Dynamic Programming

Textbook Reading

Chapters 15, 24 & 25

Overview

Design principle

- Recursively break the problem into smaller subproblems.
- Avoid repeatedly solving the same subproblems by caching their solutions.

Important tool

- Recurrence relations

Problems

- Weighted interval scheduling
- Sequence alignment
- Optimal binary search trees
- Shortest paths

Weighted Interval Scheduling

Given:

A set of activities competing for time intervals on a certain resource
(E.g., classes to be scheduled competing for a classroom)

Goal:

Schedule non-conflicting activities so that the total time the resource is in use is maximized.

W. I. S.: A Naïve Solution

- Try all possible subsets.
- Check each subset for conflicts.
- Out of the non-conflicting ones, remember the one with maximal total length.

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Cost:

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Cost: $O(2^n \cdot n^2)$

W. I. S.: Towards a Better Solution

General idea:

- Try to make one choice at a time, just as in a greedy algorithm.
- In each step, what are the options we can choose from?
- What can we say about the subproblem we obtain after choosing each option?

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If the maximal-length subset of $\{I_1, I_2, \ldots, I_n\}$ does not include I_n, then it must be a maximal-length subset of $\{I_1, I_2, \ldots, I_{n-1}\}$.

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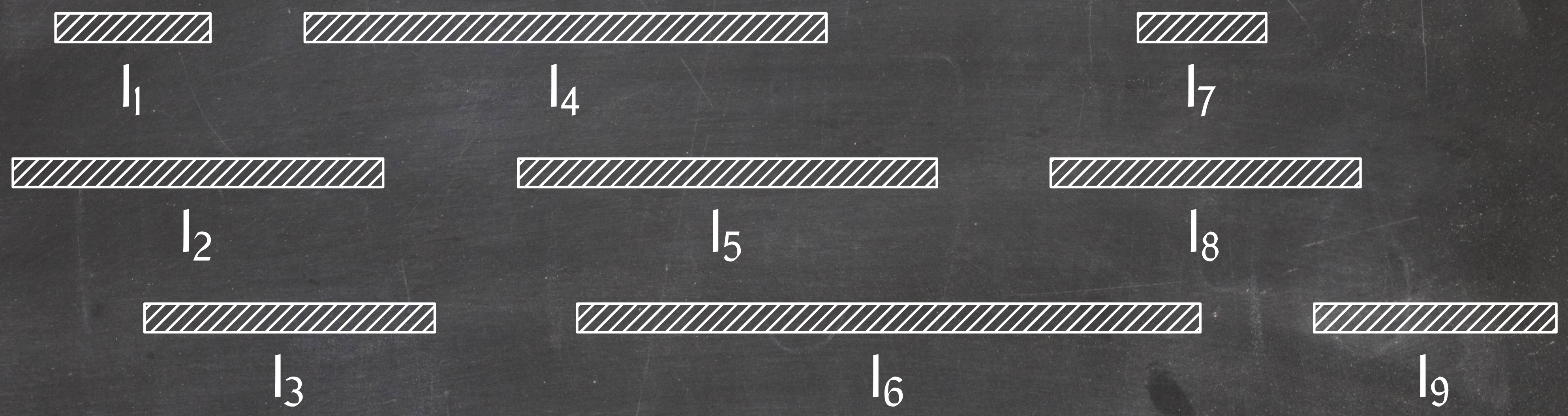
If the maximal-length subset of $\{I_1, I_2, \dots, I_n\}$ includes I_n, then it must be $O \cup \{I_n\}$, where O is a maximal-length subset of all intervals in $\{I_1, I_2, \dots, I_n\}$ that do not overlap I_n.

W. I. S.: Cleaning Up the Model

Number the intervals by increasing ending times:



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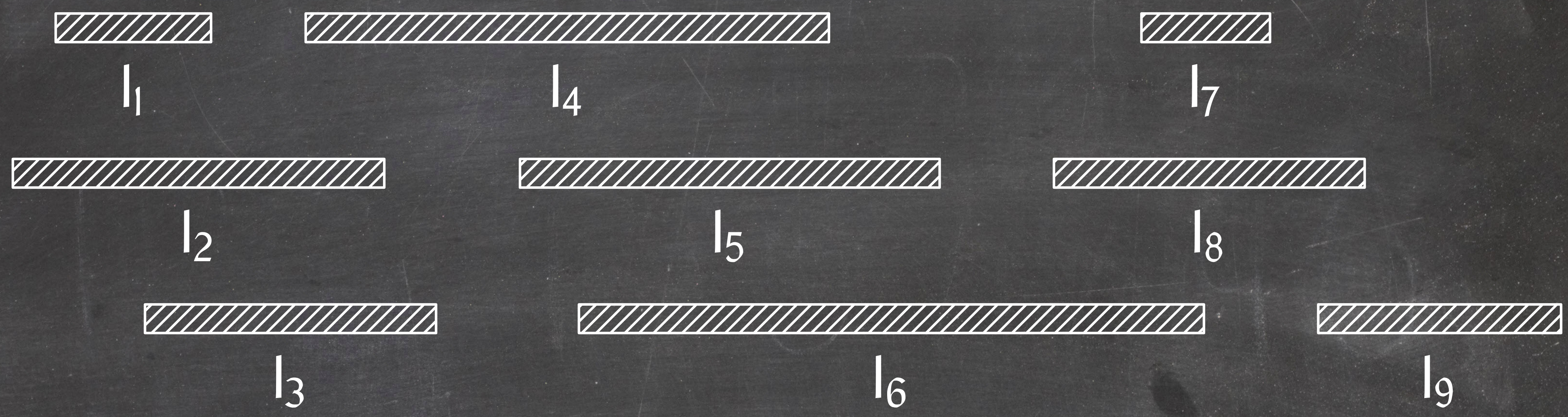

For $1 \leq j \leq n$, let $p_j = \max(\{0\} \cup \{k \mid 1 \leq k < j \text{ and } I_k \text{ does not overlap } I_j\})$.

j	1	2	3	4	5	6	7	8	9
p_j	0	0	0	1	3	3	5	5	7

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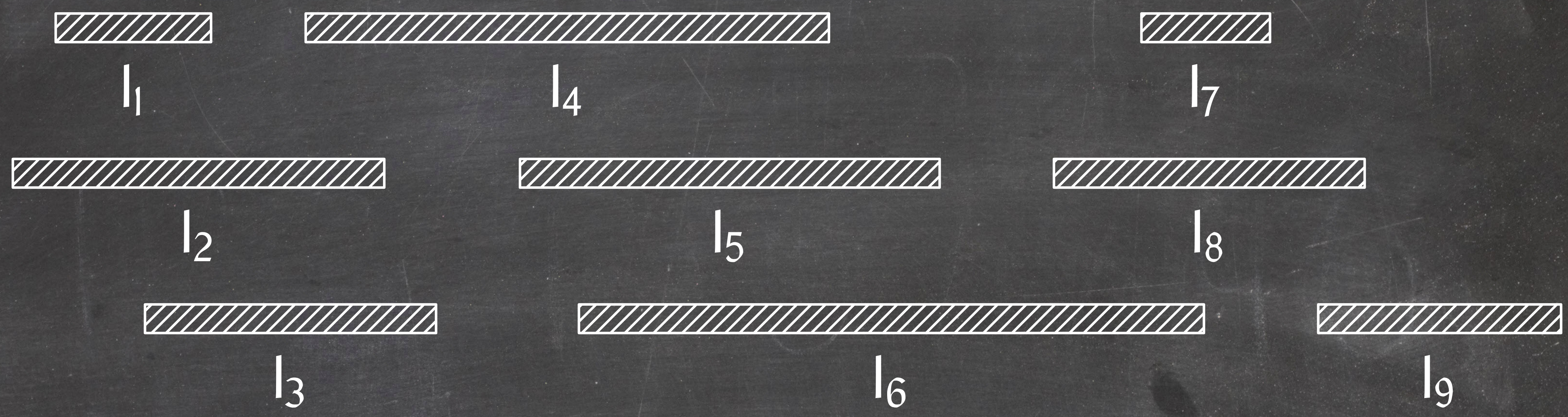

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If the maximal-length subset of $\{I_1, I_2, \ldots, I_n\}$ includes I_n, then it is O_{p_n}, where O_{p_n} is a maximal-length subset of the intervals $\{I_1, I_2, \ldots, I_{p_n}\}$.

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Let $|I_j|$ be the length of interval I_j.

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$$\ell(j) = \begin{cases} 0 & j = 0 \\ \max(\ell(j-1), |I_j| + \ell(p_j)) & j > 0 \end{cases}$$

W. I. S.: A Recursive Algorithm

FindScheduleLength(I, p, j)

```
if j = 0
   then return 0
   else  return max(FindScheduleLength(I, p, p[j]) + |I[j]|,
                    FindScheduleLength(I, p, j − 1))
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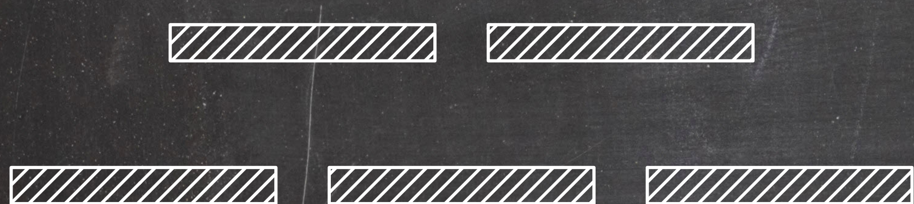

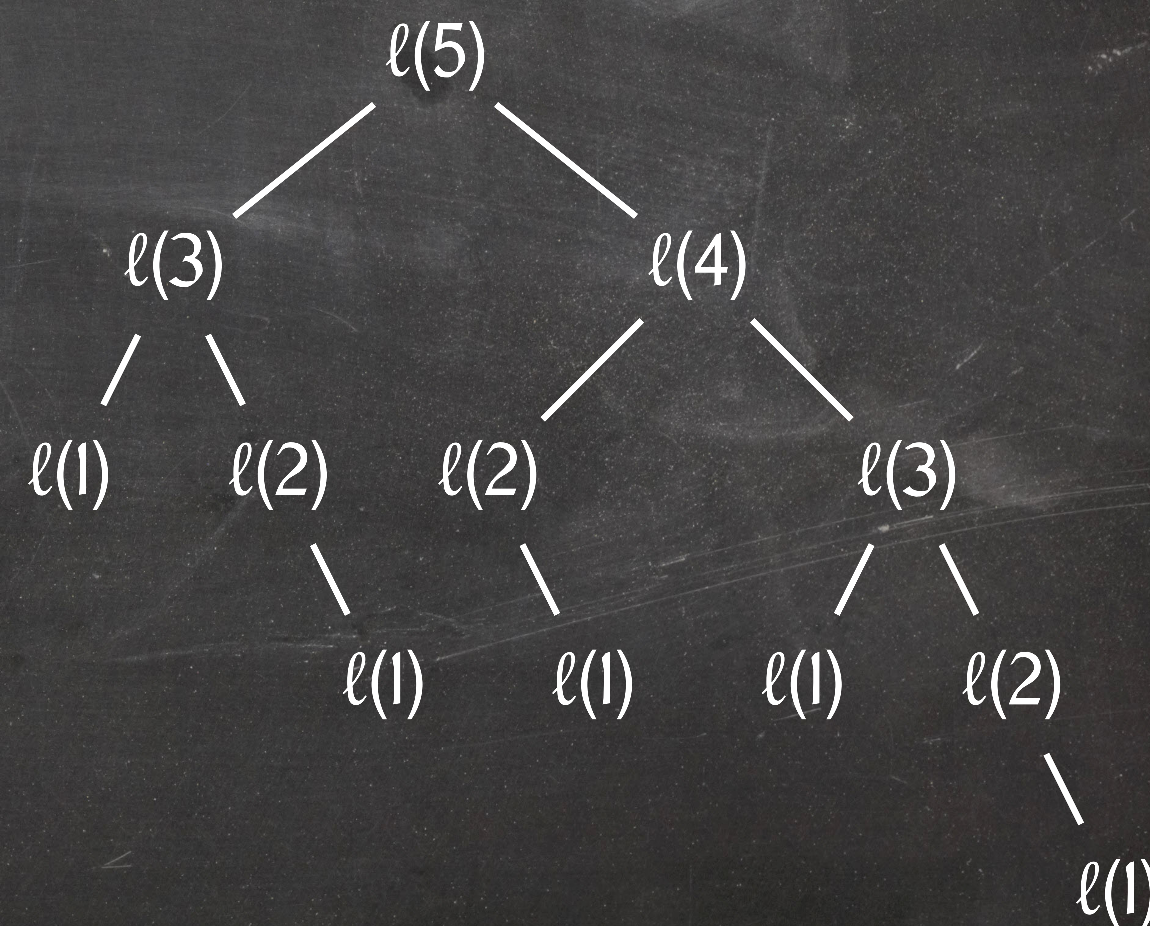

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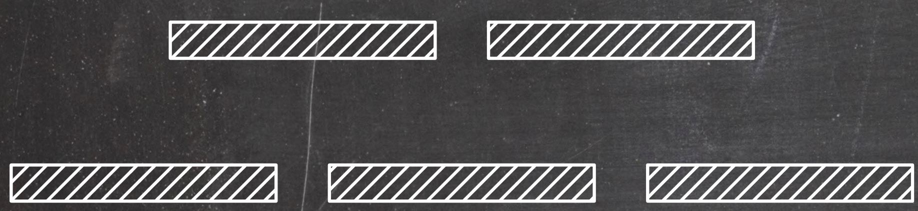

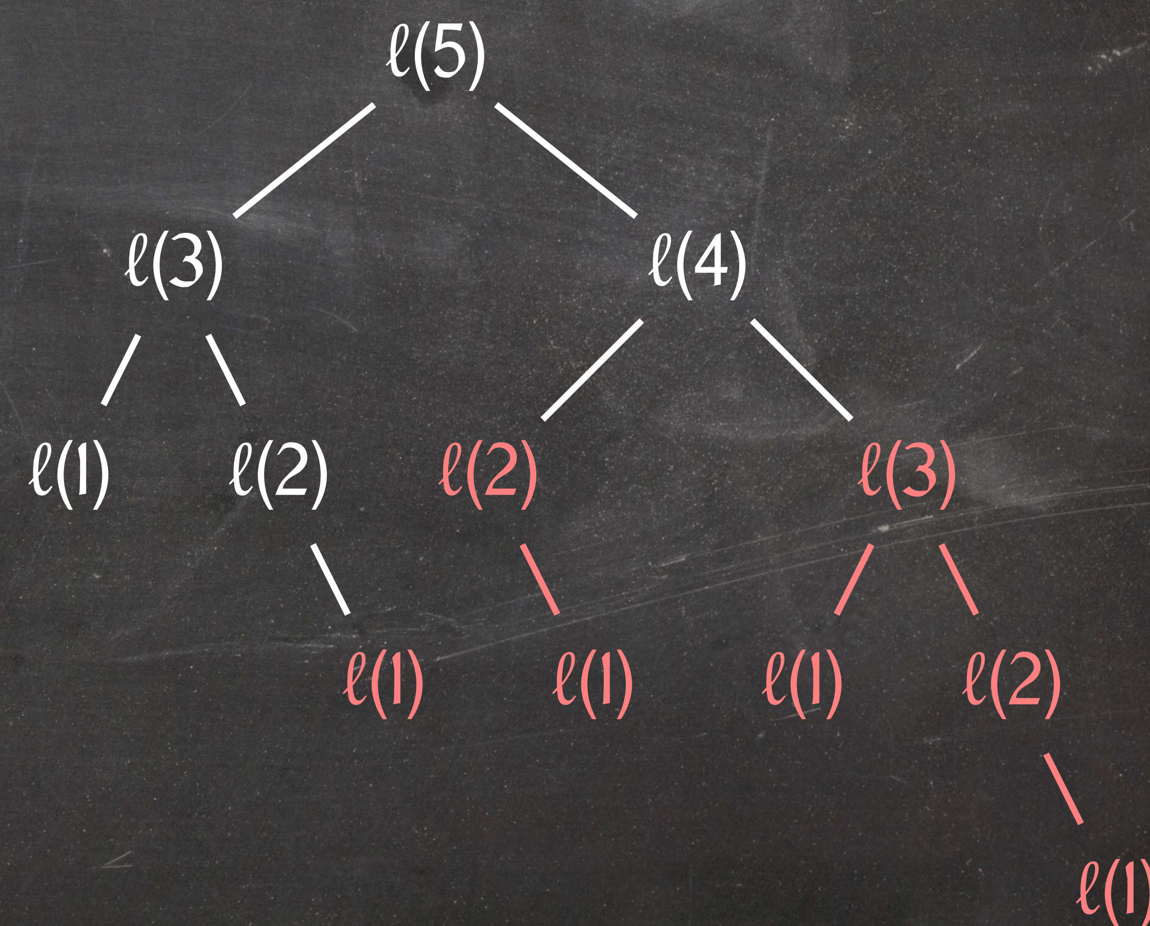

The recursive algorithm computes many values repeatedly.

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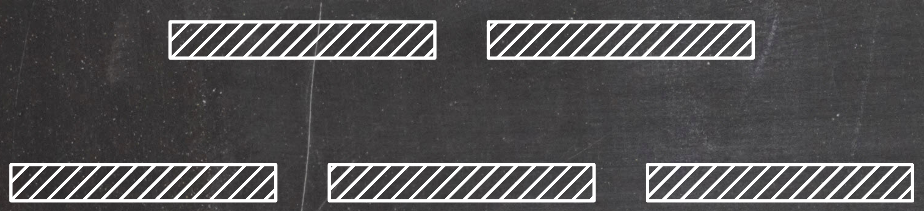

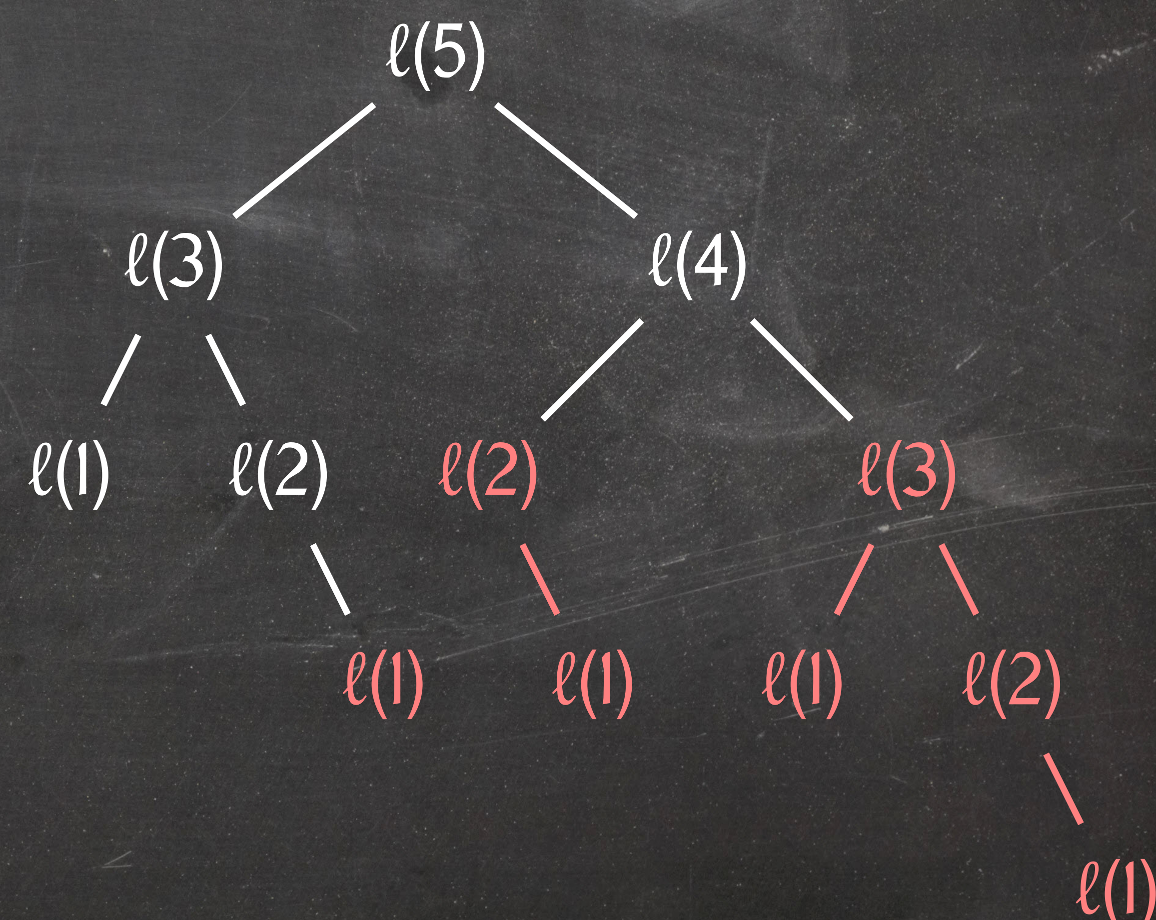

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There are only n values to compute!

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Memoization: Store already computed values in a table to avoid recomputing them.

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FindScheduleLength(I, ℓ, p, j)

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if j = 0
   then return 0
   else if ℓ[j] < 0
           then ℓ[j] = max(FindScheduleLength(I, p, p[j]) + |I[j]|,
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Running time: O(n)

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Advantage over memoization:

- No need for recursion.
- Algorithm is often simpler.

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- No need for recursion.
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Disadvantage over memoization:

- Need to worry about the order in which the table entries are computed:
  - All entries needed to compute the current entry need to be computed first.
- Memoization computes table entries as needed.

W. I. S.: Computing the Set of Intervals

FindSchedule(I, p)

```
ℓ[0] = 0
S[0] = []
for j = 1 to n
    do if ℓ[j − 1] > ℓ[p[j]] + |I[j]|
          then ℓ[j] = ℓ[j − 1]
               S[j] = S[j − 1]
          else ℓ[j] = ℓ[p[j]] + |I[j]|
               S[j] = [I[j]] ++ S[p[j]]
return S[n]
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Running time: $O(n)$

This computes the sequence of intervals ordered from last to first.

This list is of course easy to reverse in linear time.

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What's missing?

- Sort the intervals by their ending times.
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Solution:

- Sorting is easily done in $O(n \lg n)$ time.
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Theorem: The weighted interval scheduling problem can be solved in $O(n \lg n)$ time.

The Dynamic Programming Technique

The technique:

- Develop a recurrence expressing the optimal solution for a given problem instance in terms of optimal solutions for smaller problem instances:
- Evaluate this recurrence
  - Recursively using memoization or
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For this to work, the problem must exhibit the optimal substructure property: The optimal solution to a problem instance must be composed of optimal solutions to smaller problem instances.

A speed-up over the naïve recursive algorithm is achieved if the problem exhibits overlapping subproblems: The same subproblem occurs over and over again in the recursive evaluation of the recurrence.

Developing a Dynamic Programming Algorithm

Step 1: Think top-down:

- Consider an optimal solution (without worrying about how to compute it).
- Identify how the optimal solution of any problem instance decomposes into optimal solutions to smaller problem instances.
- Write down a recurrence based on this analysis.

Step 2: Formulate the algorithm, which computes the solution bottom-up:

- Since an optimal solution depends on optimal solutions to smaller problem instances, we need to compute those first.

Sequence Alignment

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Can Google read your mind? No!

They use a clever algorithm to match your mistyped query against the phrases they have in their database.

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What's a good similarity criterion?

Sequence Alignment

Problem: Given two strings $X = x_1x_2 \cdots x_m$ and $Y = y_1y_2 \cdots y_n$, extend them to two strings $X' = x_1'x_2' \cdots x_t'$ and $Y' = y_1'y_2' \cdots y_t'$ of the same length by inserting gaps so that the following dissimilarity measure $D(X', Y')$ is minimized:

$$D(X', Y') = \sum_{i=1}^{t} d(x_i', y_i')$$

$$d(x, y) = \begin{cases} \delta & x = \textvisiblespace \text{ or } y = \textvisiblespace \text{ (gap penalty)} \\ \mu_{x,y} & \text{otherwise (mismatch penalty)} \end{cases}$$

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$$D(X', Y') = \sum_{i=1}^{t} d(x'_i, y'_i)$$

$$d(x, y) = \begin{cases} \delta & x = \text{␣} \text{ or } y = \text{␣} \text{ (gap penalty)} \\ \mu_{x,y} & \text{otherwise (mismatch penalty)} \end{cases}$$

Example:

```
Dalh␣usy␣
Dalhousie
```

$$D(X', Y') = 2\delta + \mu_{iy}$$

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Another (more important?) application:
DNA sequence alignment to measure the similarity between different DNA samples.

Sequence Alignment: Problem Analysis

Assume $(x'_1 x'_2 \cdots x'_t, y'_1 y'_2 \cdots, y'_t)$ is an optimal alignment for $(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_n)$.

What choices do we have for the final pair (x'_t, y'_t)?

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- $x'_t = x_m$ and $y'_t = y_n$
- $x'_t = x_m$ and $y'_t = \text{␣}$

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- $x'_t = x_m$ and $y'_t = \sqcup$
- $x'_t = \sqcup$ and $y'_t = y_n$

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  Assume there's a better alignment $(x''_1 x''_2 \cdots x''_s, y''_1 y''_2 \cdots y''_s)$ with dissimilarity

$$\sum_{i=1}^{s} d(x''_i, y''_i) < \sum_{i=1}^{t-1} d(x'_i, y'_i).$$

Sequence Alignment: Problem Analysis

Assume $(x_1'x_2'\cdots x_t', y_1'y_2'\cdots, y_t')$ is an optimal alignment for $(x_1x_2\cdots x_m, y_1y_2\cdots y_n)$.

What choices do we have for the final pair (x_t', y_t')?

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  $$\sum_{i=1}^{s} d(x_i'', y_i'') < \sum_{i=1}^{t-1} d(x_i', y_i').$$

  Then $(x_1''x_2''\cdots x_s''x_t', y_1''y_2''\cdots y_s''y_t')$ is an aligment for $(x_1x_2\cdots x_m, y_1y_2\cdots y_n)$ with dissimilarity

  $$\sum_{i=1}^{s} d(x_i'', y_i'') + d(x_t', y_t') < \sum_{i=1}^{t-1} d(x_i', y_i') + d(x_t', y_t') = \sum_{i=1}^{t} d(x_i', y_i'),$$

  a contradiction.

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Sequence Alignment: Problem Analysis

Assume $(x'_1x'_2\cdots x'_t, y'_1y'_2\cdots, y'_t)$ is an optimal alignment for $(x_1x_2\cdots x_m, y_1y_2\cdots y_n)$.

What choices do we have for the final pair (x'_t, y'_t)?

- $x'_t = x_m$ and $y'_t = y_n$

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Let $D(i, j)$ be the dissimilarity of the strings $x_1 x_2 \cdots x_i$ and $y_1 y_2 \cdots y_j$.

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Recurrence:

$$D(i, j) = \begin{cases} \delta \cdot j & i = 0 \\ \delta \cdot i & j = 0 \\ \min(D(i-1, j-1) + \mu_{x_i, y_j}, D(i, j-1) + \delta, D(i-1, j) + \delta) & \text{otherwise} \end{cases}$$

Sequence Alignment: The Algorithm

SequenceAlignment(X, Y, μ, δ)

```
D[0, 0] = 0
A[0, 0] = [ ]
for i = 1 to m
    do D[i, 0] = D[i − 1, 0] + δ
       A[i, 0] = [(X[i], ␣)] ++ A[i − 1, 0]
for j = 1 to n
    do D[0, j] = D[0, j − 1] + δ
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           do D[i, j] = D[i − 1, j − 1] + μ[X[i], Y[j]]
              A[i, j] = [(X[i], Y[j])] ++ A[i − 1, j − 1]
              if D[i, j] > D[i − 1, j] + δ
                 then D[i, j] = D[i − 1, j] + δ
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              if D[i, j] > D[i, j − 1] + δ
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```

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Again, the sequence alignment is reported back-to-front and can be reversed in $O(m + n)$ time.

Optimal Binary Search Trees

Balanced binary search trees (red-black trees, AVL trees, ...) guarantee $O(\lg n)$ time to find an element.

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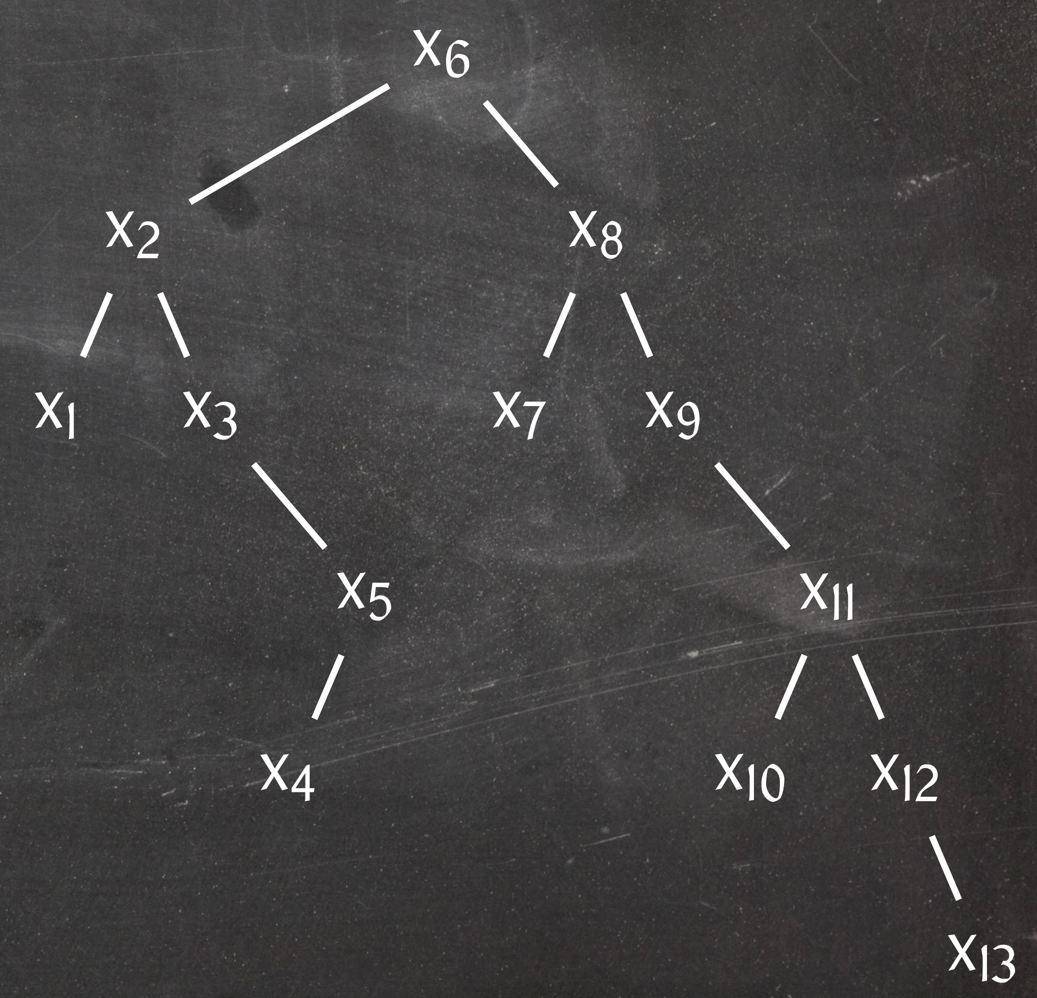

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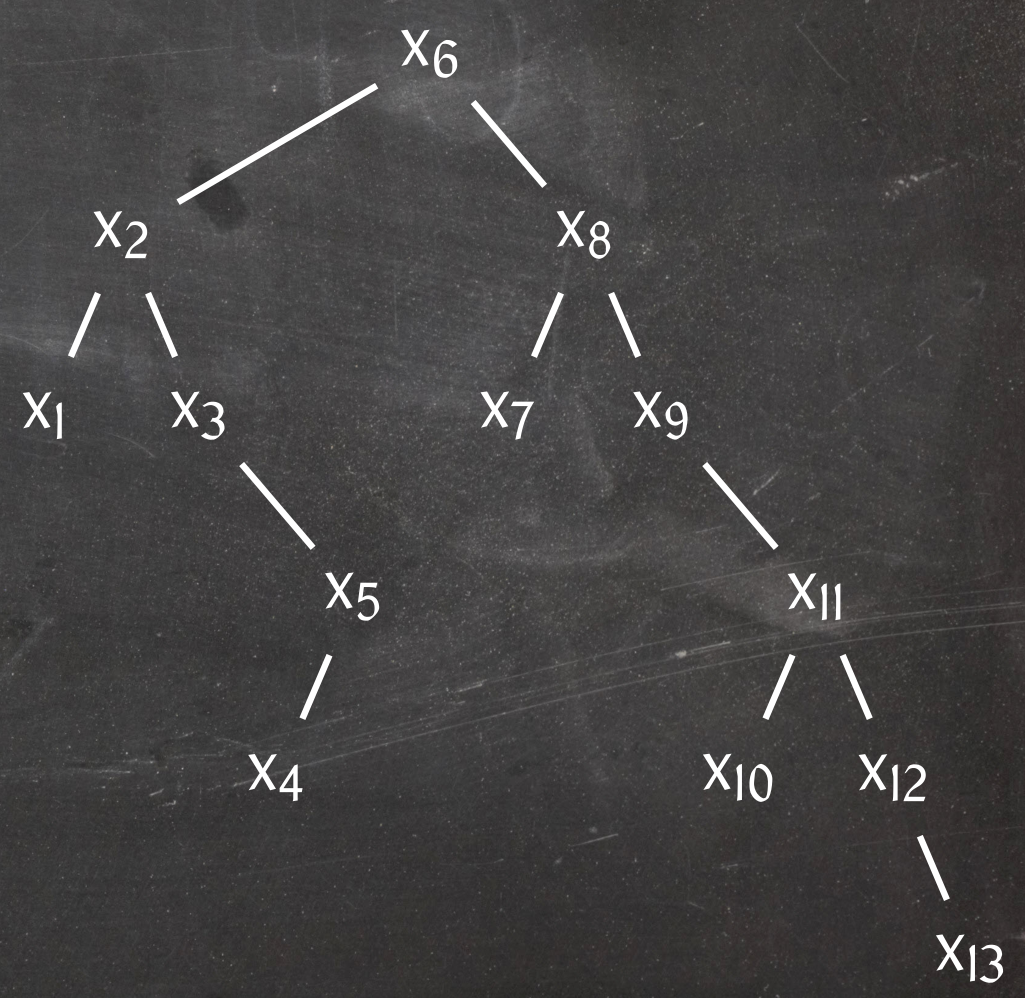

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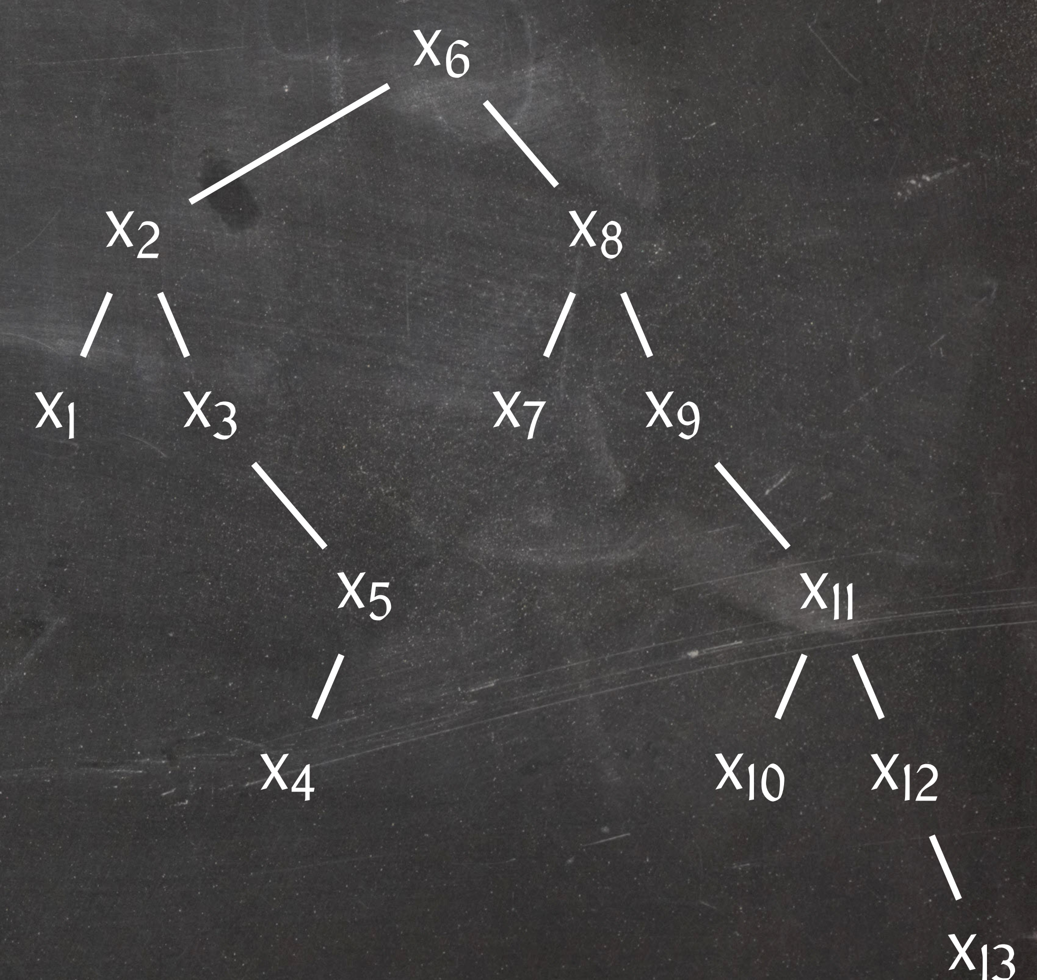

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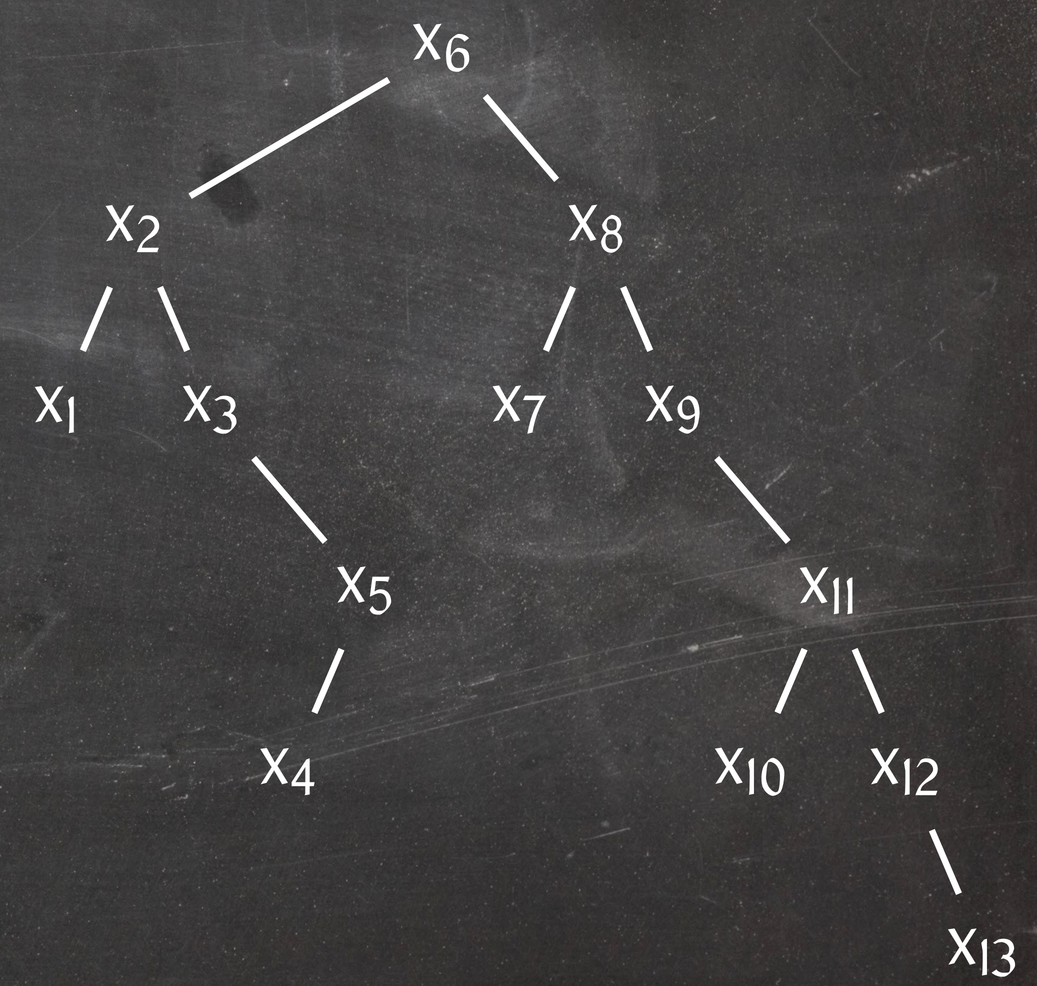
The cost of searching for element x_i is in $O(d_T(x_i))$.



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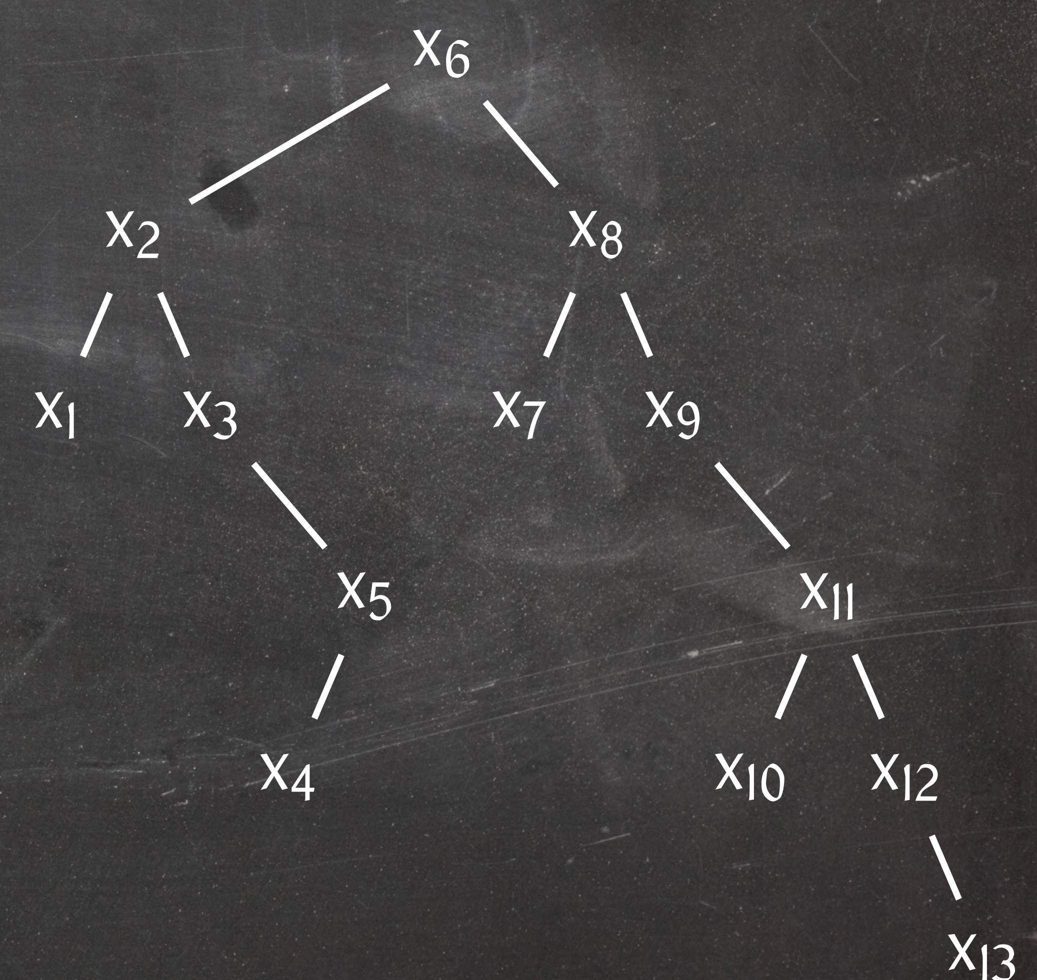

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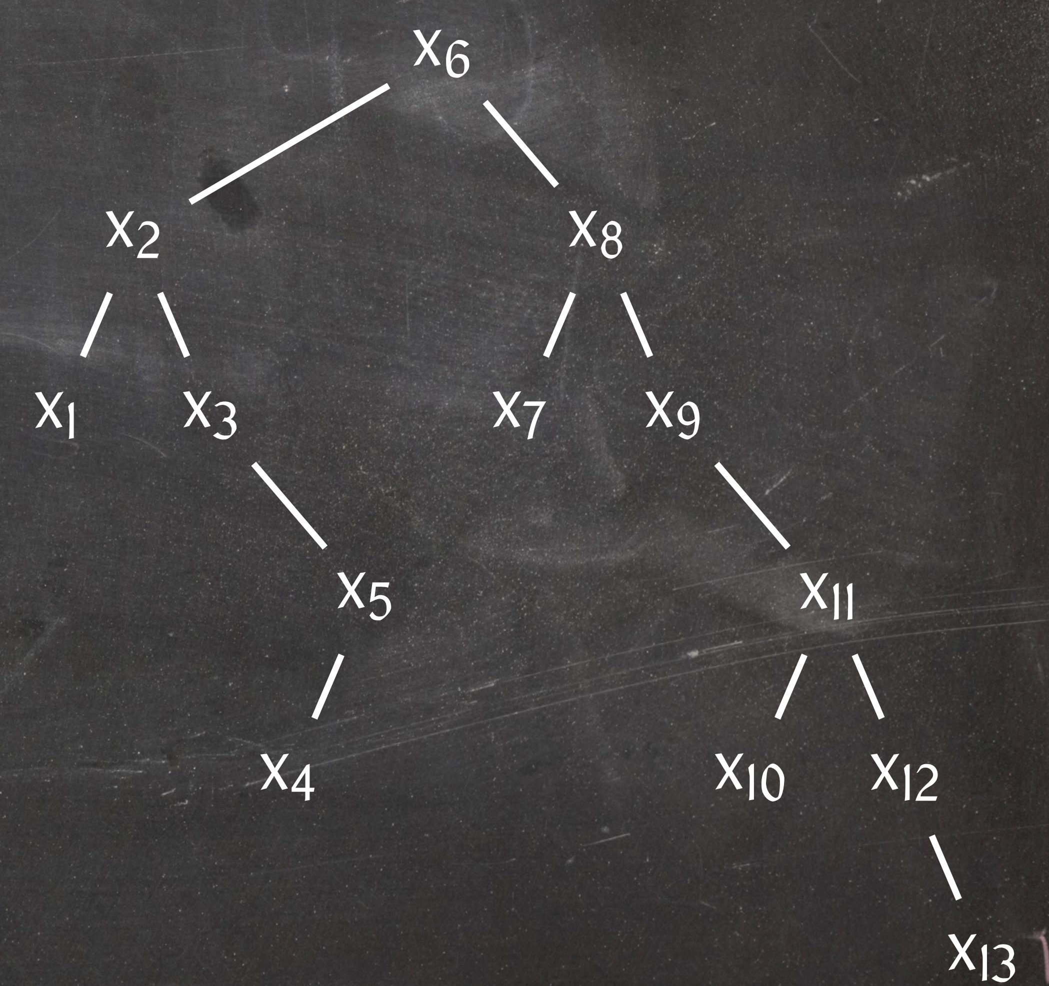

An optimal binary search tree is a binary search tree T that minimizes $C_P(T)$.

Balancing Is Not Necessarily Optimal

Assume $n = 2^k - 1$ and $p_i = 2^{-i}$ for all $1 \leq i \leq n - 1$ and $p_n = 2^{-n+1}$.

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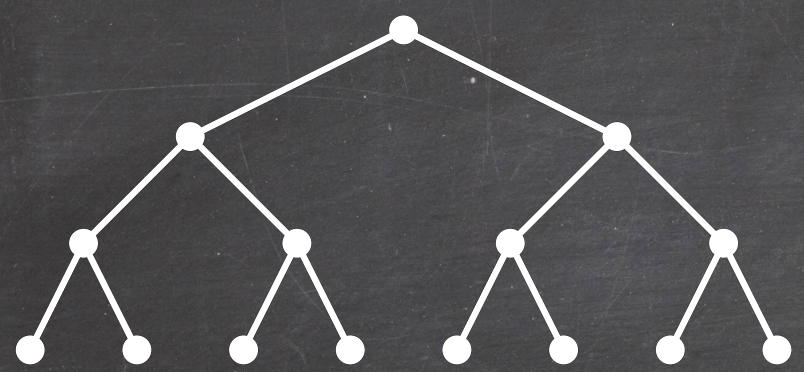

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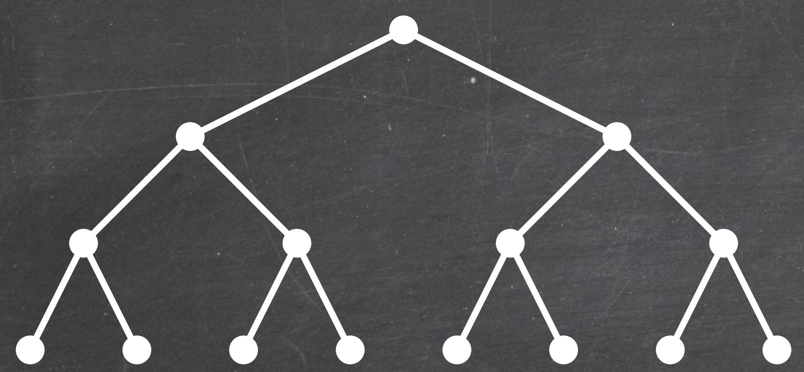

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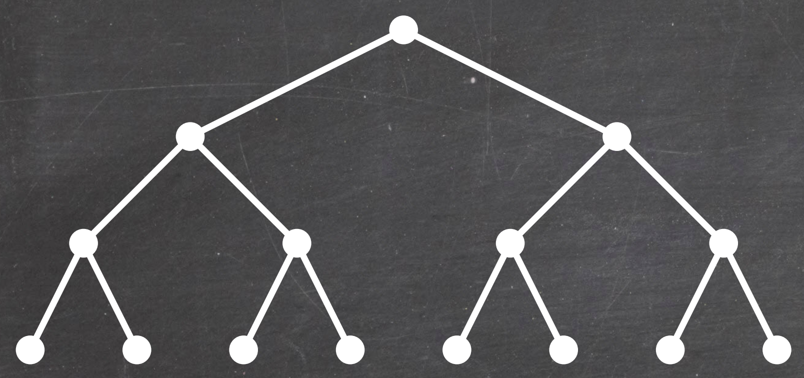

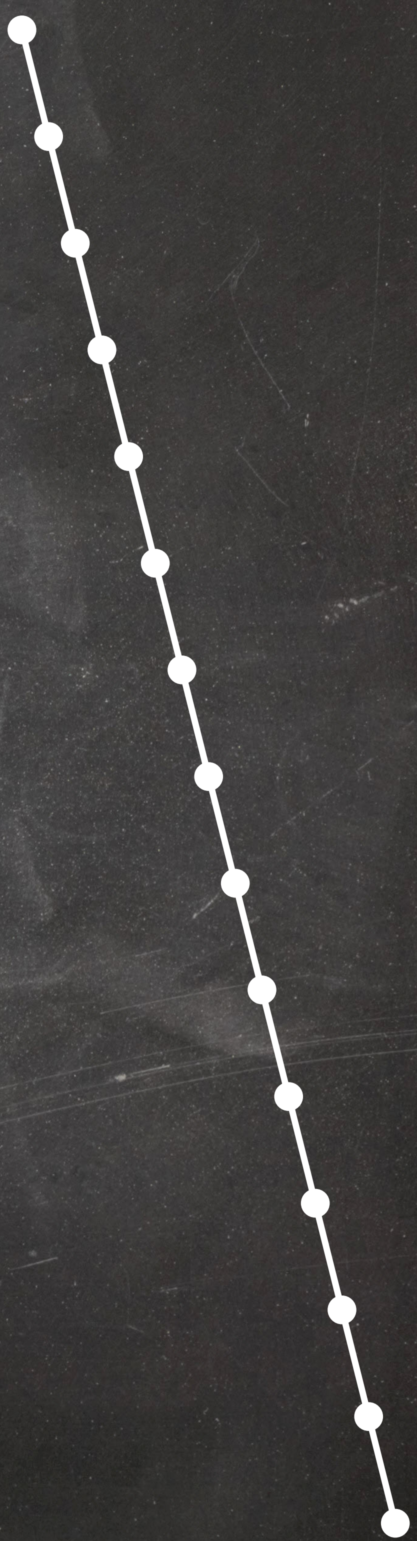

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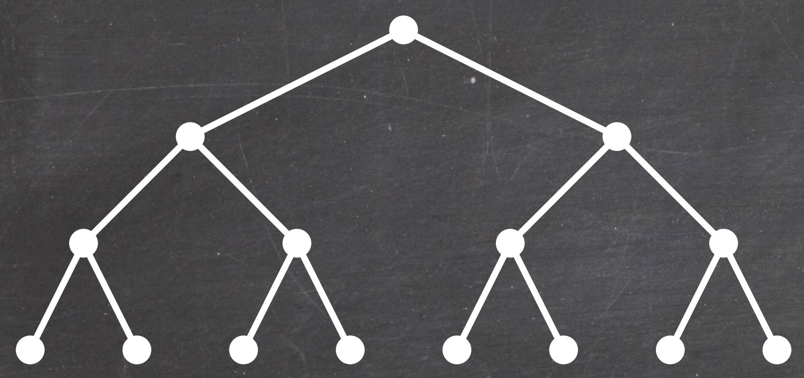

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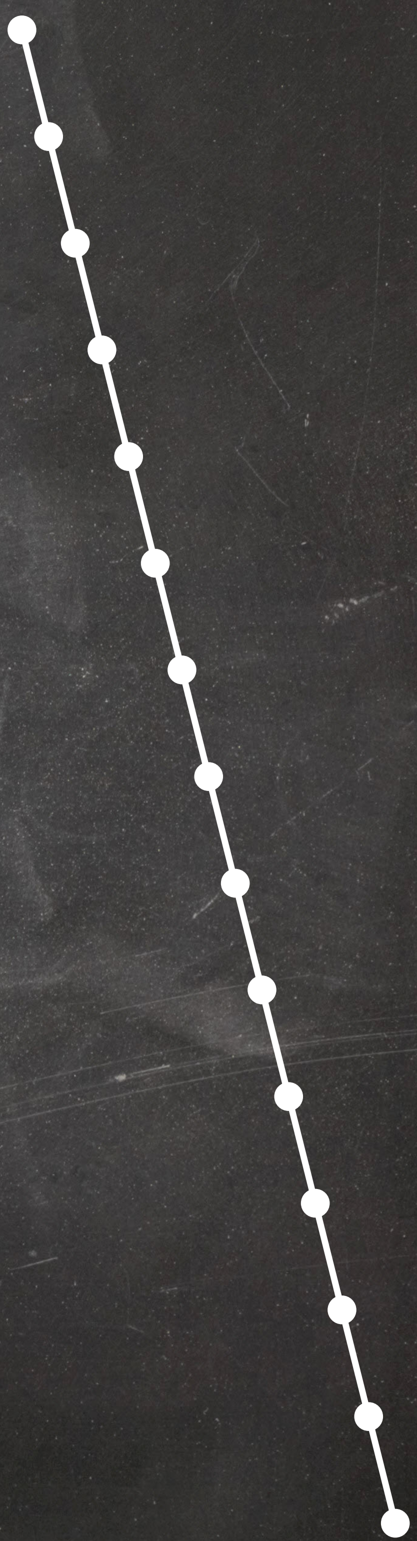

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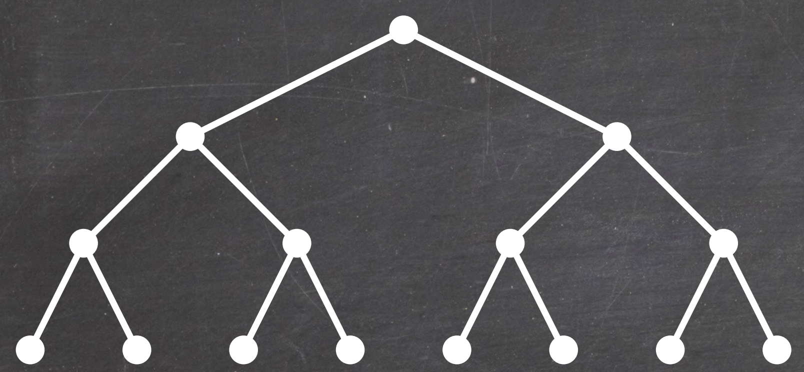

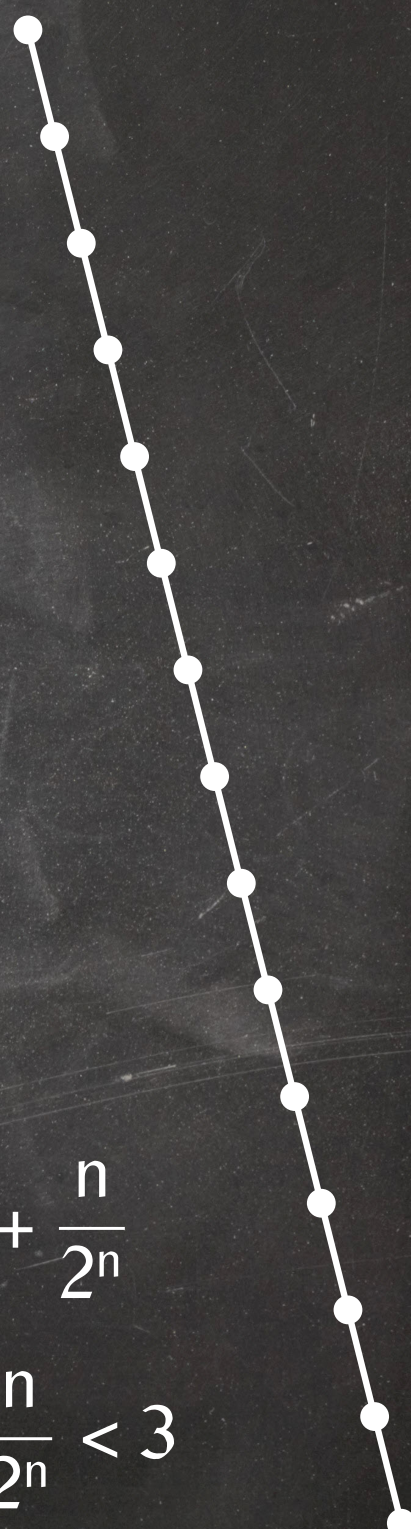

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Depth of x_i is i.

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$$= \sum_{i=1}^{n} \frac{i}{2^i} + \frac{n}{2^n} < \sum_{i=1}^{\infty} \frac{i}{2^i} + \frac{n}{2^n}$$

$$= \frac{1/2}{(1 - 1/2)^2} + \frac{n}{2^n} = 2 + \frac{n}{2^n} < 3$$

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The structure of a binary search tree:

Assume we want to store elements $x_\ell, x_{\ell+1}, \ldots, x_r$.

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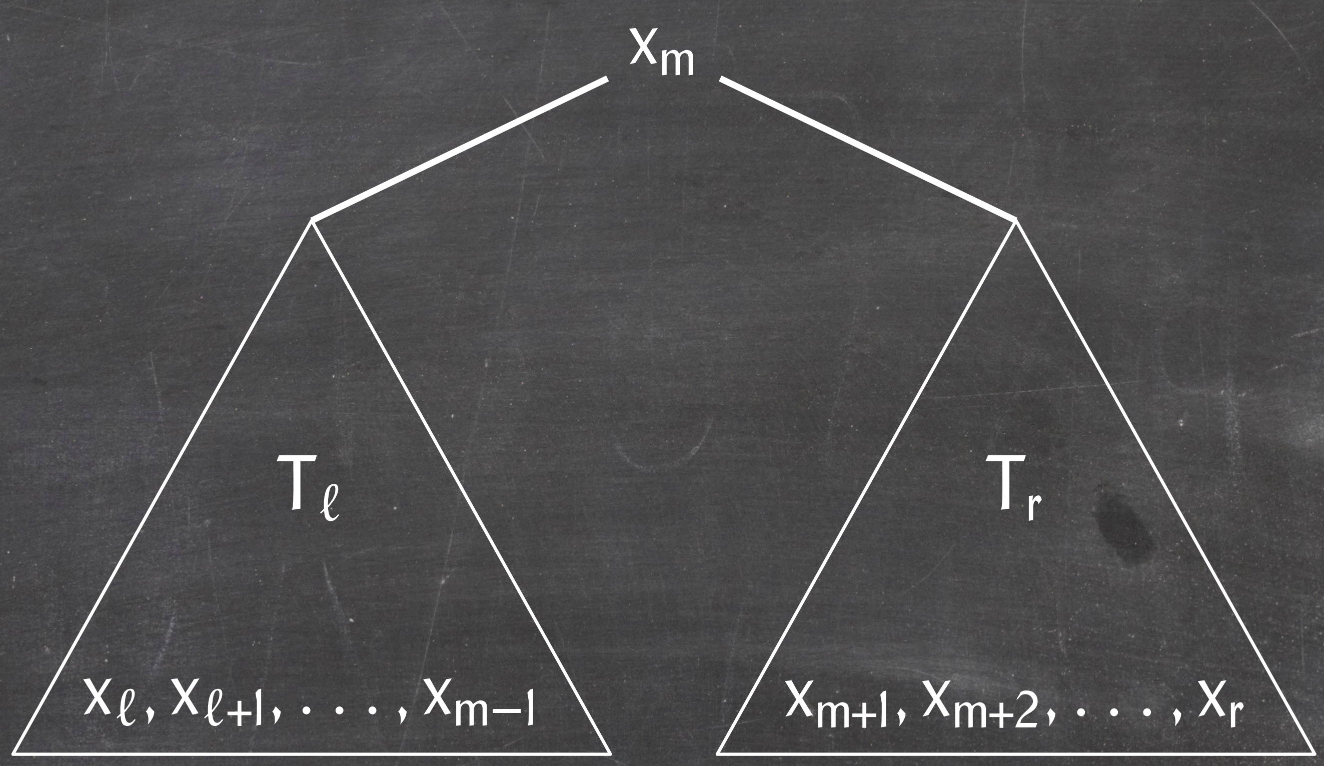

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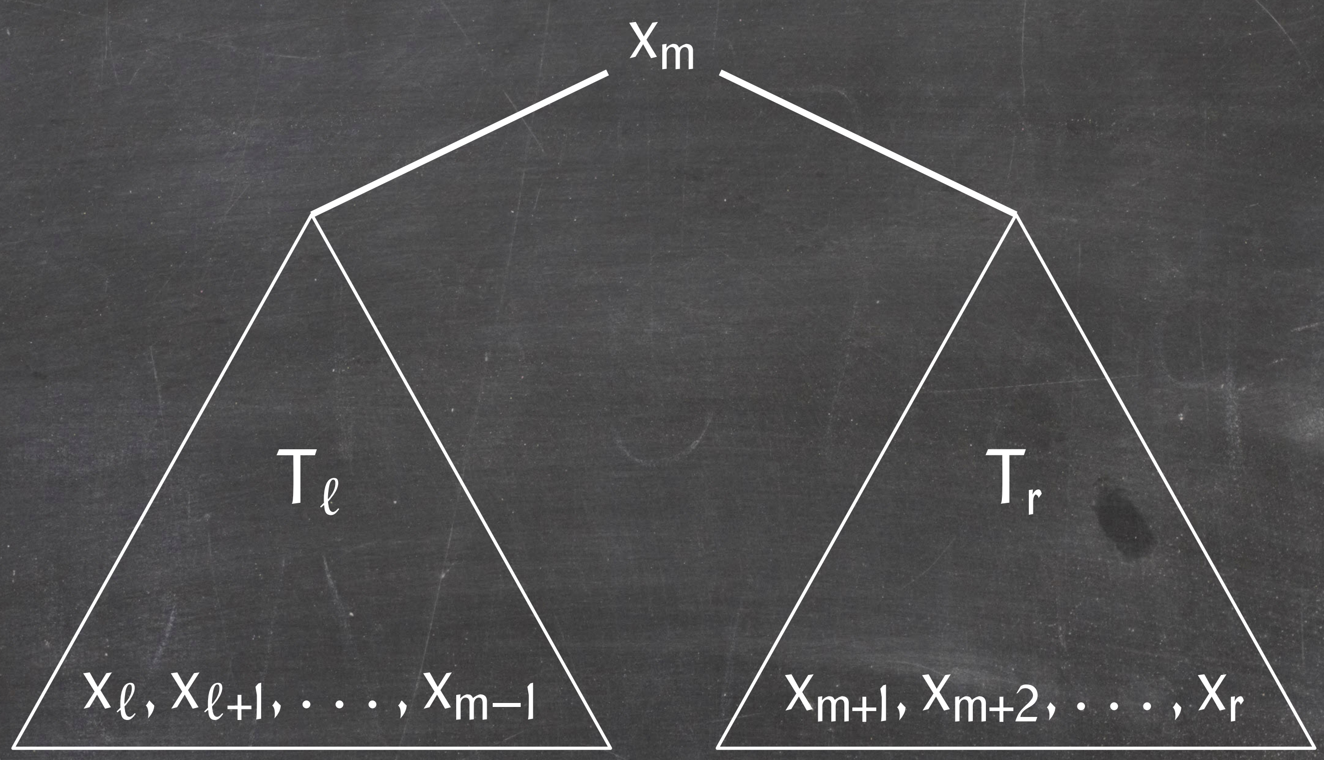

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$$C_P(T) = p_{\ell,r} + C_P(T_\ell) + C_P(T_r)$$

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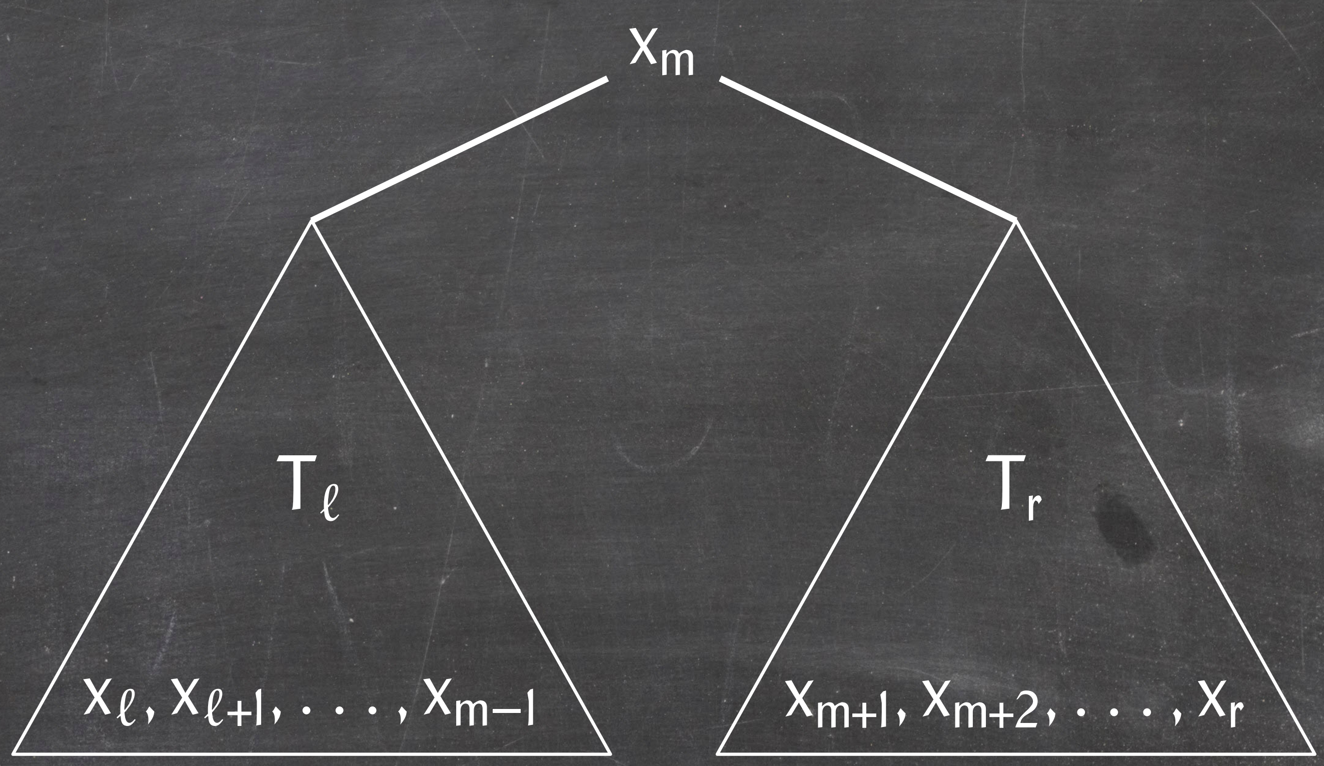

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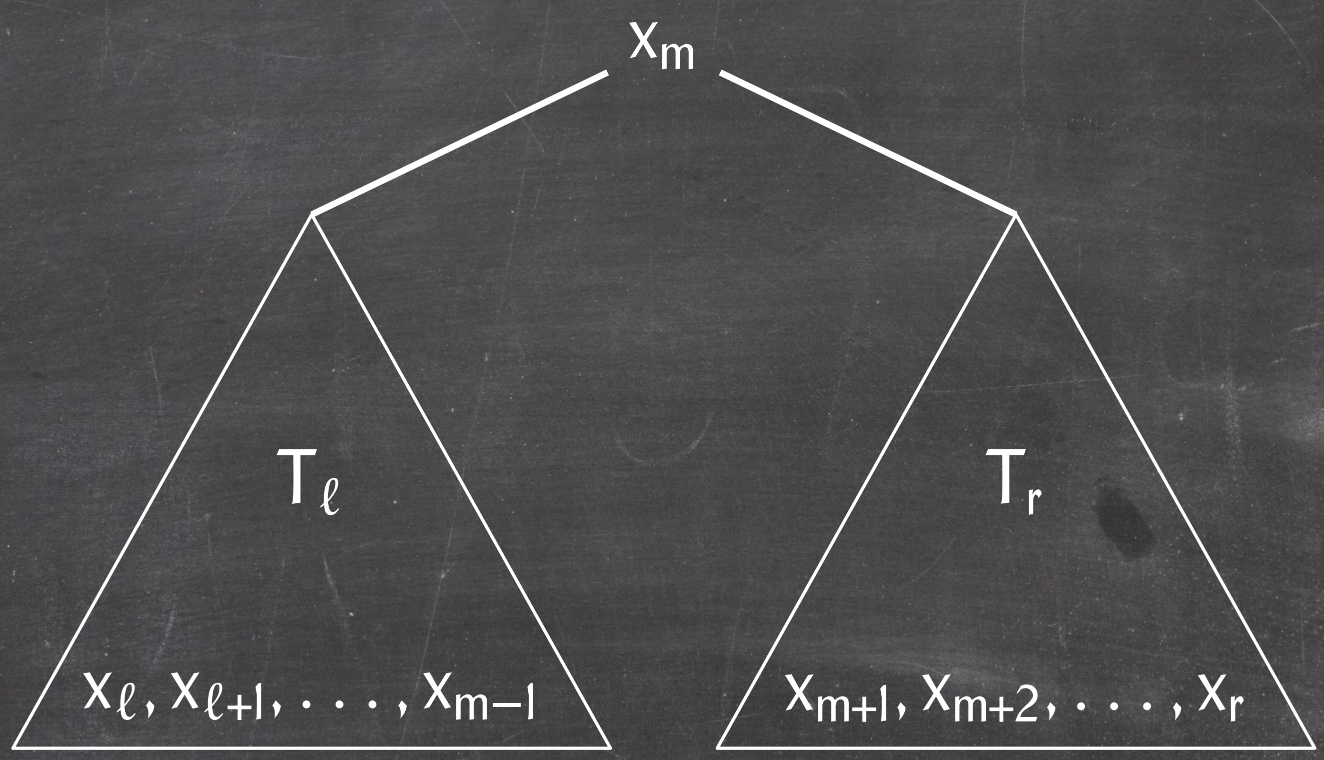

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We need to figure out which element to store at the root!

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Let $C(\ell, r)$ be the cost of an optimal binary search tree for $x_\ell, x_{\ell+1}, \ldots, x_r$.

We are interested in $C(1, n)$.

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Let $C(\ell, r)$ be the cost of an optimal binary search tree for $x_\ell, x_{\ell+1}, \ldots, x_r$.

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$$C(\ell, r) = \begin{cases} 0 & r < \ell \\ p_{\ell,r} + \min_{\ell \le m \le r}(C_{\ell,m-1} + C_{m+1,r}) & \text{otherwise} \end{cases}$$

Optimal Binary Search Trees: The Algorithm

OptimalBinarySearchTree(X, P)

```
for i = 1 to n
    do P′[i, i] = P[i]
        for j = i + 1 to n
            do P′[i, j] = P′[i, j − 1] + P[j]
for i = 1 to n + 1
    do C[i, i − 1] = 0
        T[i, i − 1] = ∅
for ℓ = 0 to n − 1
    do for i = 1 to n − ℓ
            do C[i, i + ℓ] = ∞
                for j = i to i + ℓ
                    do if C[i, i + ℓ] > C[i, j − 1] + C[j + 1, i + ℓ]
                            then C[i, i + ℓ] = C[i, j − 1] + C[j + 1, i + ℓ]
                                 T[i, i + ℓ] = new node storing X[j]
                                 T[i, i + ℓ].left = T[i, j − 1]
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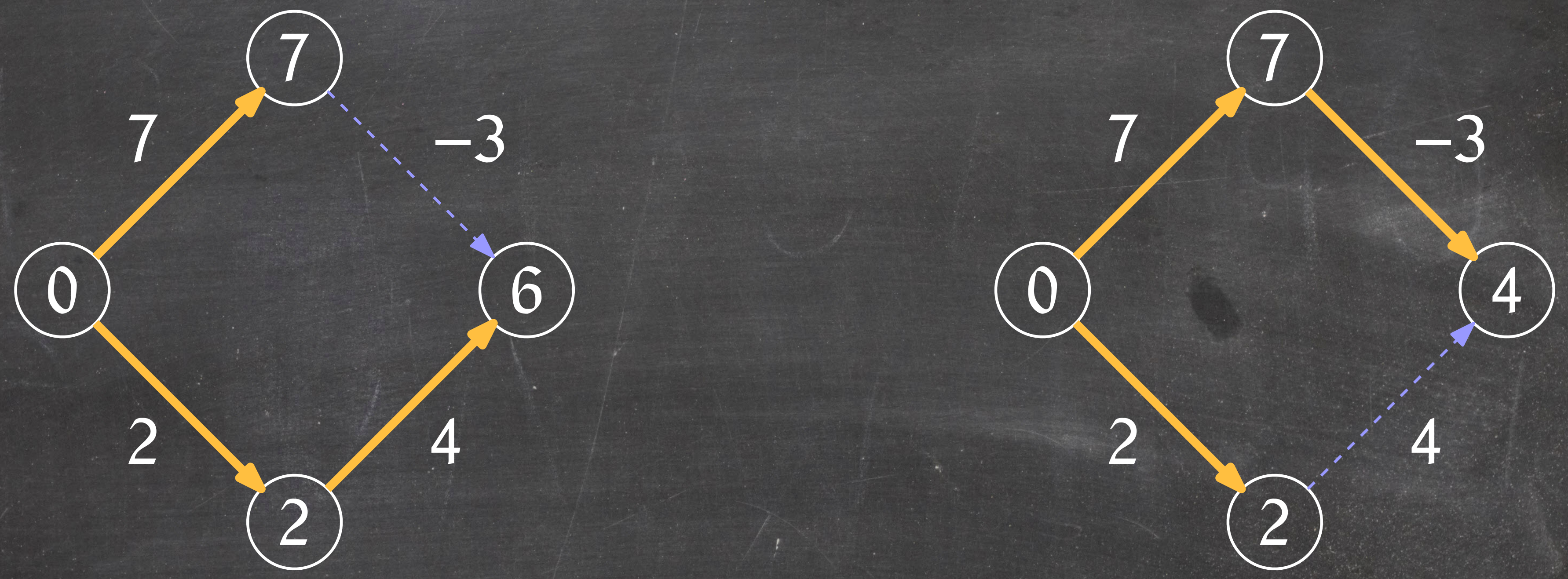
Lemma: An optimal binary search tree for n elements can be computed in $O(n^3)$ time.

Single-Source Shortest Paths

Dijkstra's algorithm may fail in the presence of negative-weight edges:



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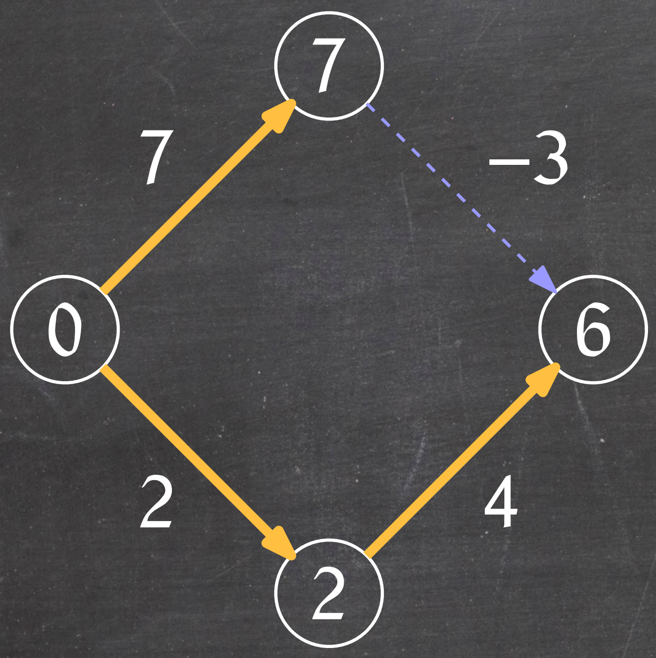

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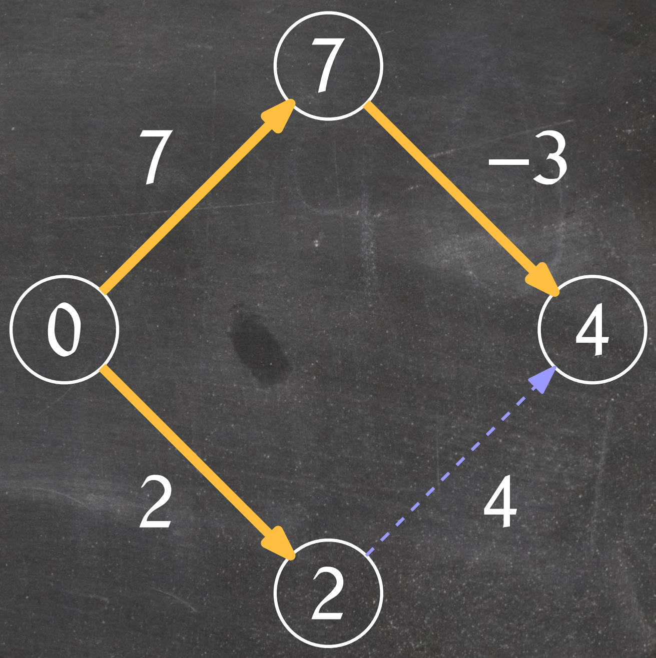
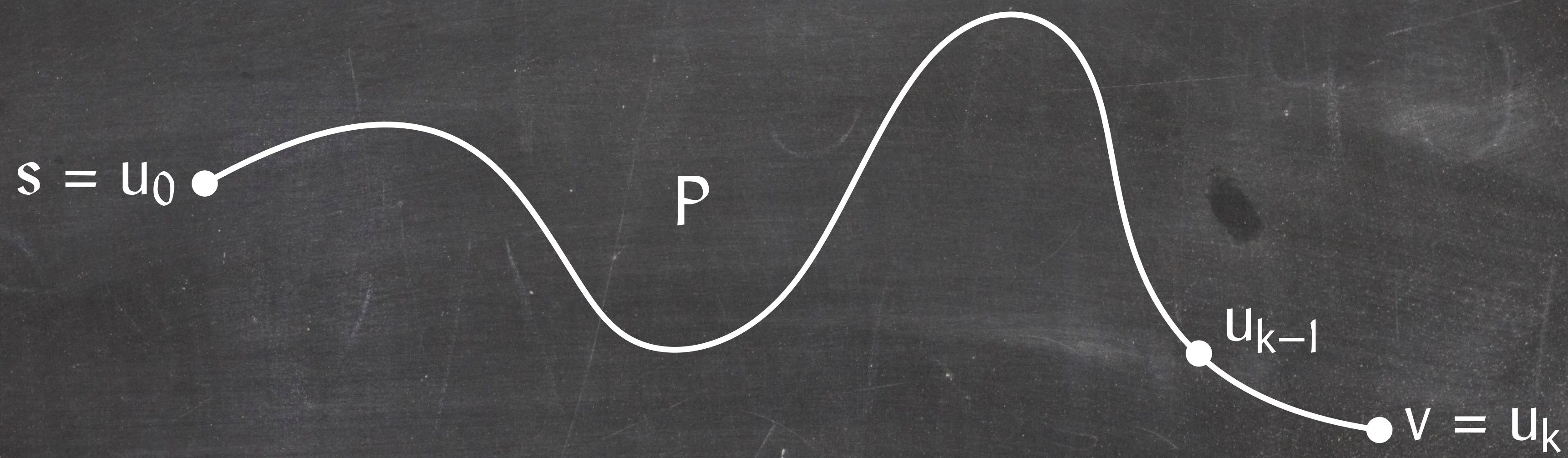


Correct

We need an algorithm that can deal with negative-length edges.

Single-Source Shortest Paths: Problem Analysis

Lemma: If $P = \langle u_0, v_1, \ldots, u_k \rangle$ is a shortest path from $u_0 = s$ to $u_k = v$, then $P' = (u_0, u_1, \ldots, u_{k-1})$ is a shortest path from u_0 to u_{k-1}.



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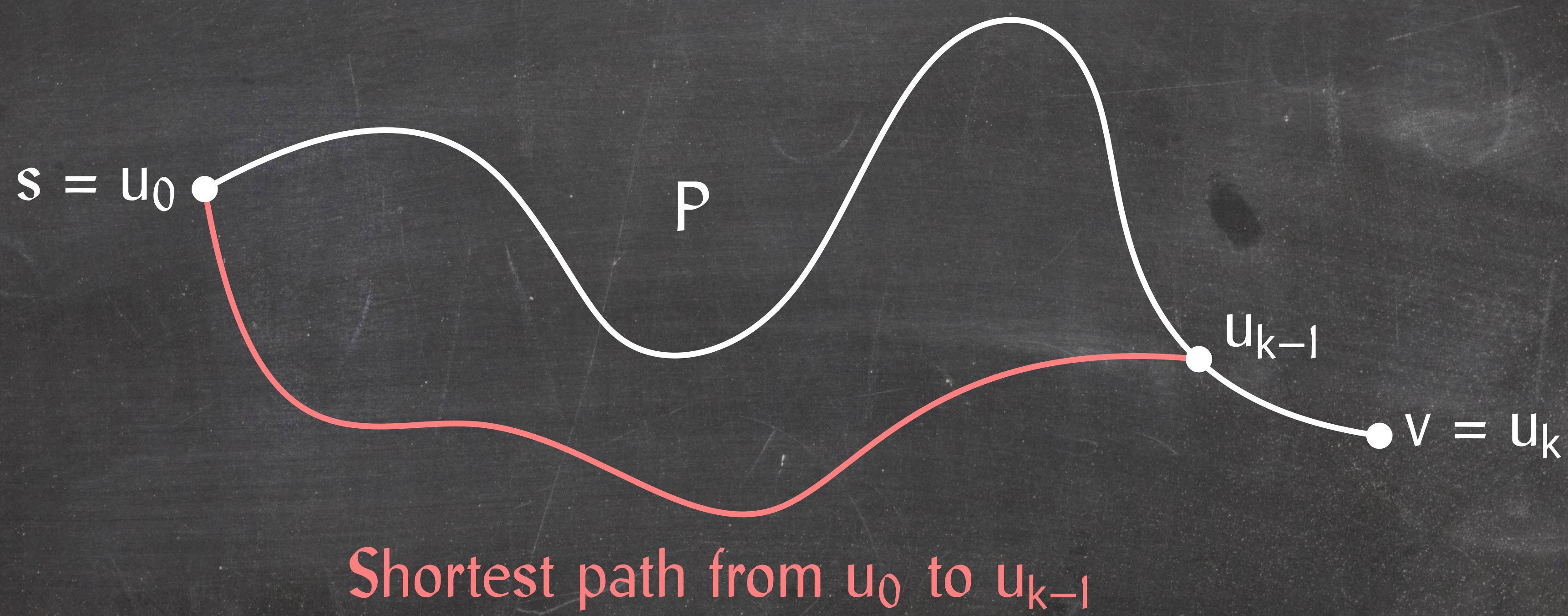

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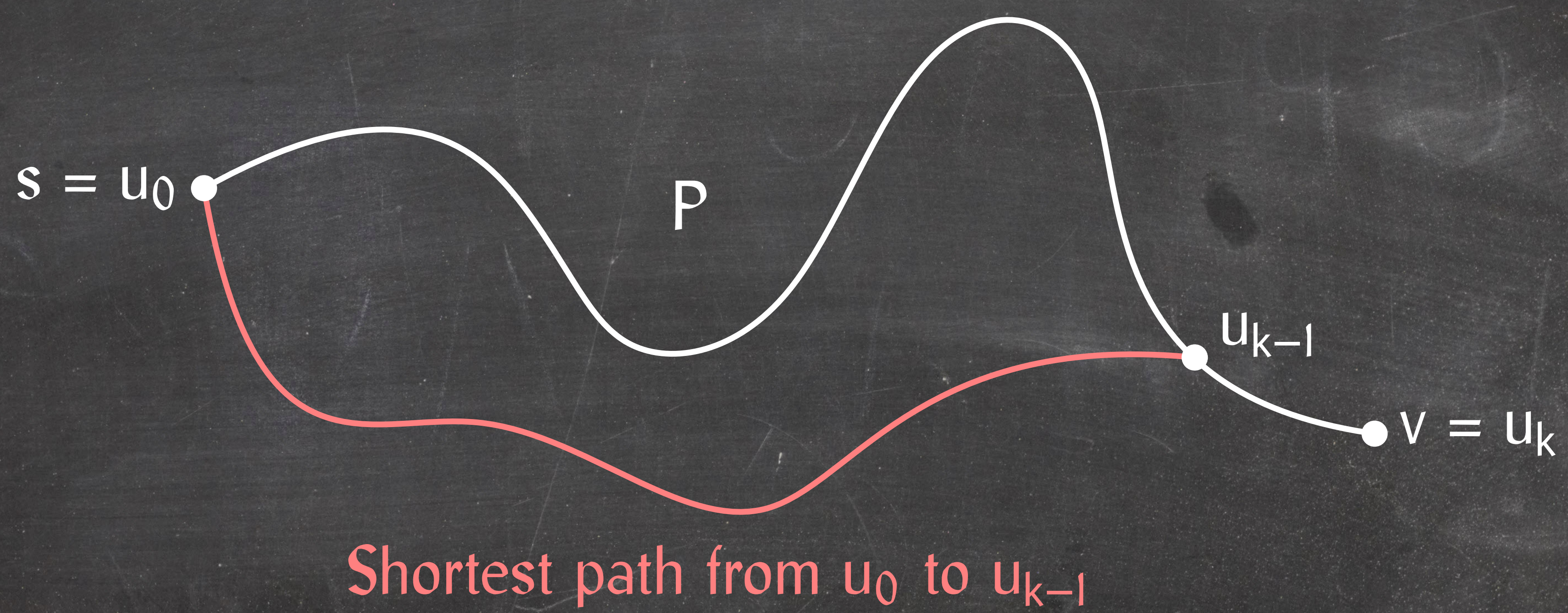

Observation: P' has one less edge than P.

Single-Source Shortest Paths: The Recurrence

Let $d_i(s, v)$ be the length of the shortest path $P_i(s, v)$ from s to v that has at most i edges.

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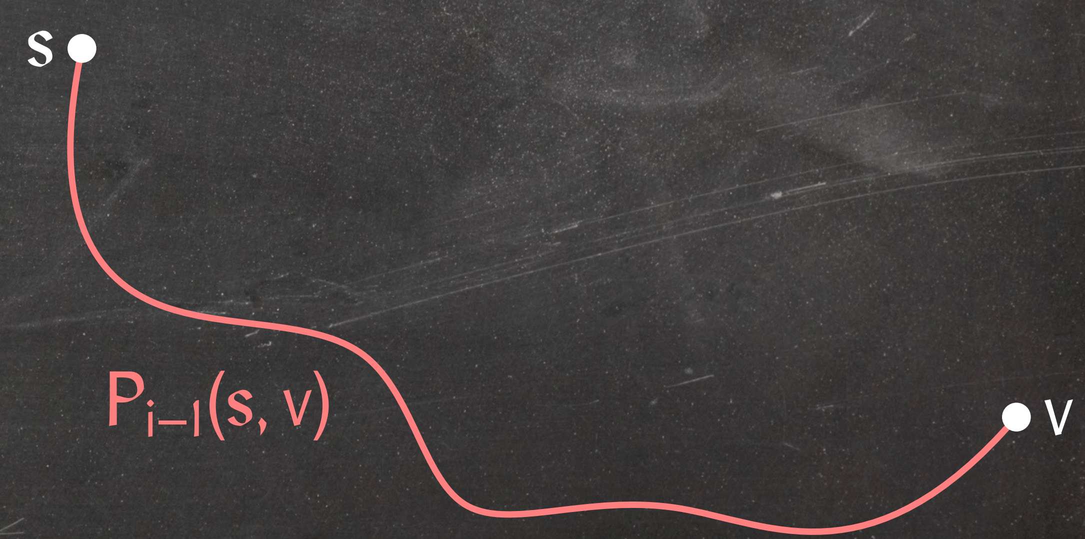

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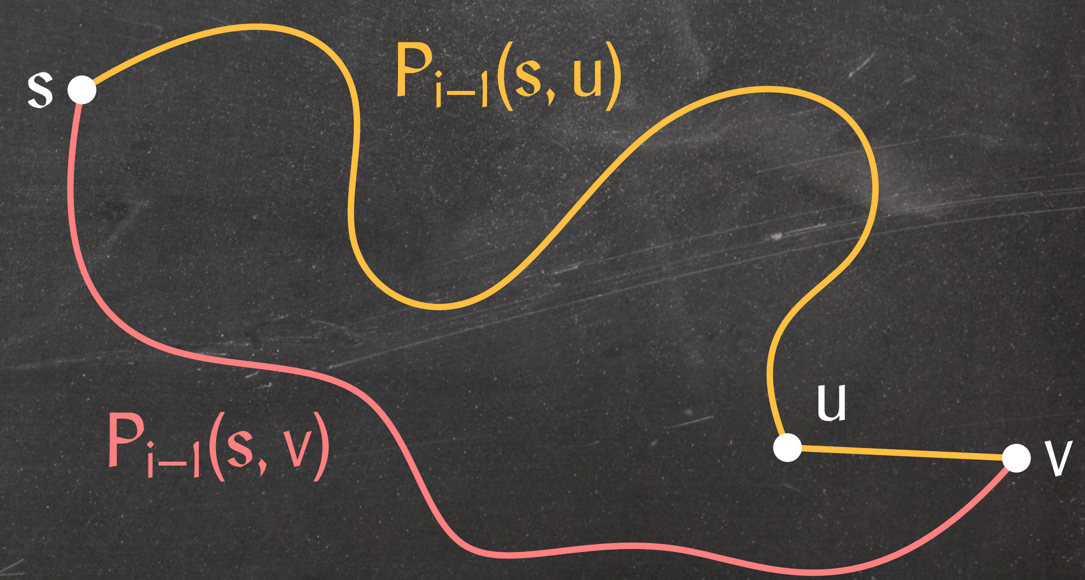

- $P_i(s, v)$ has at most $i - 1$ edges or

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- $P_i(s, v)$ has i edges.

$\Rightarrow$ $P_i(s, v) = P_{i-1}(s, u) \circ \langle(u, v)\rangle$ for some in-neighbour u of v.

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If $i > 0$, then

$$d_i(s, v) = \min(d_{i-1}(s, v), \min\{d_{i-1}(s, u) + w(u, v) \mid (u, v) \in E\})$$

Single-Source Shortest Paths: The Bellman-Ford Algorithm

BellmanFord(G, s)

```
for every vertex v ∈ G
   do d[v] = ∞
      P[v] = ∅
d[s] = 0
P[s] = [s]
for i = 1 to n − 1
   do for every vertex v ∈ G
         do for every in-edge e of v
               do if d[e.tail] + e.weight < d[v]
                     then d[v] = d[e.tail] + e.weight
                          P[v] = [v] ++ P[e.tail]
      return (d, P)
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Lemma: The single-source shortest paths problem can be solved in $O(nm)$ time on any weighted graph, provided there are no negative cycles.

All-Pairs Shortest Paths

Goal: Compute the distance $d(u, v)$ (and the corresponding shortest path), for every pair of vertices $u, v \in G$.

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Improved algorithms:

- Floyd-Warshall: $O(n^3)$
- Johnson: $O(n^2 \lg n + nm)$ (really cool!)
  - Run Bellman-Ford from an arbitrary vertex s in $O(nm)$ time.
  - Change edge weights so they are all non-negative but shortest paths don't change!
  - Run Dijkstra n times.

All-Pairs Shortest Paths: The Recurrence

Number the vertices $1, 2, \ldots, n$.

Let $d_i(u, v)$ be the length of the shortest path $P_i(u, v)$ that visits only vertices in $\{1, 2, \ldots, i\} \cup \{u, v\}$.

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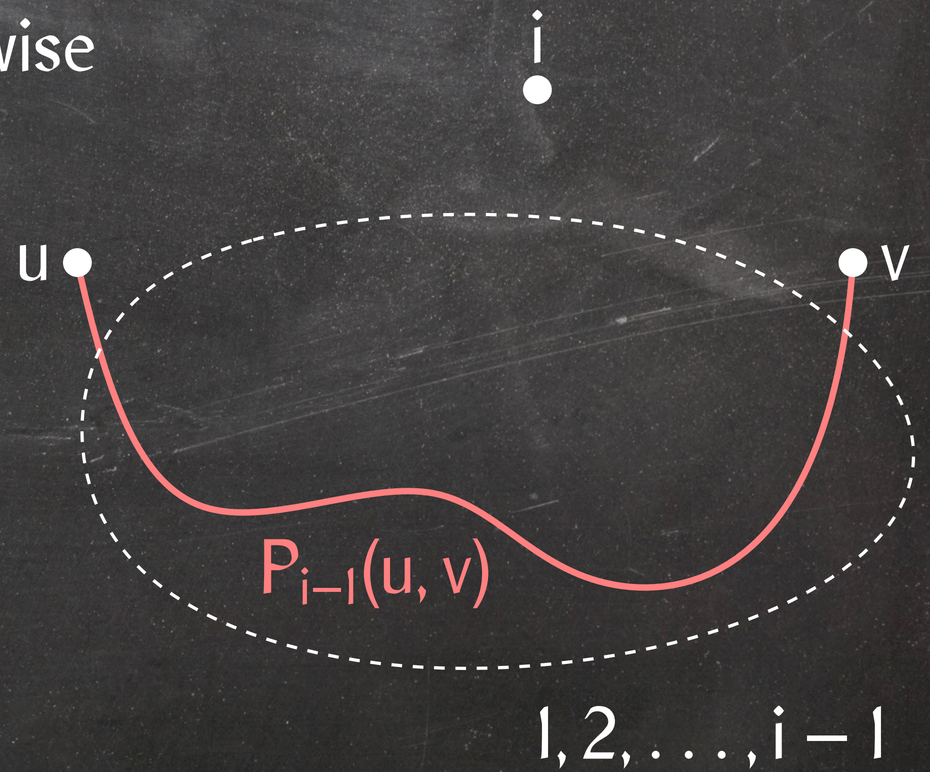

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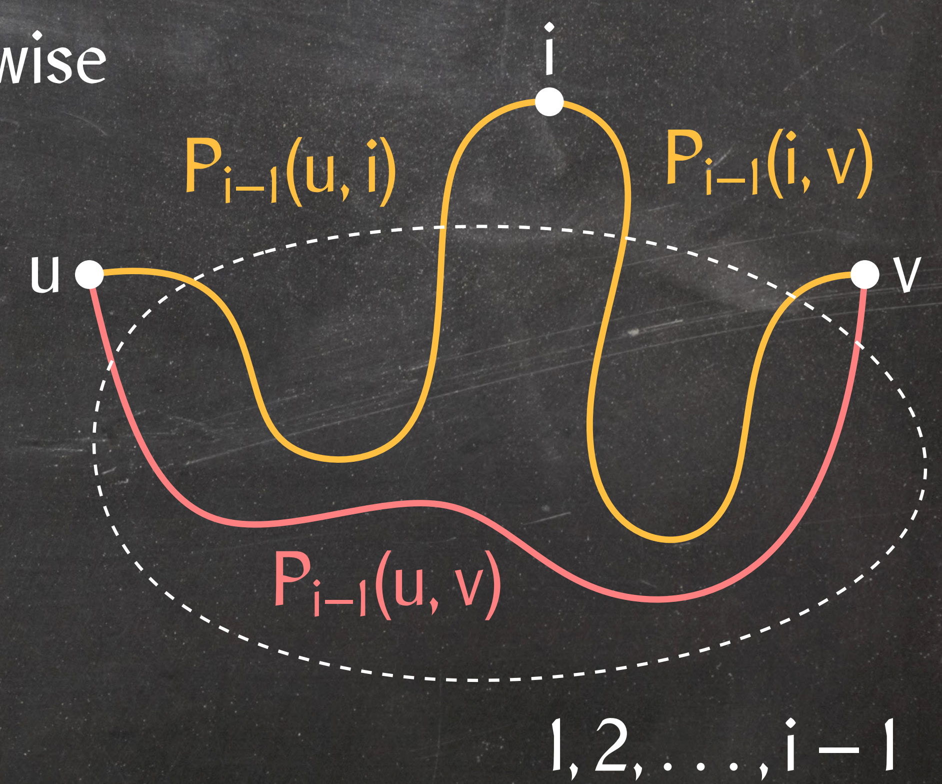

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All-Pairs Shortest Paths: The Floyd-Warshall Algorithm

FloydWarshall(G)

```
for every pair of vertices u, v ∈ G
    do d[u, v] = ∞
       p[u, v] = Nothing
for every vertex v ∈ G
    do d[v, v] = 0
       p[v, v] = v
for every edge e ∈ G
    do d[e.tail, e.head] = e.weight
       p[e.tail, e.head] = e.tail
for i = 1 to n
    do for every pair of vertices u, v ∈ G such that i ∉ {u, v}
           do if d[u, v] > d[u, i] + d[i, v]
                 then d[u, v] = d[u, i] + d[i, v]
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return (d, p)
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ReportPath(p, u, v)

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if p[u, v] = Nothing
    then return Nothing
P = [v]
while v ≠ u
    do v = p[u, v]
       P.prepend(v)
return P
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Dynamic programming exhaustively explores all possible choices and chooses the one that gives the best solution.

Dynamic programming yields a faster solution than the naïve recursive algorithm when there are lots of overlapping subproblems.

Summary

The design of a dynamic programming algorithm proceeds in two phases:

1. Analyze the structure of an optimal solution to develop a recurrence for the cost of an optimal solution.

2. Develop an algorithm that uses the recurrence to compute an optimal solution
   - Recursively using memoization or
   - Iteratively by populating a table with the costs of the solutions to all possible subproblems.

   Both types of algorithms compute optimal solutions bottom-up.