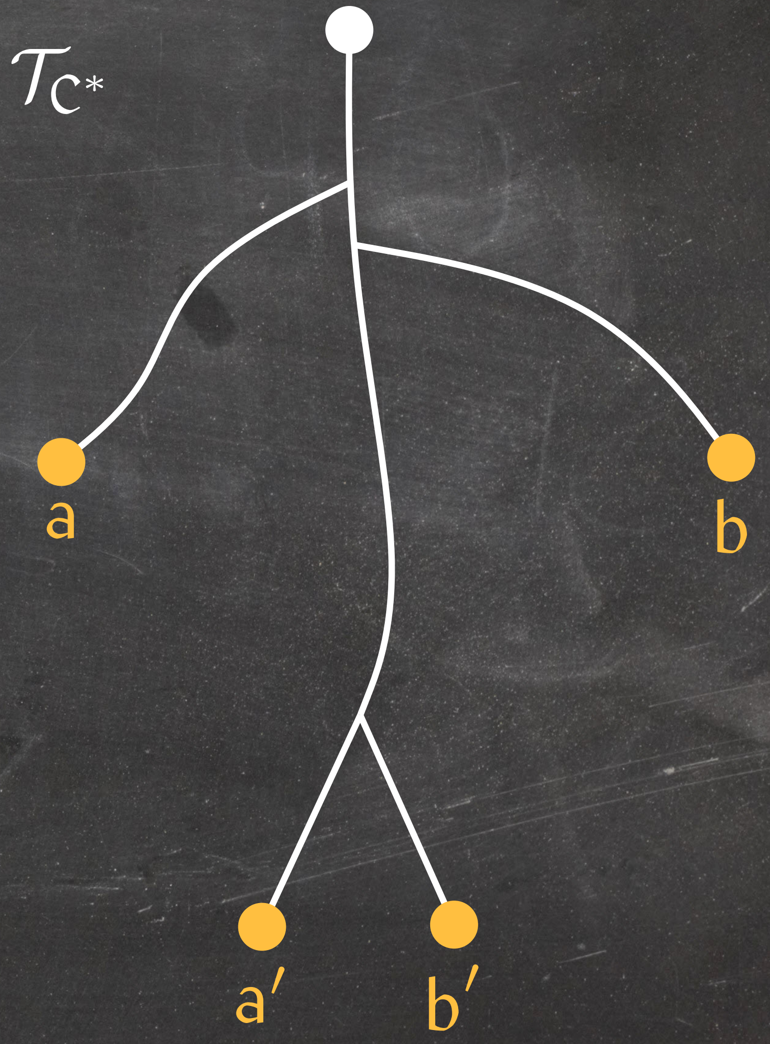

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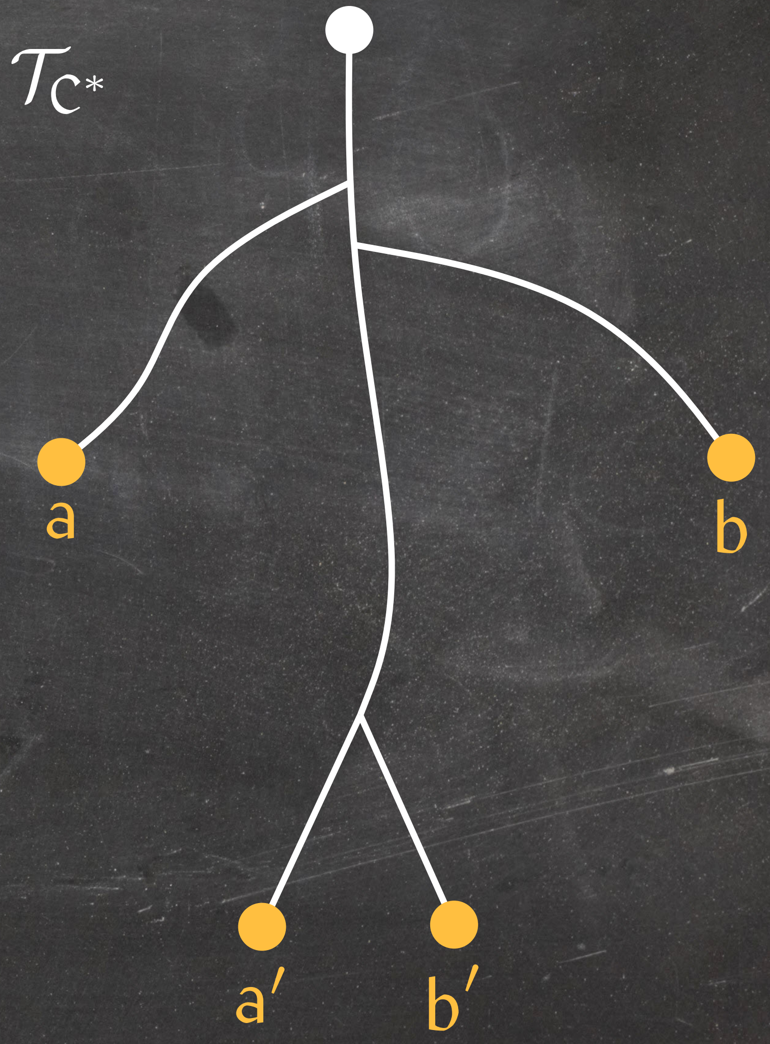

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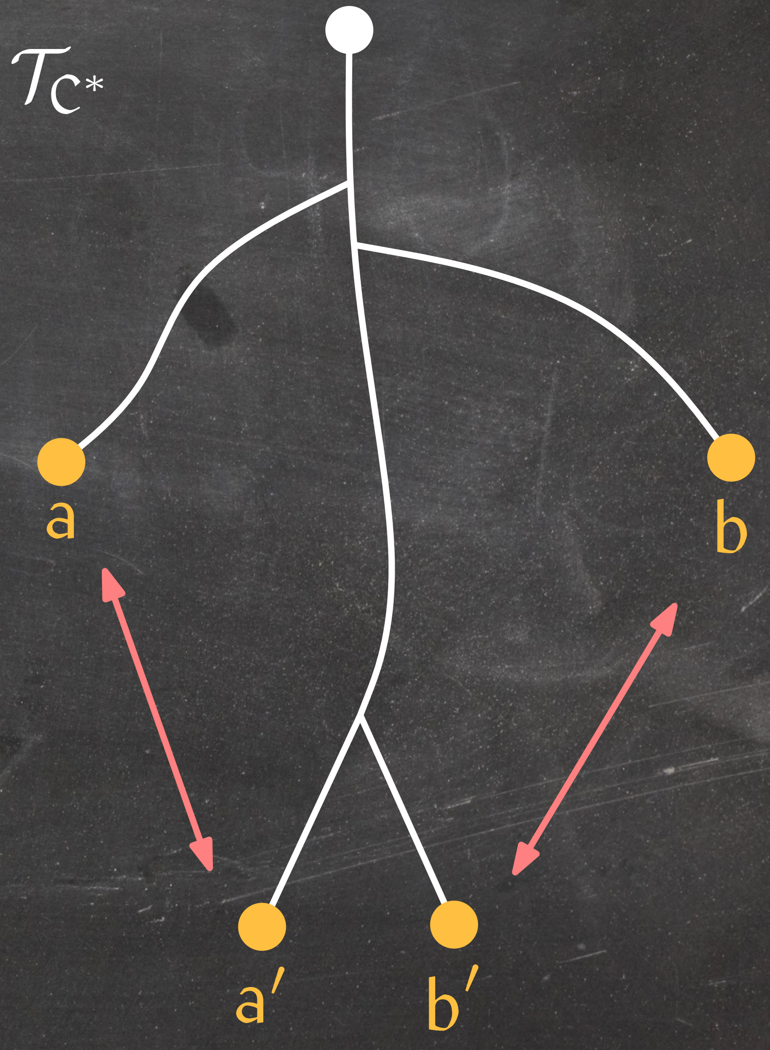

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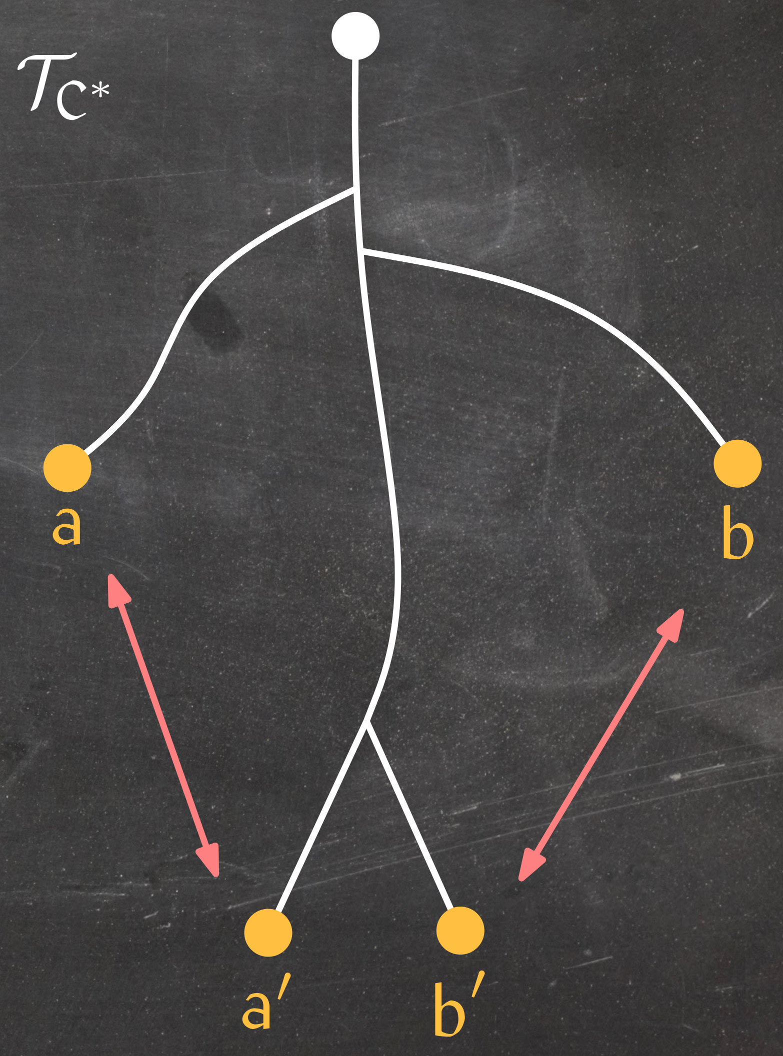

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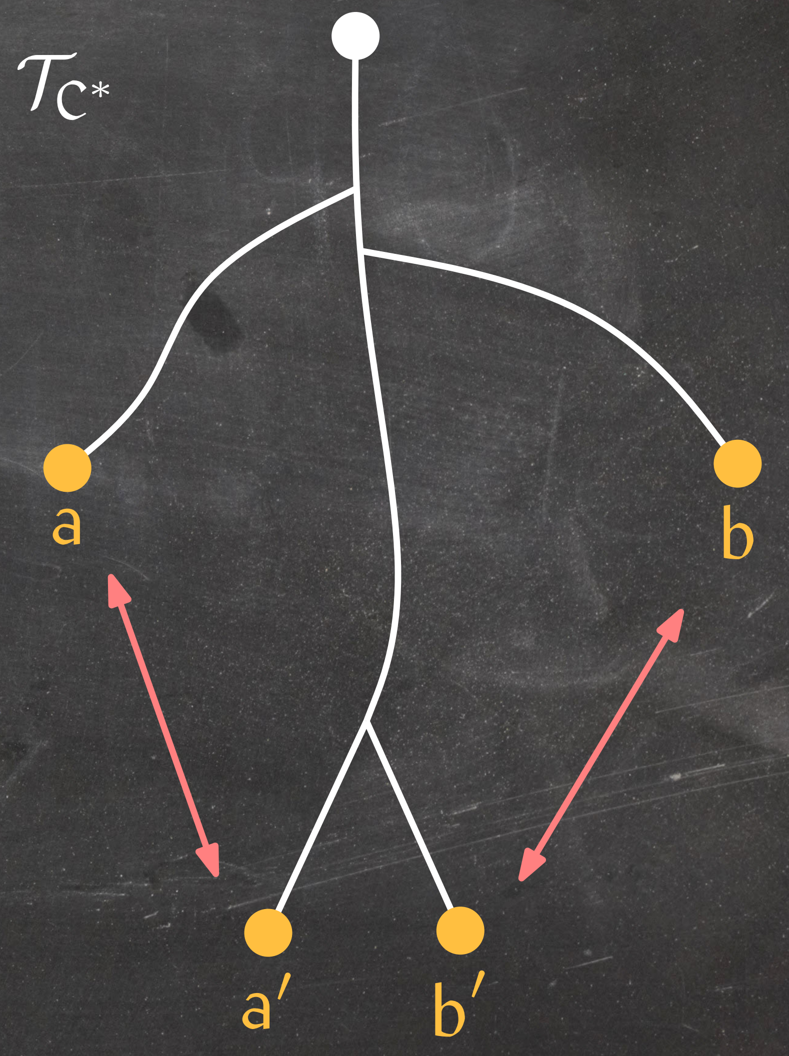
Since $a$ and $b$ are siblings in $\mathcal{T}_C$, this proves the claim.



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$$\begin{aligned}|C(T)| - |C^*(T)| = {} & f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\ & f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')|\end{aligned}$$

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$$\begin{aligned}
|C(T)| - |C^*(T)| &= f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')|
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&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= (f(a) - f(a'))\,(|C^*(a')| - |C^*(a)|) + (f(b) - f(b'))\,(|C^*(b')| - |C^*(b)|)
\end{aligned}$$

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**Claim:** There exists an optimal prefix-free code $C(\cdot)$ for T such that the two least frequent characters a and b in T are siblings in $\mathcal{T}_C$.

**Given:** $|C^*(a)| \le |C^*(a')|$, $|C^*(b)| \le |C^*(b')|$, $f(a) \le f(b)$, and $f(a') \le f(b')$.

$\Rightarrow$ $f(a) \le f(a')$ and $f(b) \le f(b')$.

$$\begin{aligned}
|C(T)| - |C^*(T)| &= f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\
&\quad\; f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad\; f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0}
\end{aligned}$$

# Correctness of Huffman's Algorithm

**Claim:** There exists an optimal prefix-free code C(·) for T such that the two least frequent characters a and b in T are siblings in $\mathcal{T}_C$.

**Given:** $|C^*(a)| \le |C^*(a')|$, $|C^*(b)| \le |C^*(b')|$, $f(a) \le f(b)$, and $f(a') \le f(b')$.

$\Rightarrow$ $f(a) \le f(a')$ and $f(b) \le f(b')$.

$$\begin{aligned}
|C(T)| - |C^*(T)| &= f(a)|C(a)| + f(b)|C(b)| + f(a')|C(a')| + f(b')|C(b')| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= f(a)|C^*(a')| + f(b)|C^*(b')| + f(a')|C^*(a)| + f(b')|C^*(b)| - \\
&\quad f(a)|C^*(a)| - f(b)|C^*(b)| - f(a')|C^*(a')| - f(b')|C^*(b')| \\
&= \underbrace{(f(a) - f(a'))}_{\le 0}\underbrace{(|C^*(a')| - |C^*(a)|)}_{\ge 0} + \underbrace{(f(b) - f(b'))}_{\le 0}\underbrace{(|C^*(b')| - |C^*(b)|)}_{\ge 0} \\
&\le 0
\end{aligned}$$

# Summary

Greedy algorithms make natural local choices in their search for a globally optimal solution.

**Many good heuristics are greedy:**

- Simple
- Work well in practice

**Proof that a greedy algorithm finds an optimal solution:**

- Induction
- Exchange argument

**Useful data structures:**

- Union-find data structure
- Thin Heap

**Analysis of a sequence of data structure operations:**

- Amortized analysis
- Potential functions