Divide and Conquer

Textbook Reading

Chapters 4, 7 & 33.4

Overview

Design principle

- Divide and conquer

Proof technique

- Induction, induction, induction

Analysis technique

- Recurrence relations

Problems

- Sorting (Merge Sort and Quick Sort)
- Selection
- Matrix multiplication
- Finding the two closest points

Merge Sort

MergeSort(A, ℓ, r)

```
if r ≤ ℓ
    then return
m = ⌊(ℓ + r)/2⌋
MergeSort(A, ℓ, m)
MergeSort(A, m + 1, r)
Merge(A, ℓ, m, r)
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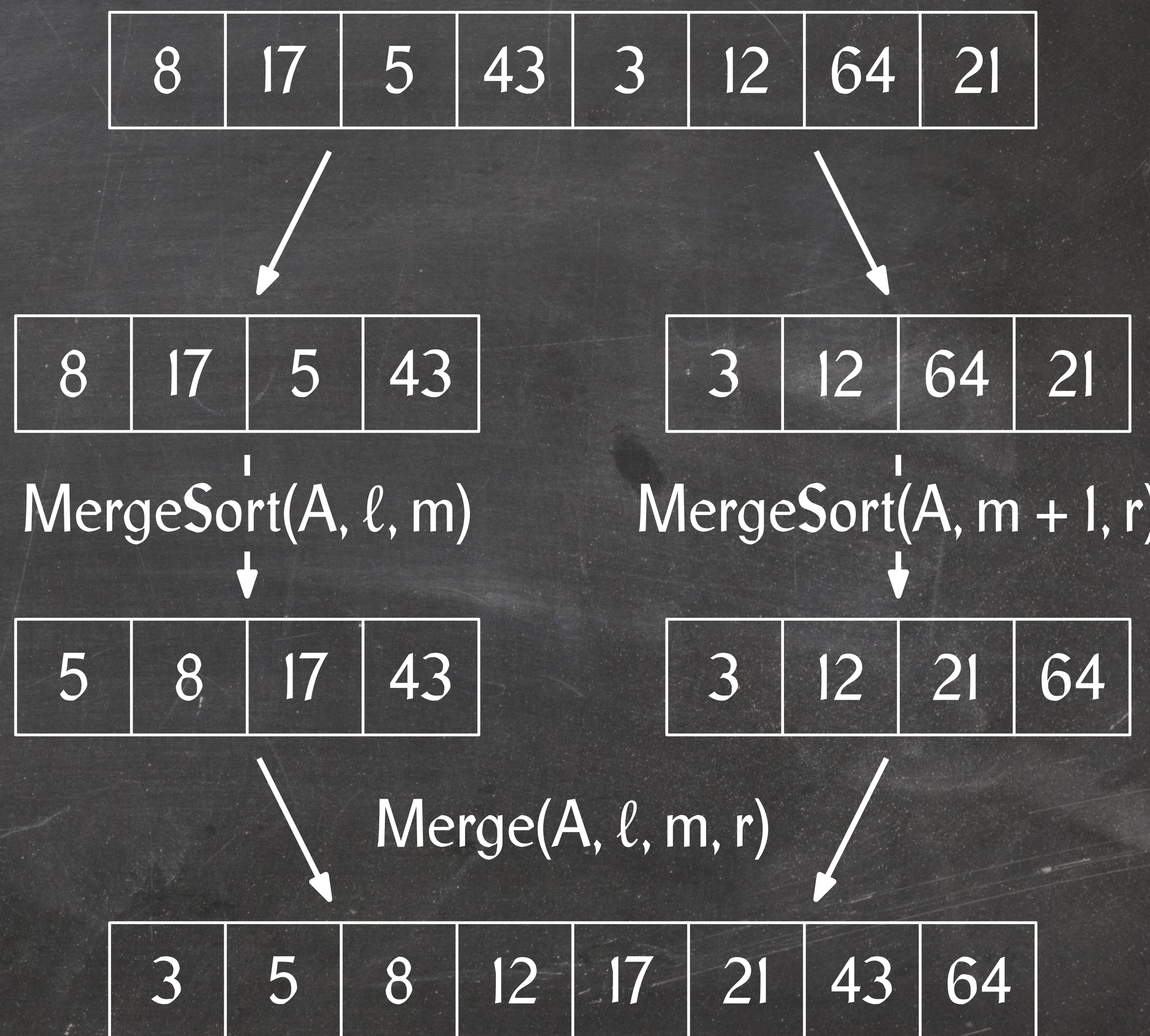

Merging Two Sorted Lists

Merge(A, ℓ, m, r)

```
n1 = m − ℓ + 1
n2 = r − m
for i = 1 to n1
    do L[i] = A[l + i − 1]
for i = 1 to n2
    do R[i] = A[m + i]
L[n1 + 1] = R[n2 + 1] = +∞
i = j = 1
for k = ℓ to r
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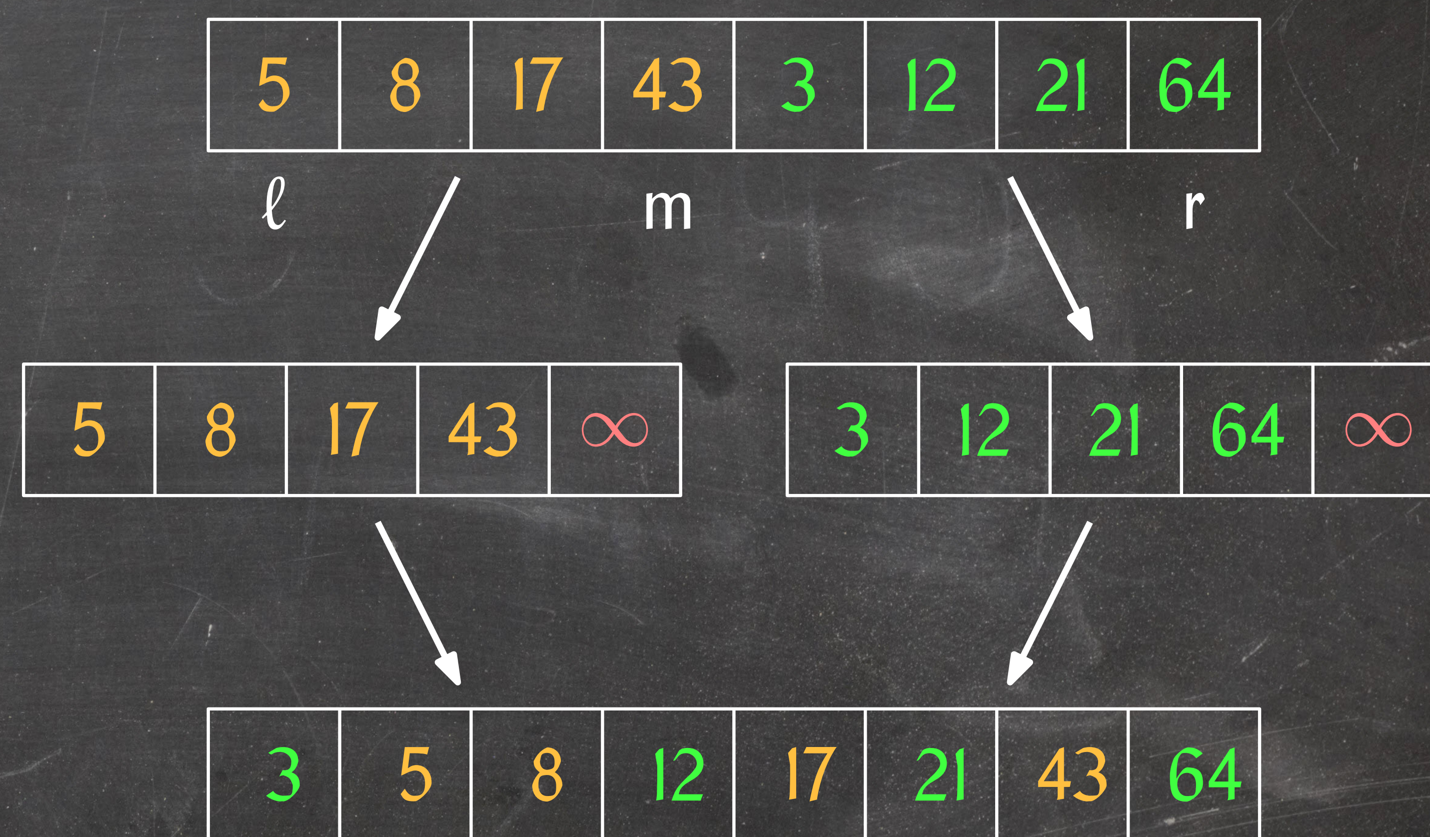
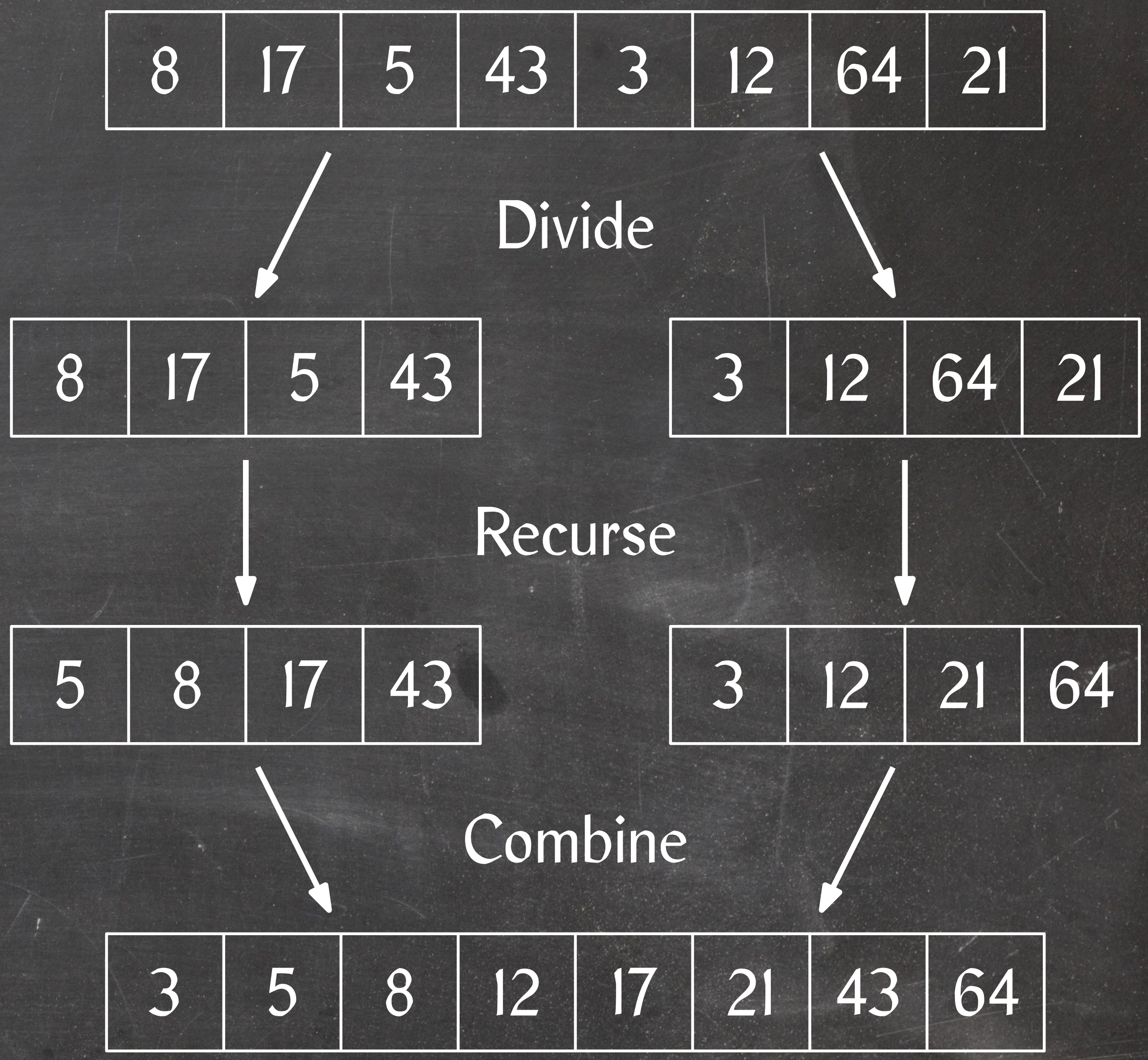


Divide and Conquer

Three steps:

- Divide the input into smaller parts.
- Recursively solve the same problem on these smaller parts.
- Combine the solutions computed by the recursive calls to obtain the final solution.



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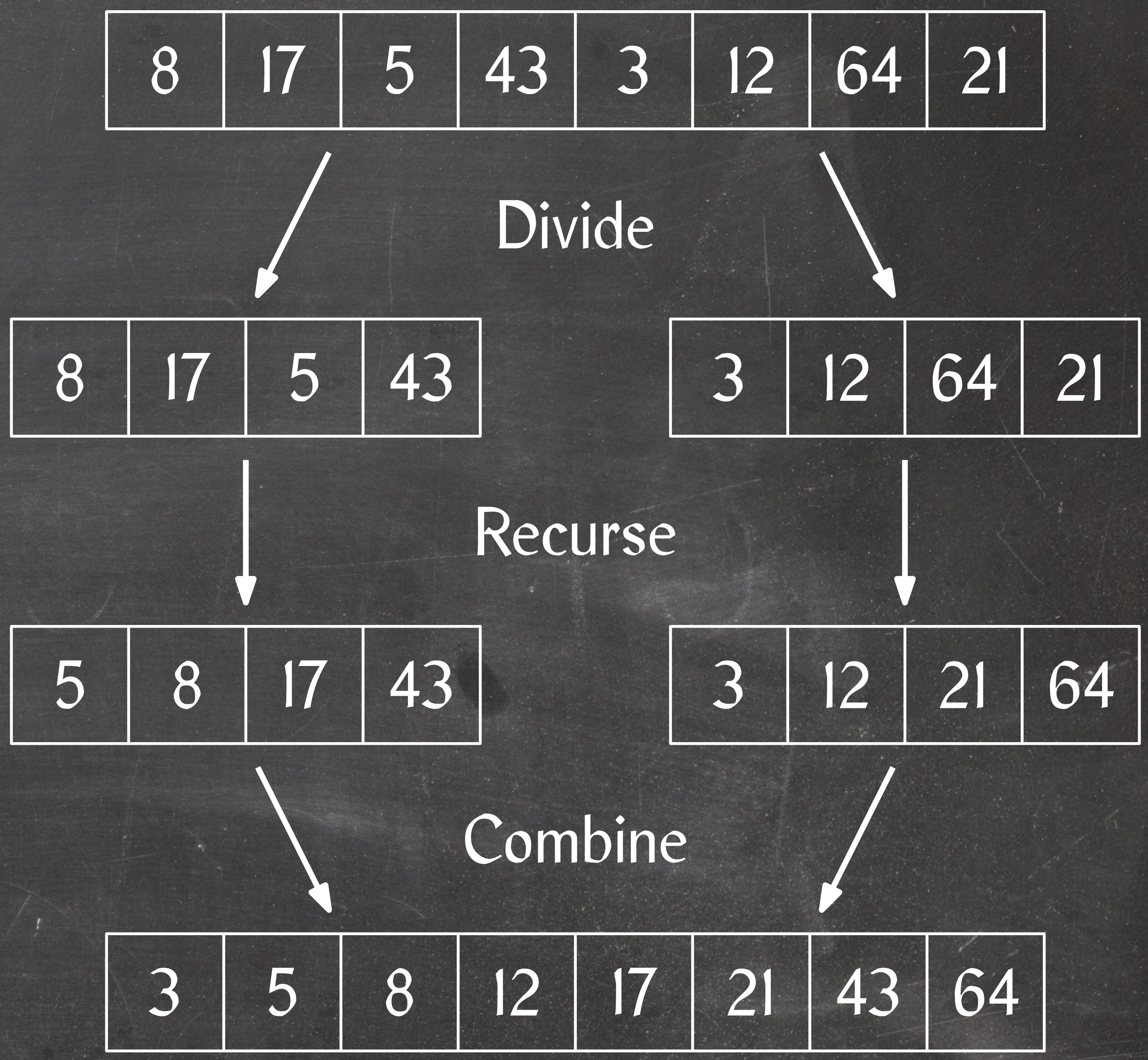

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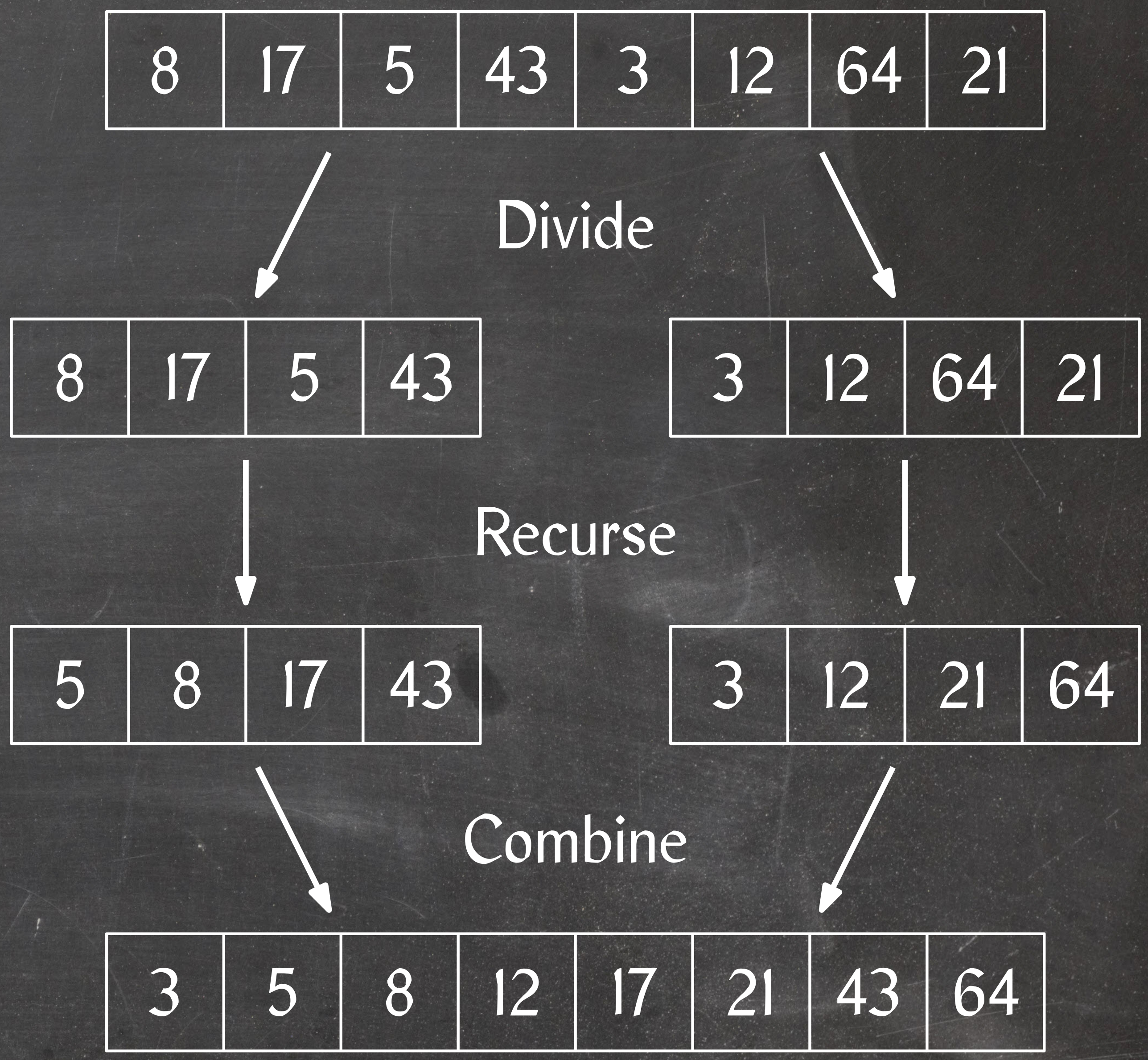

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Example: Once we unfold the recursion of Merge Sort, we're left with nothing but Merge steps. Thus, we reduce sorting to the simpler problem of merging sorted lists.

Loop Invariants

… are a technique for proving the correctness of an iterative algorithm.

The invariant states conditions that should hold before and after each iteration.

Correctness proof using a loop invariant:

Initialization: Prove the invariant holds before the first iteration.

Maintenance: Prove each iteration maintains the invariant.

Termination: Prove that the correctness of the invariant after the last iteration implies correctness of the algorithm.

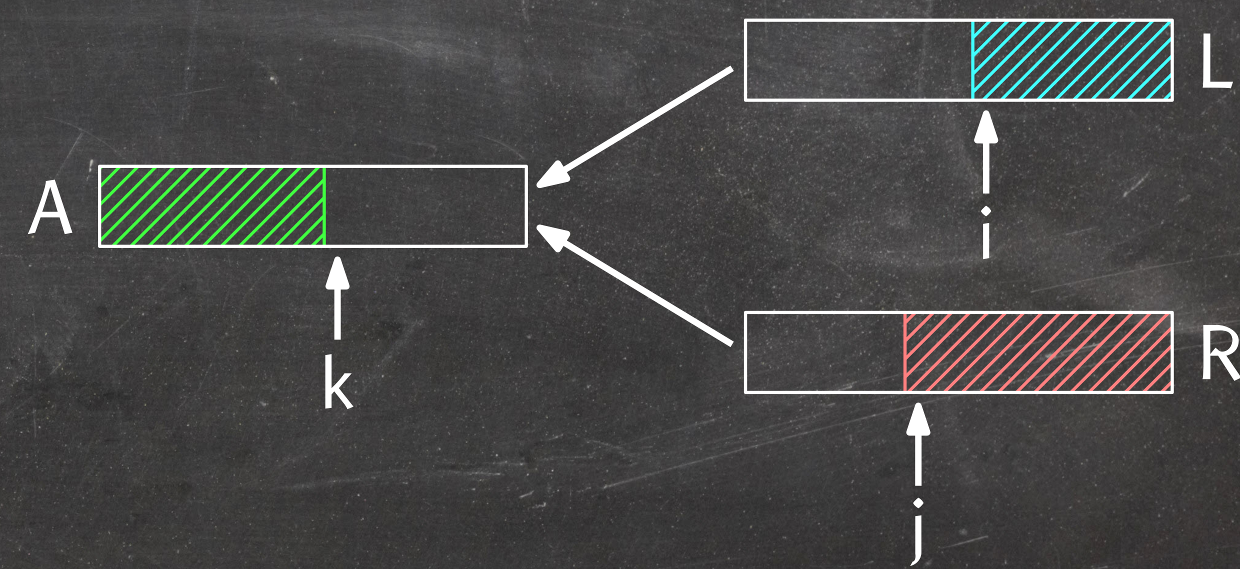
Correctness of Merge

Merge(A, ℓ, m, r)

```
n₁ = m − ℓ + 1
n₂ = r − m
for i = 1 to n₁
    do L[i] = A[l + i − 1]
for i = 1 to n₂
    do R[i] = A[m + i]
L[n₁ + 1] = R[n₂ + 1] = +∞
i = j = 1
for k = ℓ to r
    do if L[i] ≤ R[j]
          then A[k] = L[i]
               i = i + 1
          else A[k] = R[j]
               j = j + 1
```

Loop invariant:

- $A[\ell\,..\,k-1] \cup L[i\,..\,n_1] \cup R[j\,..\,n_2]$ is the set of elements originally in $A[\ell\,..\,r]$.
- $A[\ell\,..\,k-1]$, $L[i\,..\,n_1]$, and $R[j\,..\,n_2]$ are sorted.
- $x \leq y$ for all $x \in A[\ell\,..\,k-1]$ and $y \in L[i\,..\,n_1] \cup R[j\,..\,n_2]$.



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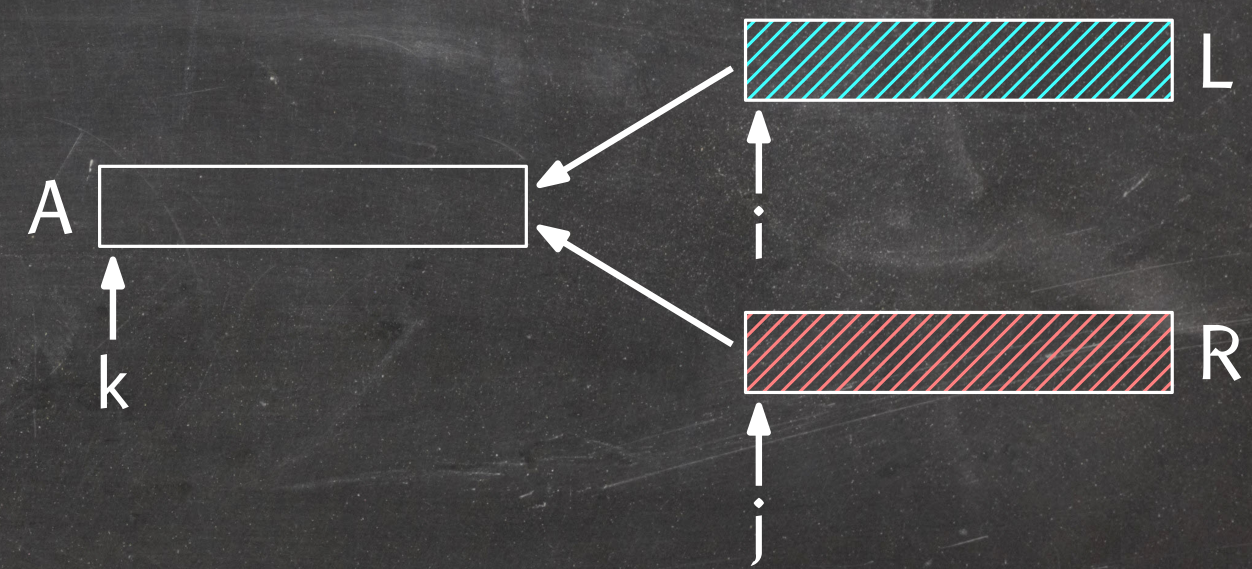
Initialization:

- $A[\ell \,..\, m]$ is copied to $L[1 \,..\, n_1]$.
- $A[m+1 \,..\, r]$ is copied to $R[1 \,..\, n_2]$.
- $i = 1$, $j = 1$, $k = 1$.



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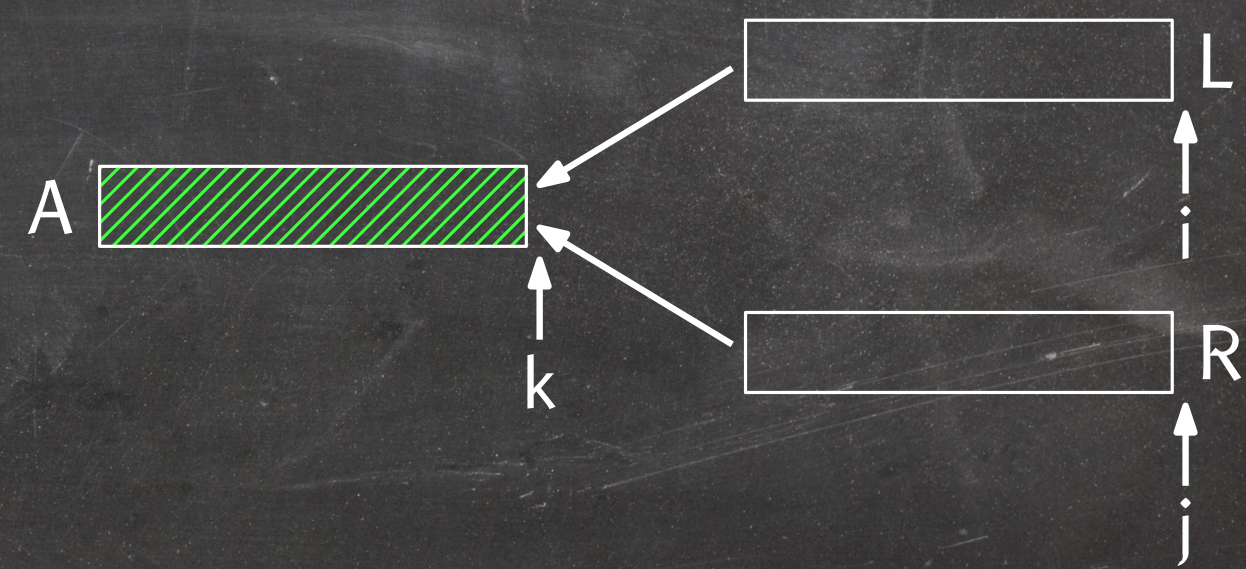
Termination:

- $k = r + 1$

$\Rightarrow$ $A[\ell \mathinner{..} r]$ contains all items it contained initially, in sorted order.



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Maintenance:

- $A[k'] \le L[i]$ for all $k' < k$
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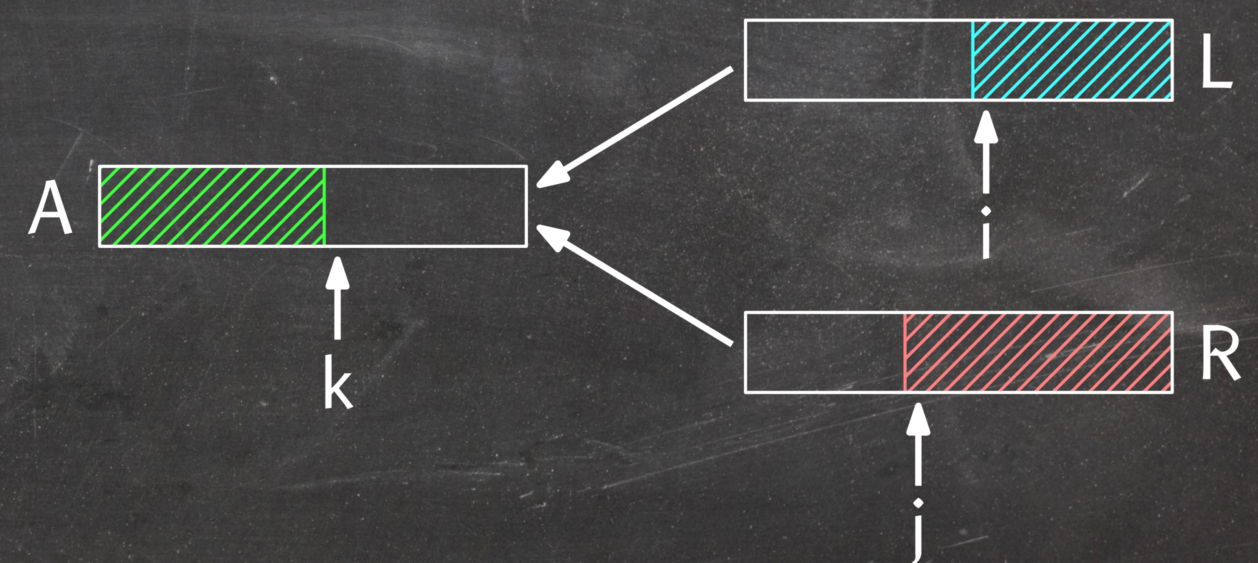

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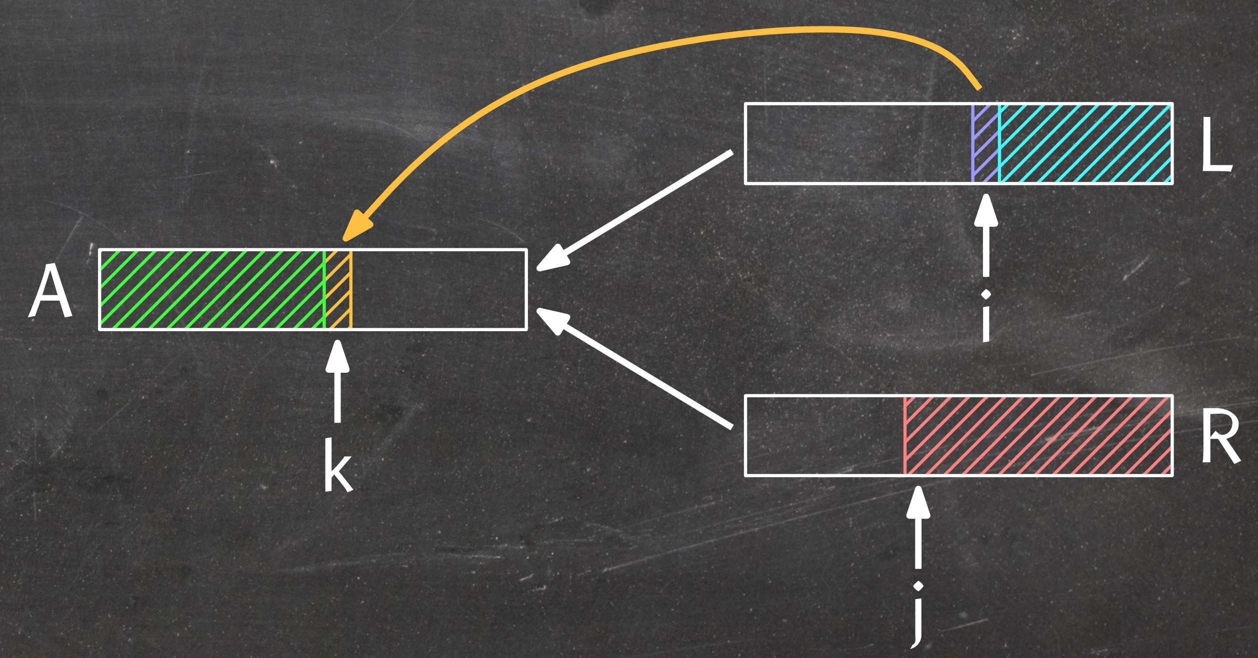

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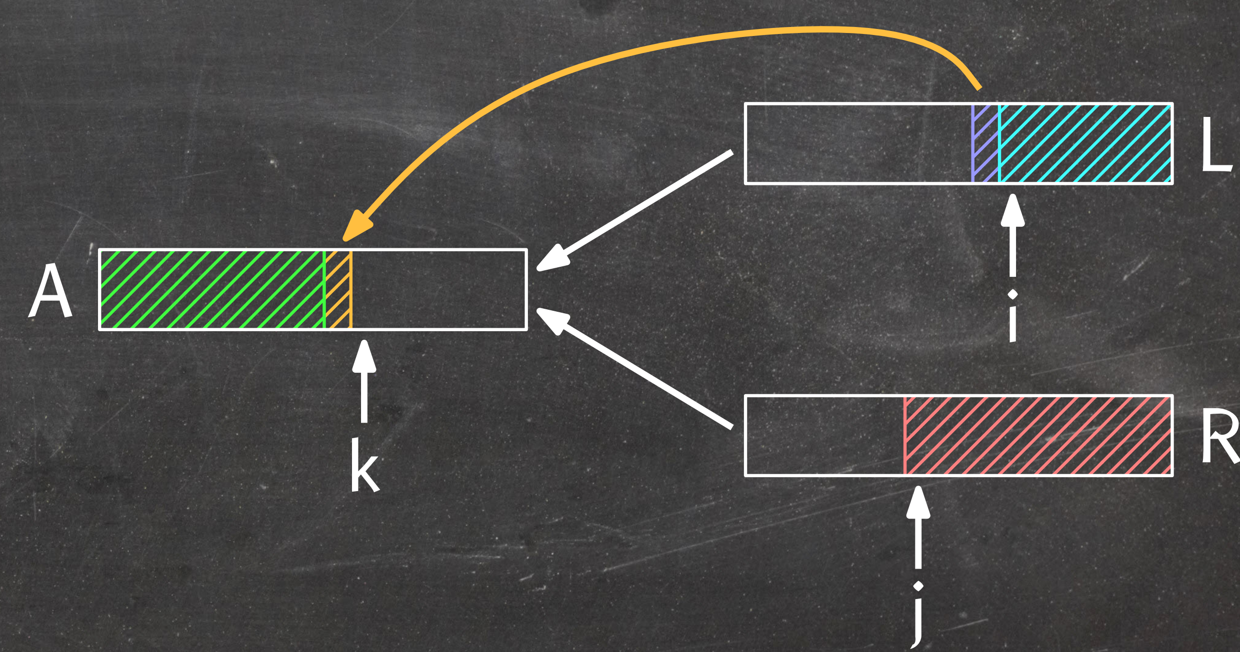

Correctness of Merge Sort

Lemma: Merge Sort correctly sorts any input array.

MergeSort(A, ℓ, r)

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if r ≤ ℓ
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m = ⌊(ℓ + r)/2⌋
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Inductive step: $(n > 1)$

- The left and right halves have size less than n each.
- By the inductive hypothesis, the recursive calls sort them correctly.
- Merge correctly merges the two sorted sequences.

MergeSort(A, ℓ, r)

if $r \leq \ell$
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$m = \lfloor (\ell + r)/2 \rfloor$
MergeSort(A, ℓ, m)
MergeSort($A, m + 1, r$)
Merge(A, ℓ, m, r)

Correctness of Divide and Conquer Algorithms

Divide and conquer algorithms are the algorithmic incarnation of induction:

Base case: Solve trivial instances directly, without recursing.

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$\Rightarrow$ Induction is the natural proof method for divide and conquer algorithms.

Recurrence Relations

A recurrence relation defines the value $f(n)$ of a function $f(\cdot)$ for argument n in terms of the values of $f(\cdot)$ for arguments smaller than n.

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Examples:

Fibonacci numbers:

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Binomial coefficients

$$B(n, k) = \begin{cases} 1 & k = 1 \text{ or } k = n \\ B(n-1, k-1) + B(n-1, k) & \text{otherwise} \end{cases}$$

A Recurrence for Merge Sort

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Recurrence:

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A Recurrence for Binary Search

BinarySearch(A, ℓ, r, x)

```
if r < ℓ
    then return "no"
m = ⌊(ℓ + r)/2⌋
if x = A[m]
    then return "yes"
if x < A[m]
    then return BinarySearch(A, ℓ, m − 1, x)
    else  return BinarySearch(A, m + 1, r, x)
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Simplified Recurrence Notation

The recurrences we use to analyze algorithms all have a base case of the form

$$T(n) \leq c \ \forall n \leq n_0.$$

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So we are lazy and write

- Merge Sort: $T(n) = 2T(n/2) + \Theta(n)$
- Binary search: $T(n) = T(n/2) + \Theta(1)$

"Solving" Recurrences

Given two algorithms A and B with running times

$$T_A(n) = 2T(n/2) + \Theta(n)$$
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$\Rightarrow$ We want a closed-form expression for $T(n)$, that is, one of the form $T(n) \in \Theta(n)$, $T(n) \in \Theta(n^2)$, ..., one that does not depend on $T(n')$ for any $n' < n$.

Methods for Solving Recurrences

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  (Intuition, experience, black magic, recursion trees, trial and error)
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Recursion trees:

- Draw a tree that visualizes how the recurrence unfolds.
- Sum up the costs of the nodes in the tree to
  - Obtain an exact answer if the analysis is done rigorously enough or
  - Obtain a guess that can then be verified rigorously using substitution.

Methods for Solving Recurrences

Substitution:

- Guess the solution.
  (Intuition, experience, black magic, recursion trees, trial and error)
- Use induction to prove that the guess is correct.

Recursion trees:

- Draw a tree that visualizes how the recurrence unfolds.
- Sum up the costs of the nodes in the tree to
  - Obtain an exact answer if the analysis is done rigorously enough or
  - Obtain a guess that can then be verified rigorously using substitution.

Master Theorem:

- Cook book recipe for solving common recurrences.
- Immediately tells us the solution after we verify some simple conditions to determine which case of the theorem applies.

Substitution: Merge Sort

Lemma: The running time of Merge Sort is in $O(n \lg n)$.

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Recurrence:

$$T(n) = 2T(n/2) + O(n), \text{ that is,}$$

$$T(n) \le 2T(n/2) + an, \text{ for some } a > 0 \text{ and all } n \ge n_0.$$

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$$T(n) \leq cn \lg n, \text{ for some } c > 0 \text{ and all } n \geq n_1.$$

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Base case:

For $2 \leq n < 4$, $T(n) \leq c' \leq c'n \leq c'n \lg n$, for some $c' > 0$.

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Inductive step: $(n \geq 4)$

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$$
\begin{aligned}
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&\leq 2 \cdot \left(\frac{cn}{2} \lg \frac{n}{2}\right) + an
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Notes:

- We only proved the upper bound. The lower bound, $T(n) \in \Omega(n \lg n)$ can be proven analogously.

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Notes:

- We only proved the upper bound. The lower bound, $T(n) \in \Omega(n \lg n)$ can be proven analogously.
- Since the base case is valid only for $n \geq 2$ and we use the inductive hypothesis for $n/2$ in the inductive step, the inductive step is valid only for $n \geq 4$. Hence, a base case for $2 \leq n < 4$.

Substitution: Binary Search

Lemma: The running time of binary search is in $O(\lg n)$.

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Guess:

$$T(n) \le c \lg n, \text{ for some } c > 0 \text{ and all } n \ge n_1.$$

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Substitution and Asymptotic Notation

Why did we expand the Merge Sort recurrence

$$T(n) = 2T(n/2) + O(n) \quad \text{to} \quad T(n) \leq 2T(n/2) + an$$

and the claim

$$T(n) \in O(n \lg n) \quad \text{to} \quad T(n) \leq cn \lg n?$$

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If we're not careful, we may "prove" nonsensical results:

Recurrence: $T(n) = T(n-1) + n$

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Note that $T(n) \in \Theta(n^2)$!

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$T(n) = T(n-1) + n \le c(n-1) + n = cn + (n - c) > cn!$

A Recursion Tree for Merge Sort

Recurrence: $T(n) = 2T(n/2) + \Theta(n) \Rightarrow T(n) = 2T(n/2) + an$

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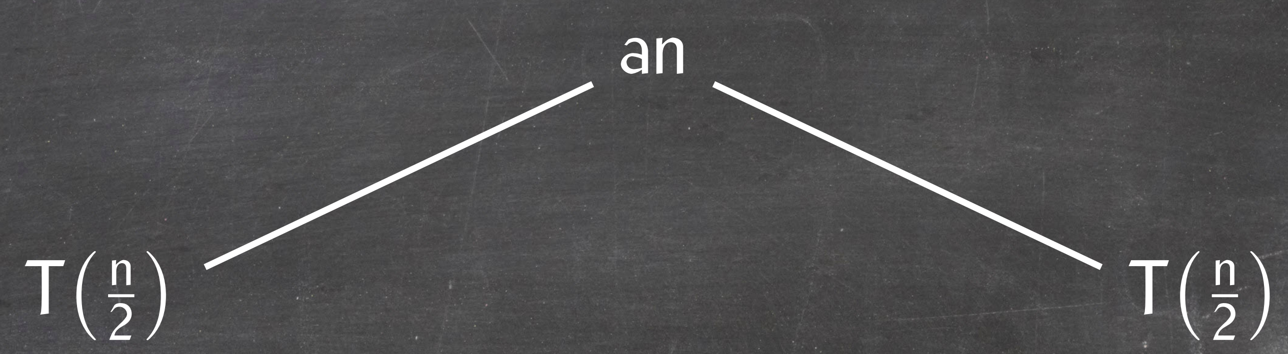

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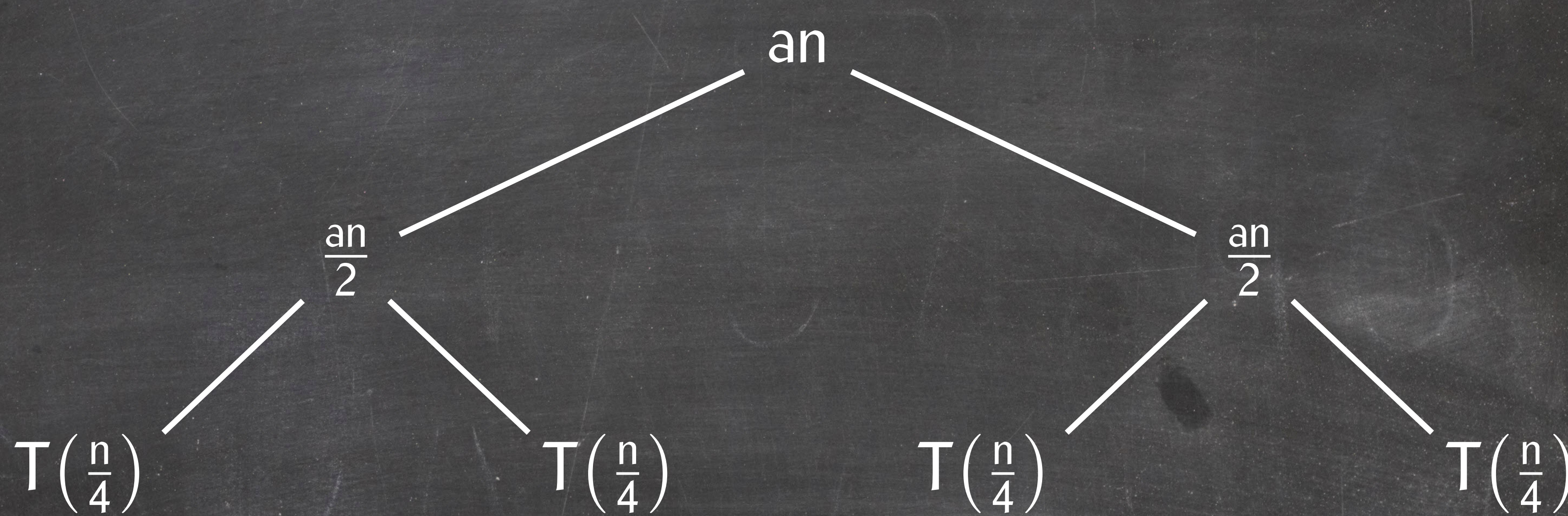

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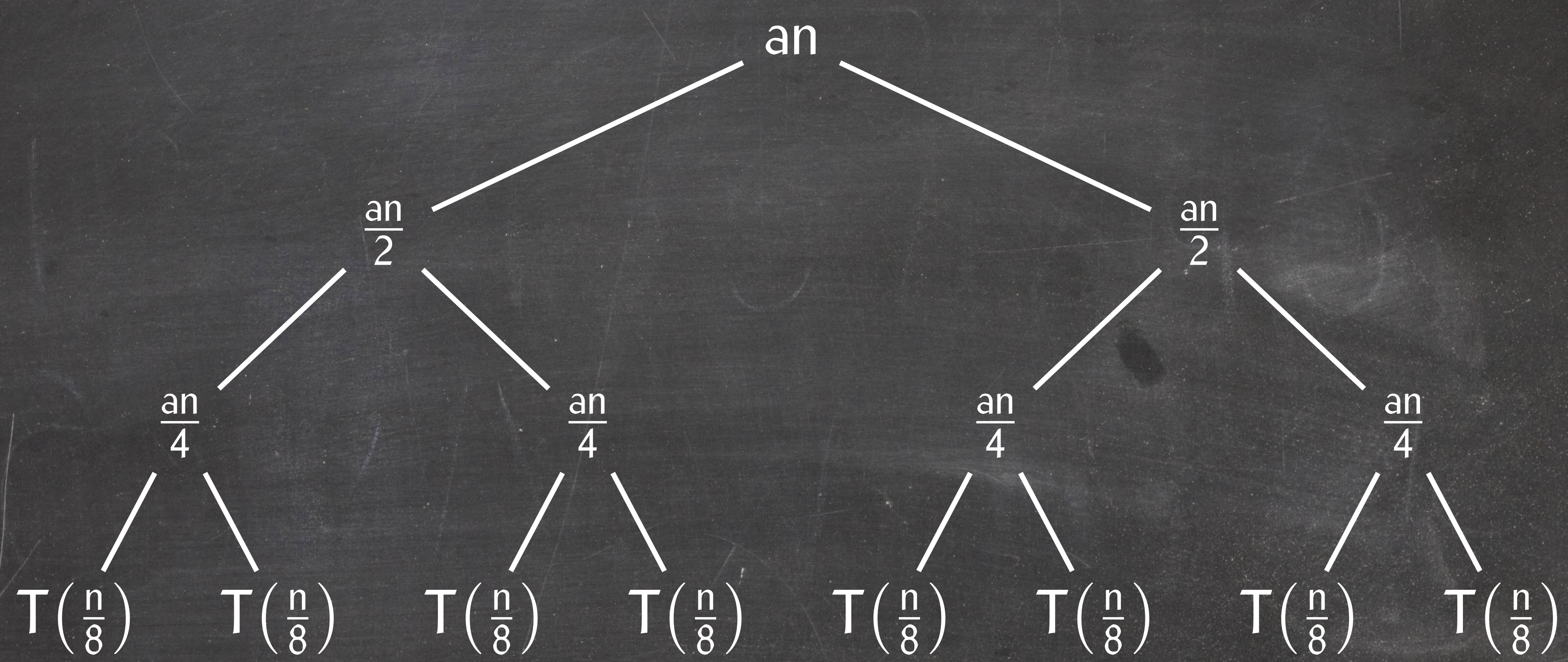

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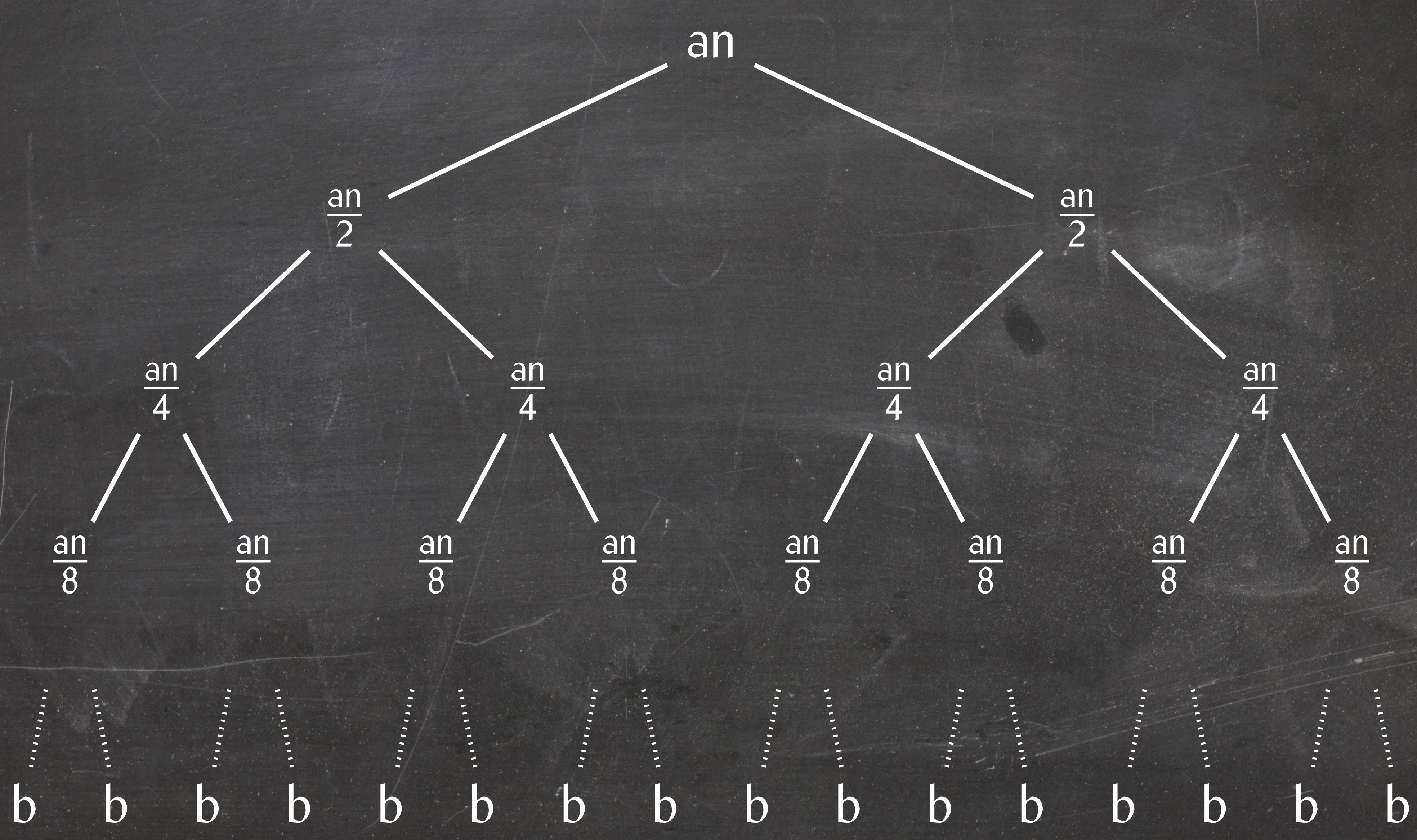

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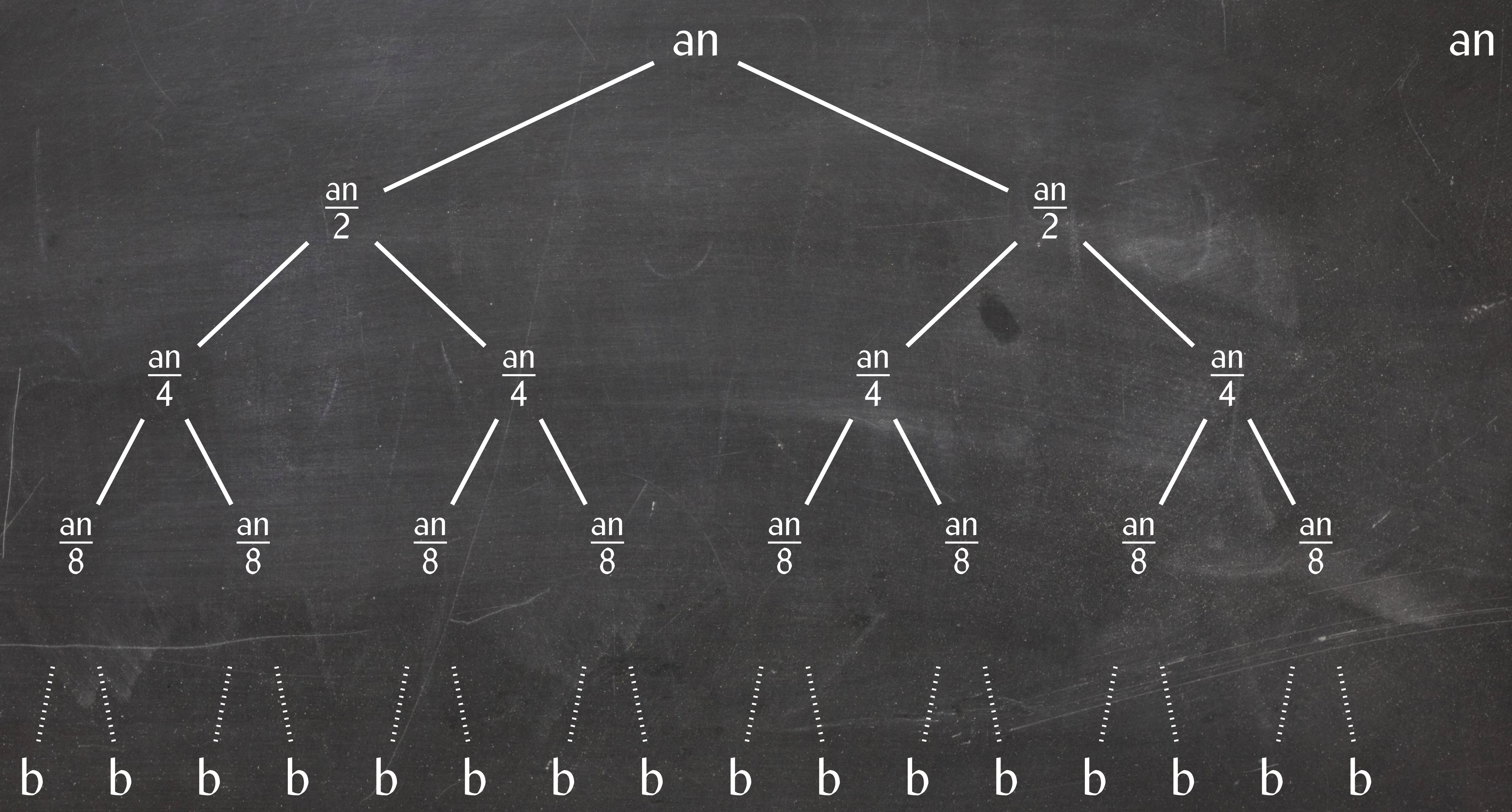

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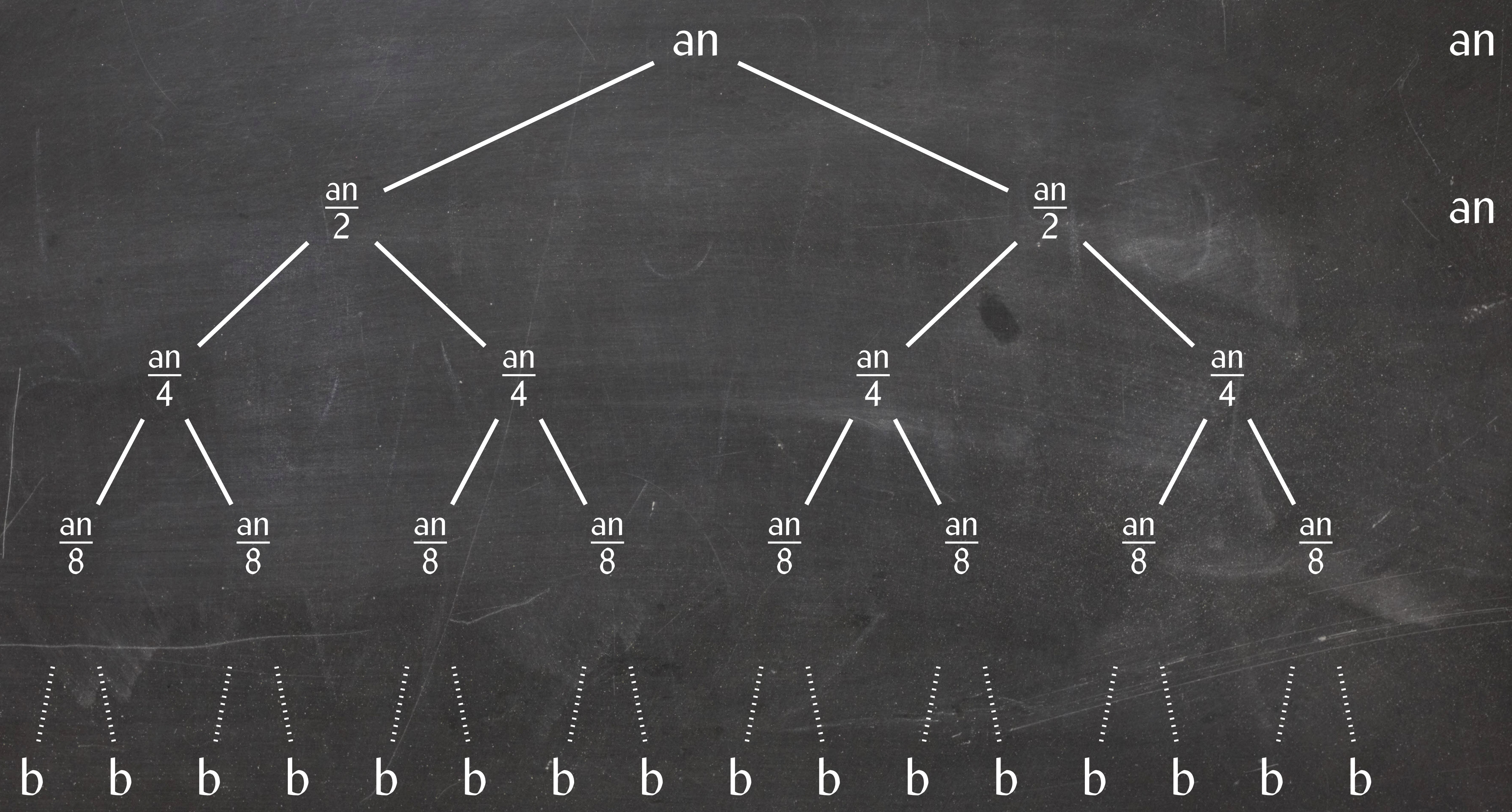

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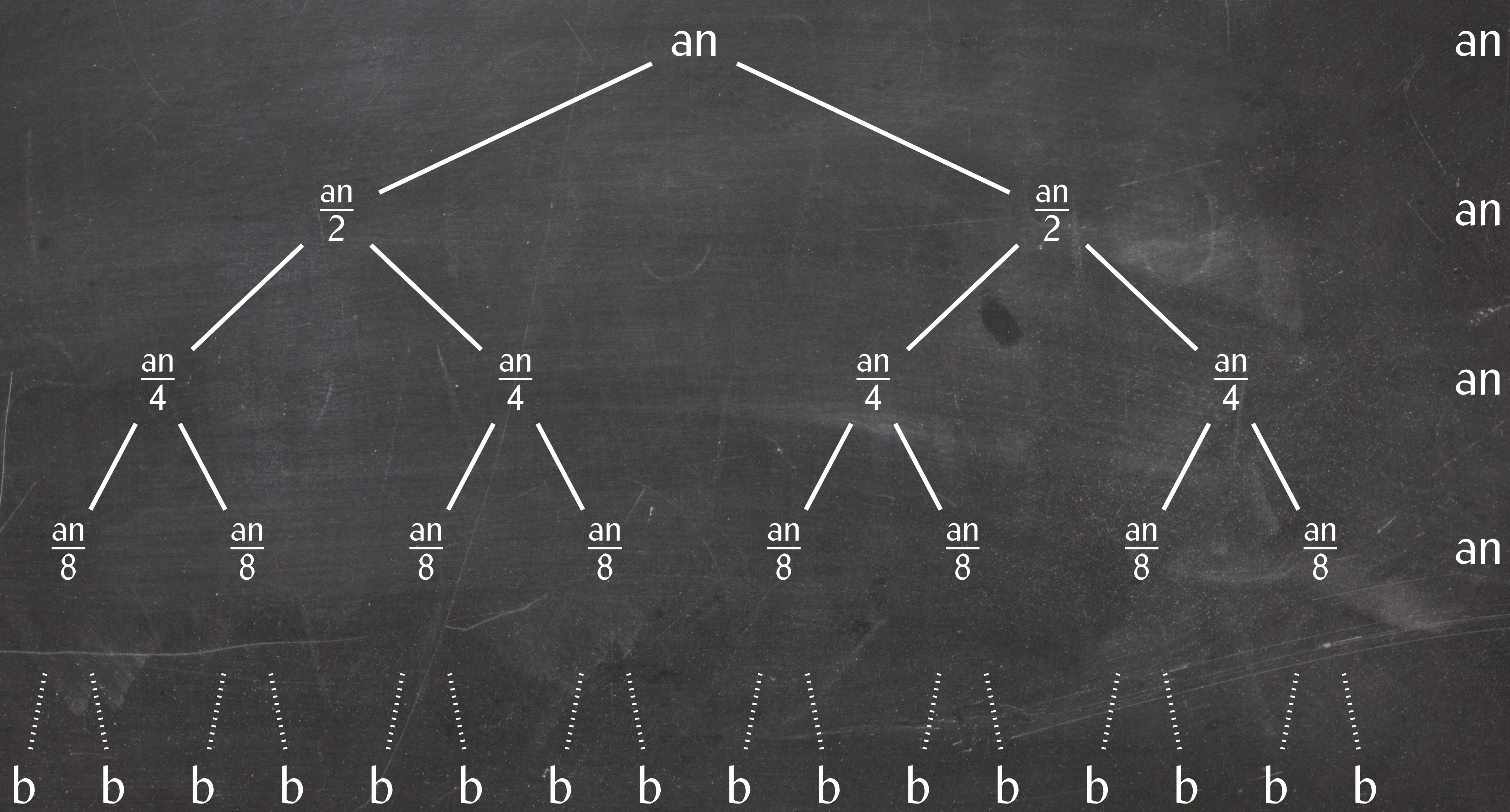

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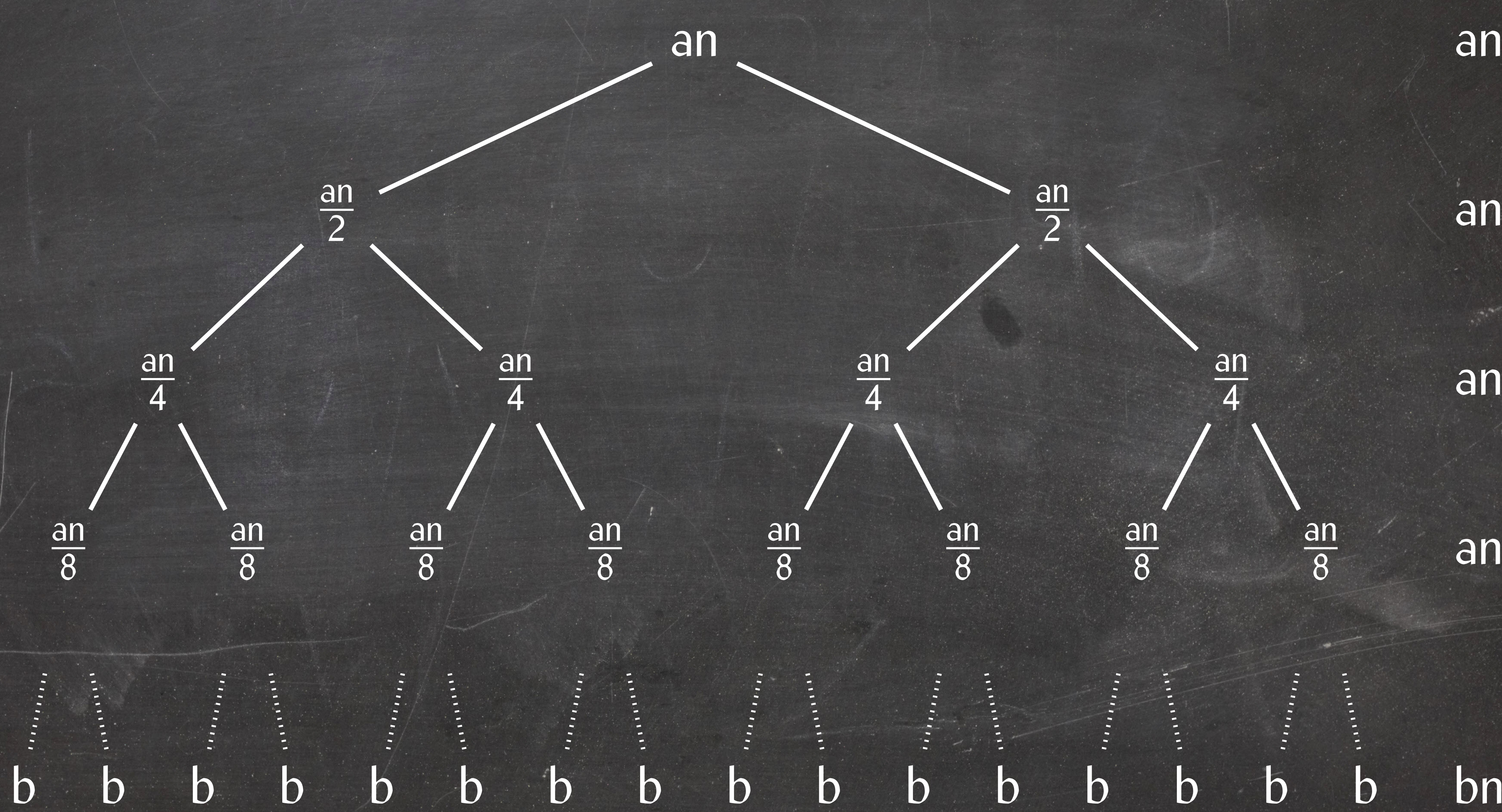

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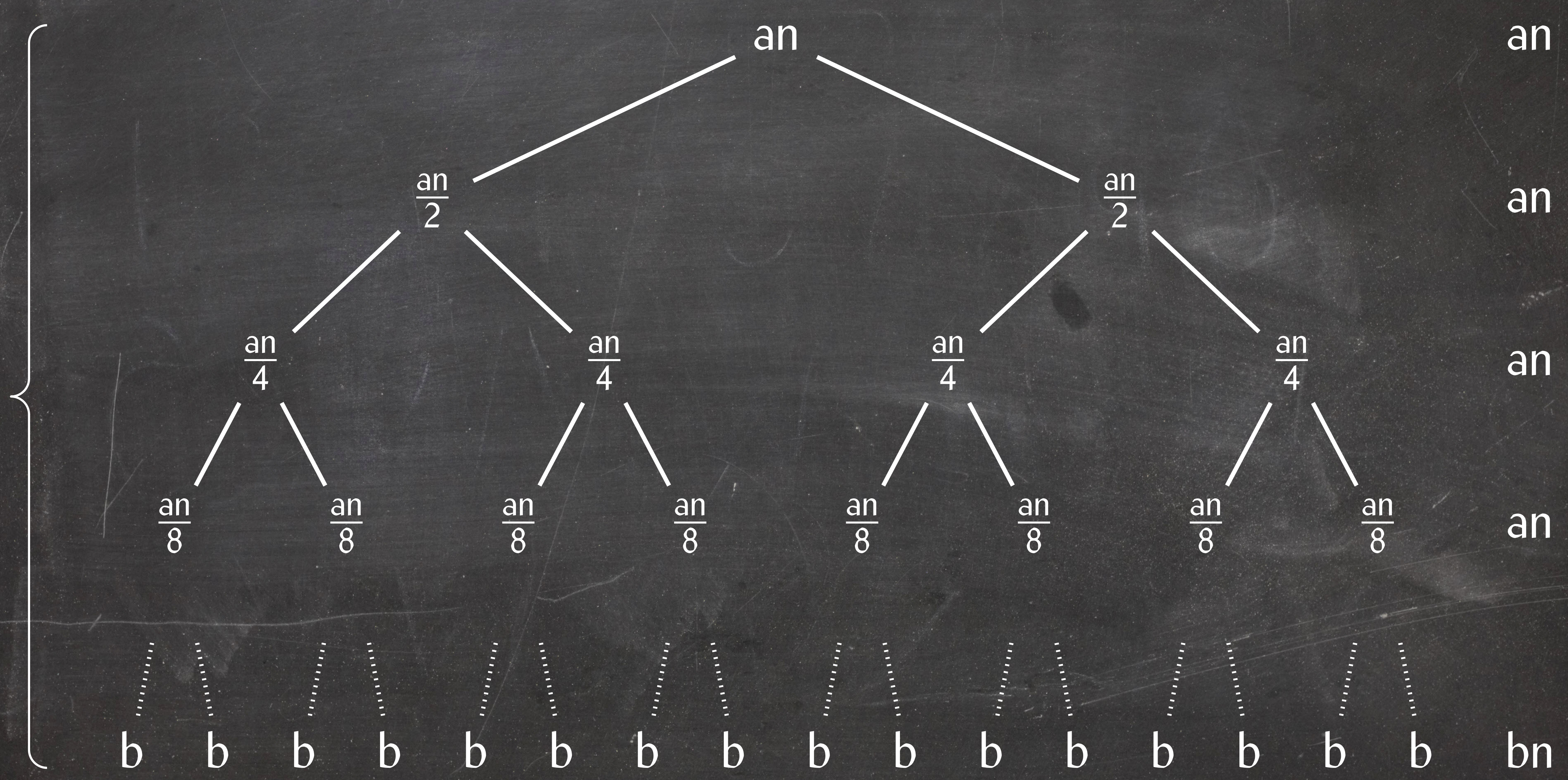

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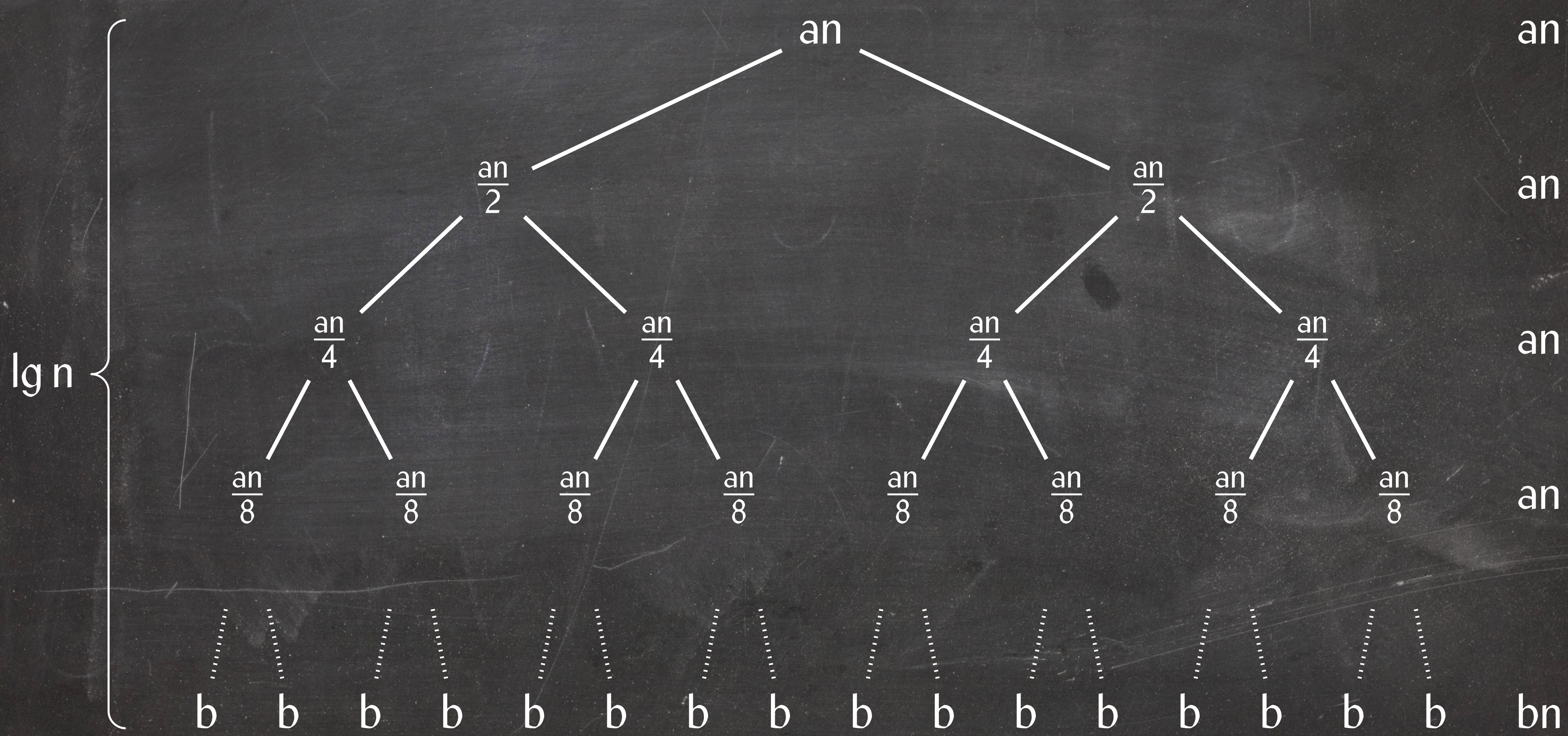

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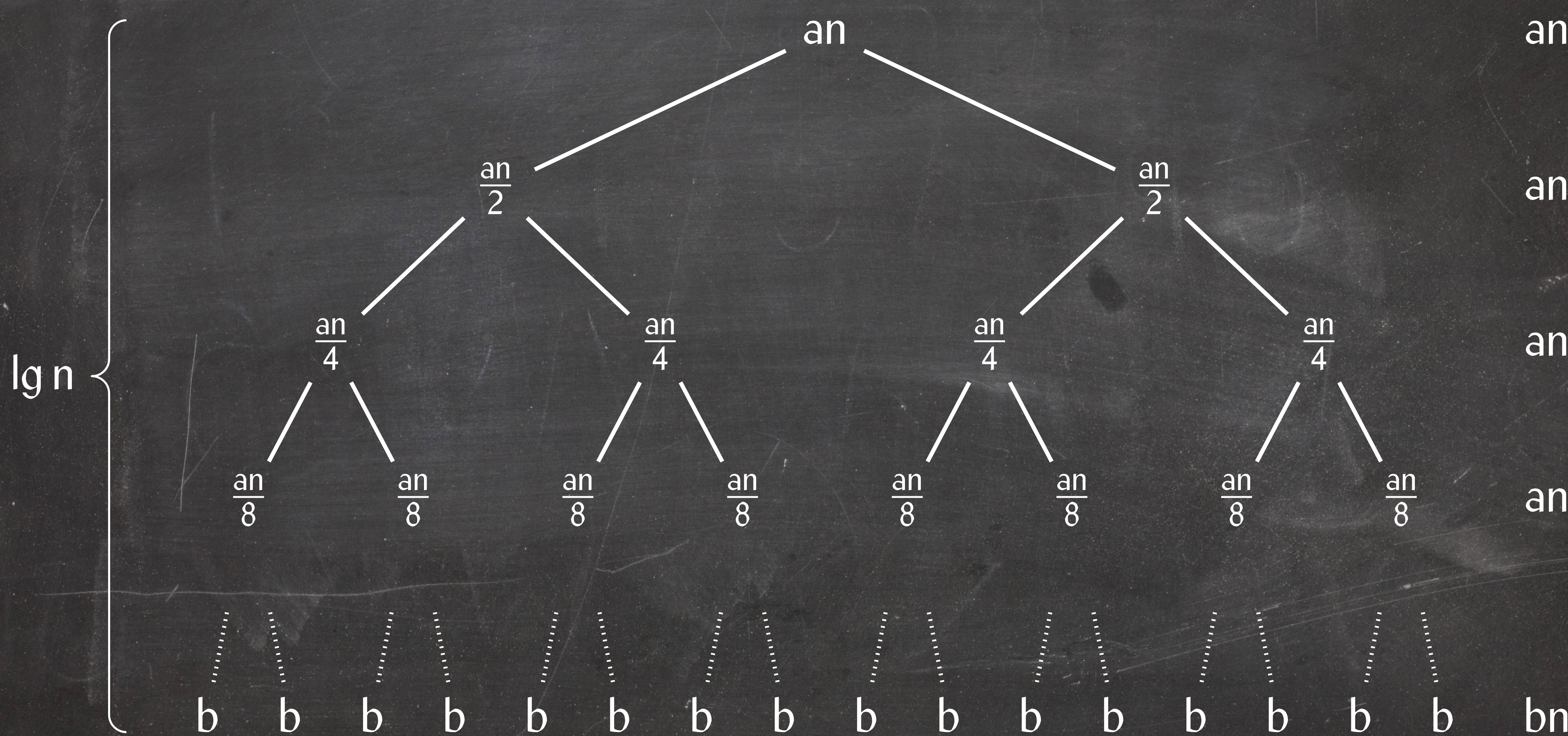

Solution: $T(n) \in \Theta(n \lg n)$

A Recursion Tree for Binary Search

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$$
\begin{array}{c}
a \\
| \\
a \\
| \\
a \\
| \\
a \\
\vdots \\
b
\end{array}
$$

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Solution: $T(n) \in \Theta(\lg n)$

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Recurrence: $T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an$

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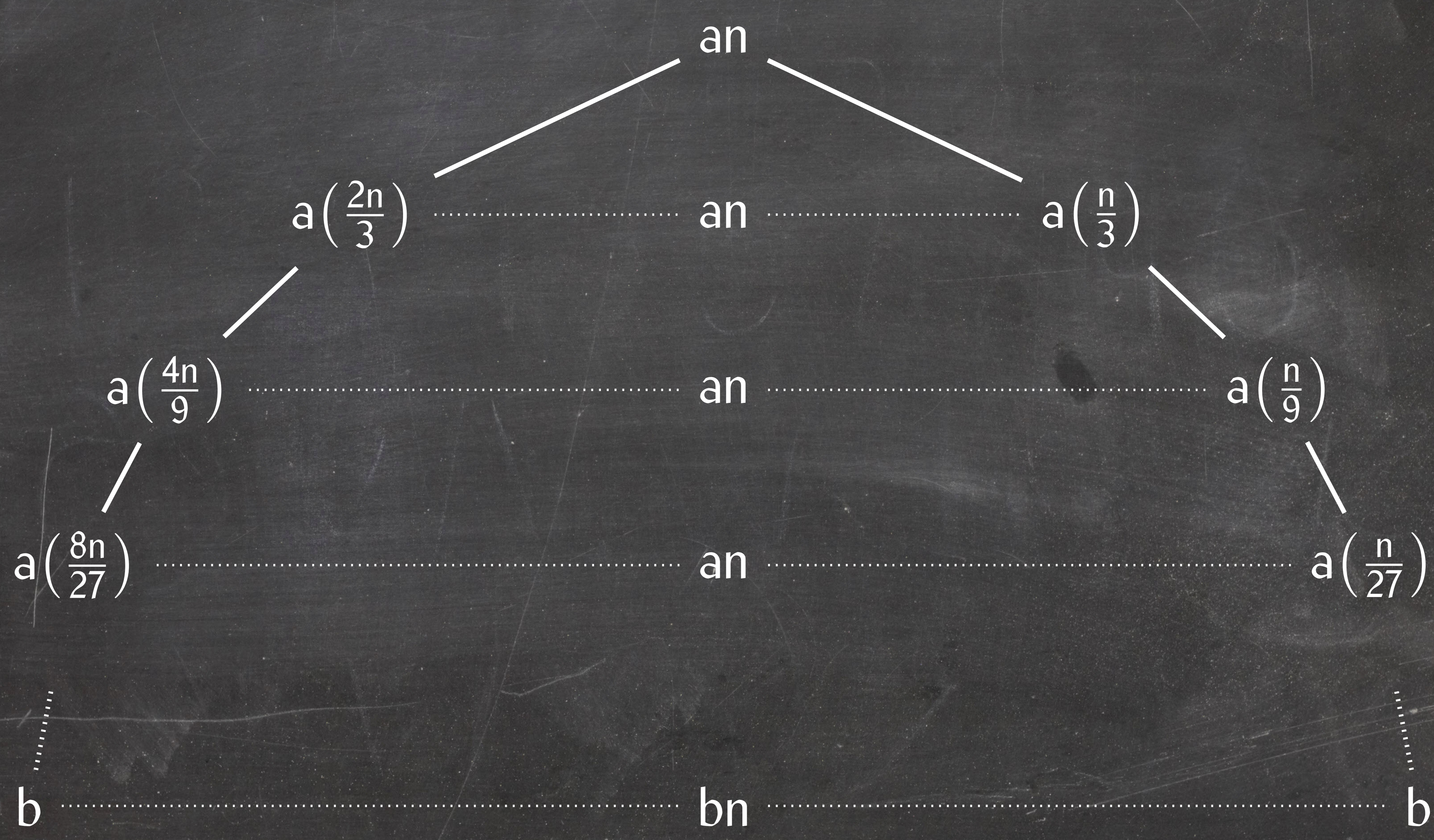

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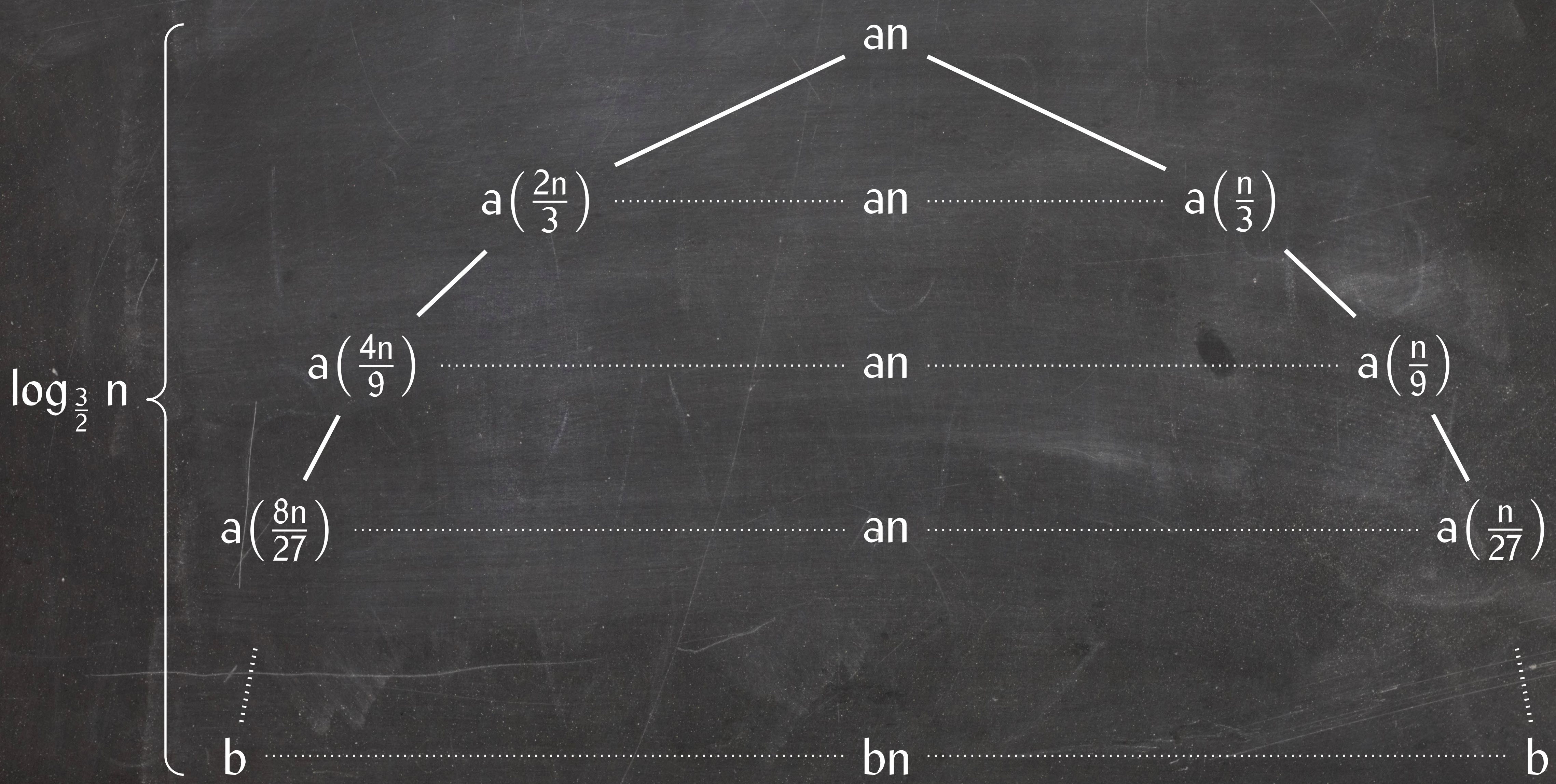

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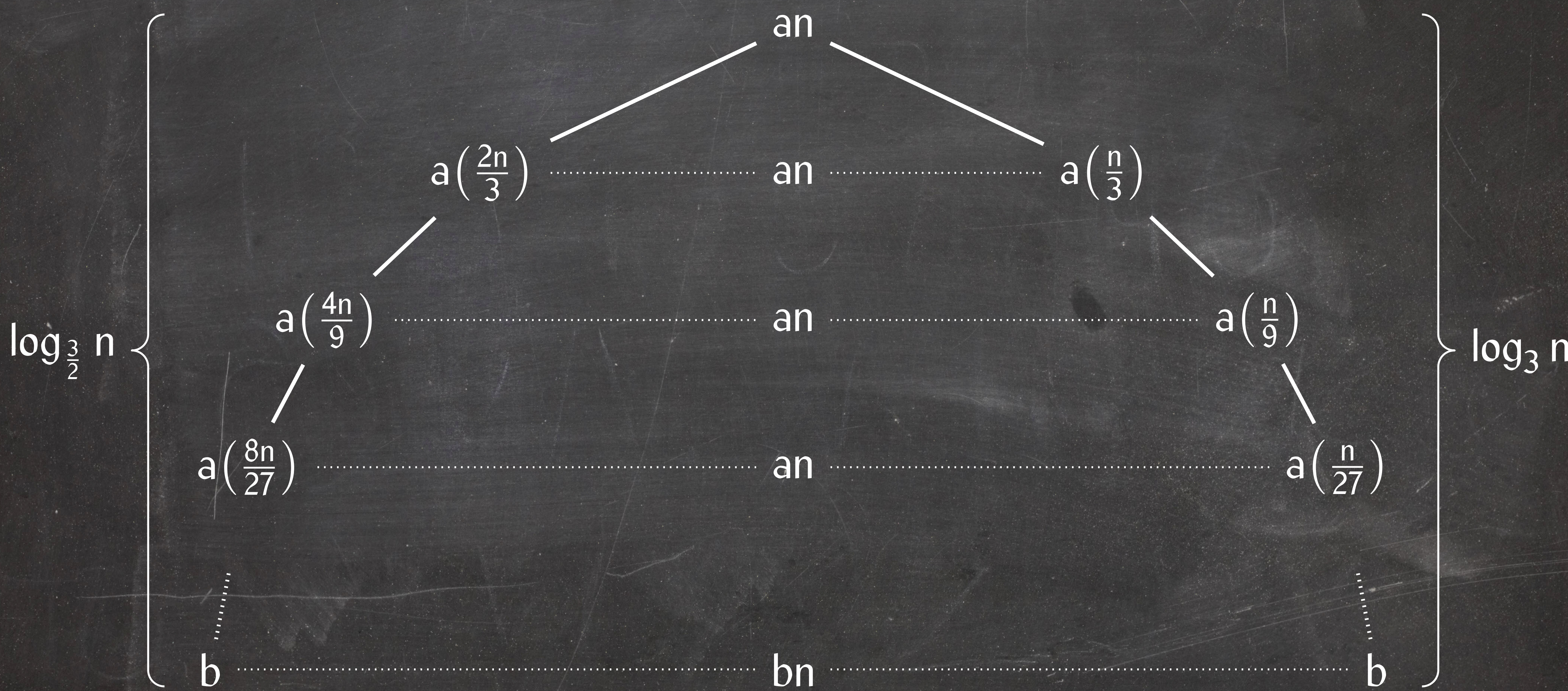

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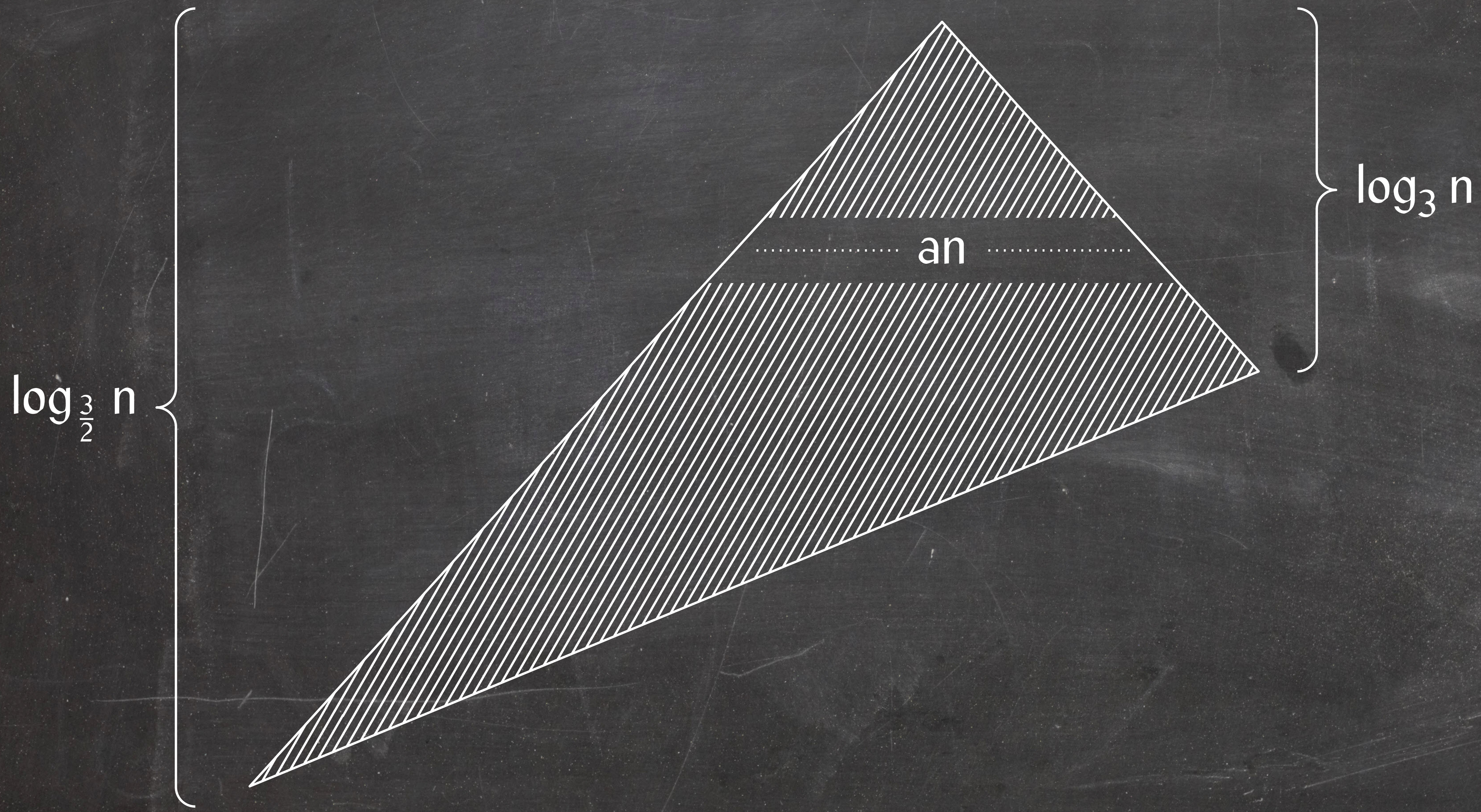

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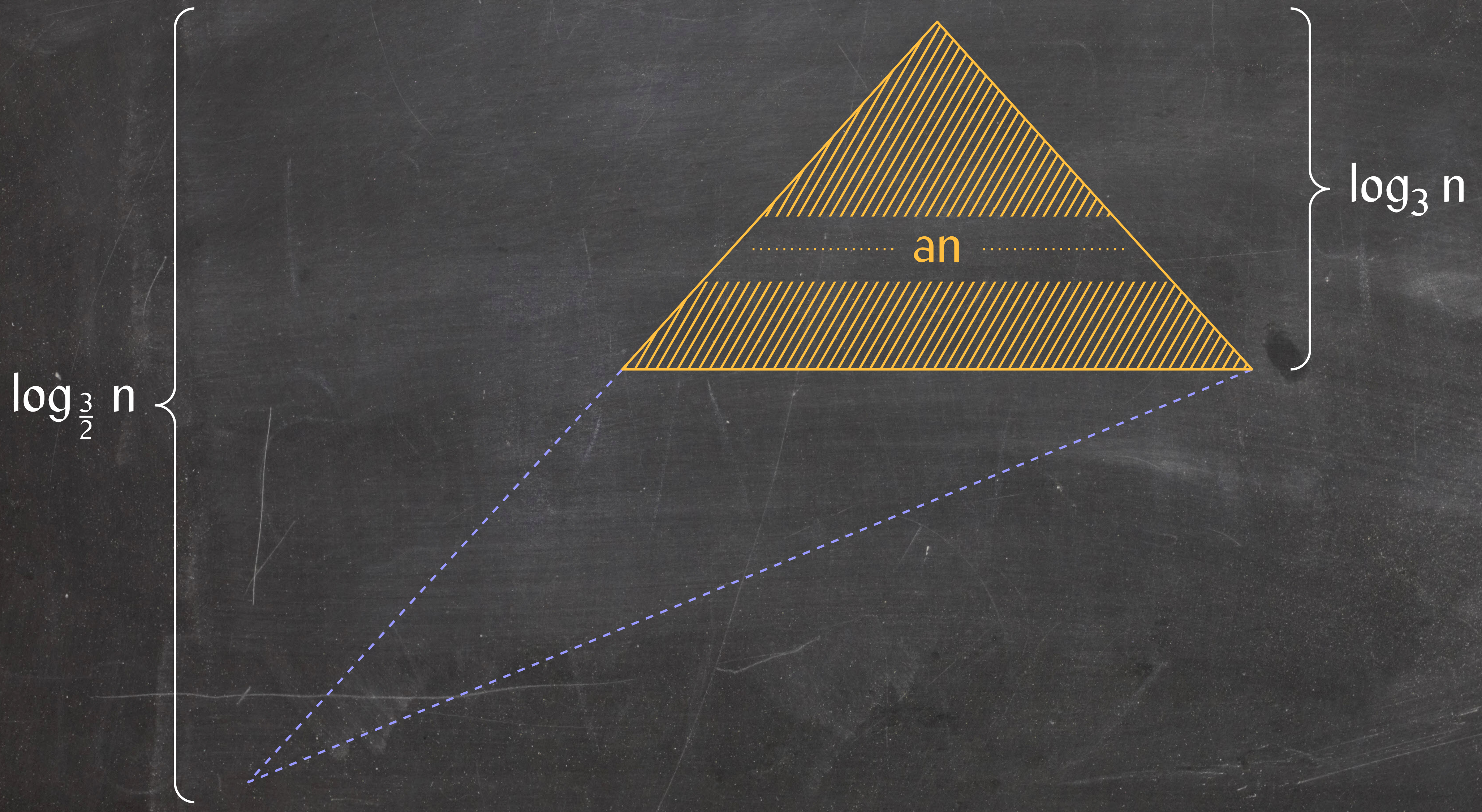

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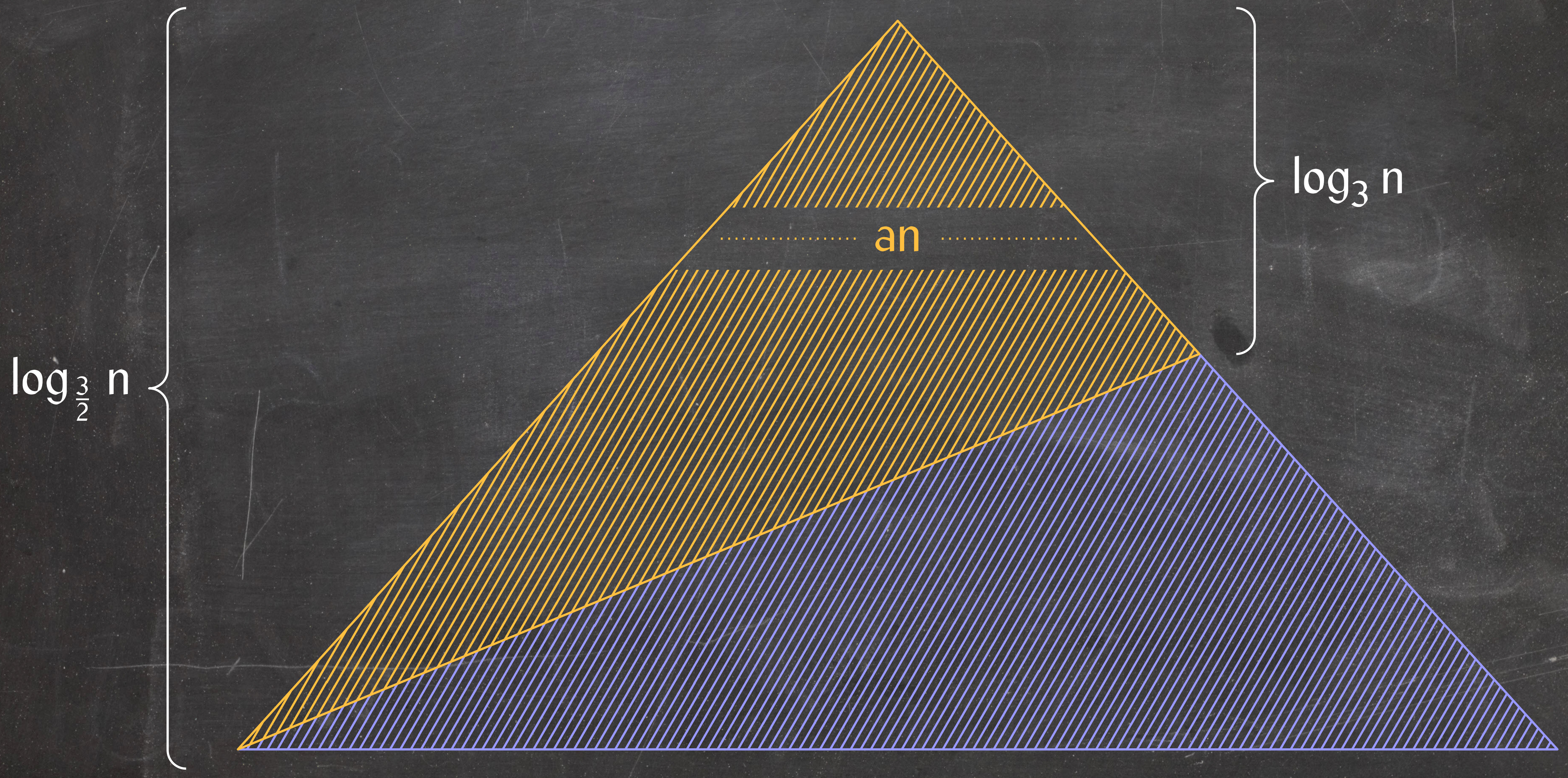
Recurrence: $T(n) = T(2n/3) + T(n/3) + \Theta(n) \Rightarrow T(n) = T(2n/3) + T(n/3) + an$



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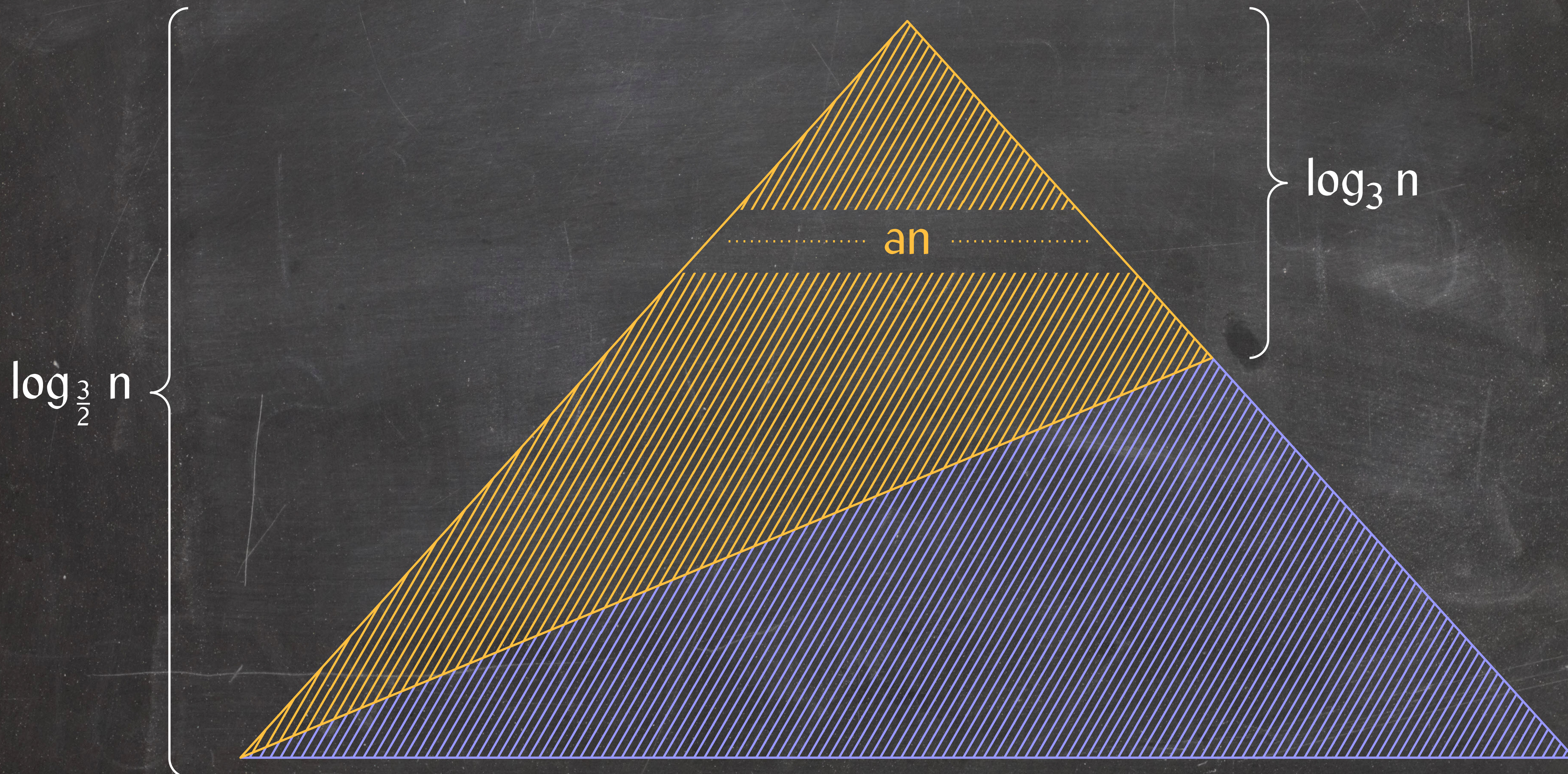

Solution: $T(n) \in \Theta(n \lg n)$

Sometimes Only Substitution Will Do

Recurrence: $T(n) = T(2n/3) + T(n/2) + \Theta(n)$

Lower bound: $T(n) \in \Omega(n^{1+\log_2(7/6)}) \approx \Omega(n^{1.22})$

Upper bound: $T(n) \in O(n^{1+\log_{3/2}(7/6)}) \approx O(n^{1.38})$

Master Theorem

Master Theorem: Let $a \geq 1$ and $b > 1$, let $f(n)$ be a positive function and let $T(n)$ be given by the following recurrence:

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n).$$

(i) If $f(n) \in O(n^{\log_b a - \epsilon})$, for some $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.

(ii) If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \lg n)$.

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Example 1: Merge Sort again

$$T(n) = 2T(n/2) + \Theta(n)$$

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Example 2: Matrix multiplication

$$T(n) = 7T(n/2) + \Theta(n^2)$$

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$f(n) = n$

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Master Theorem: Proof

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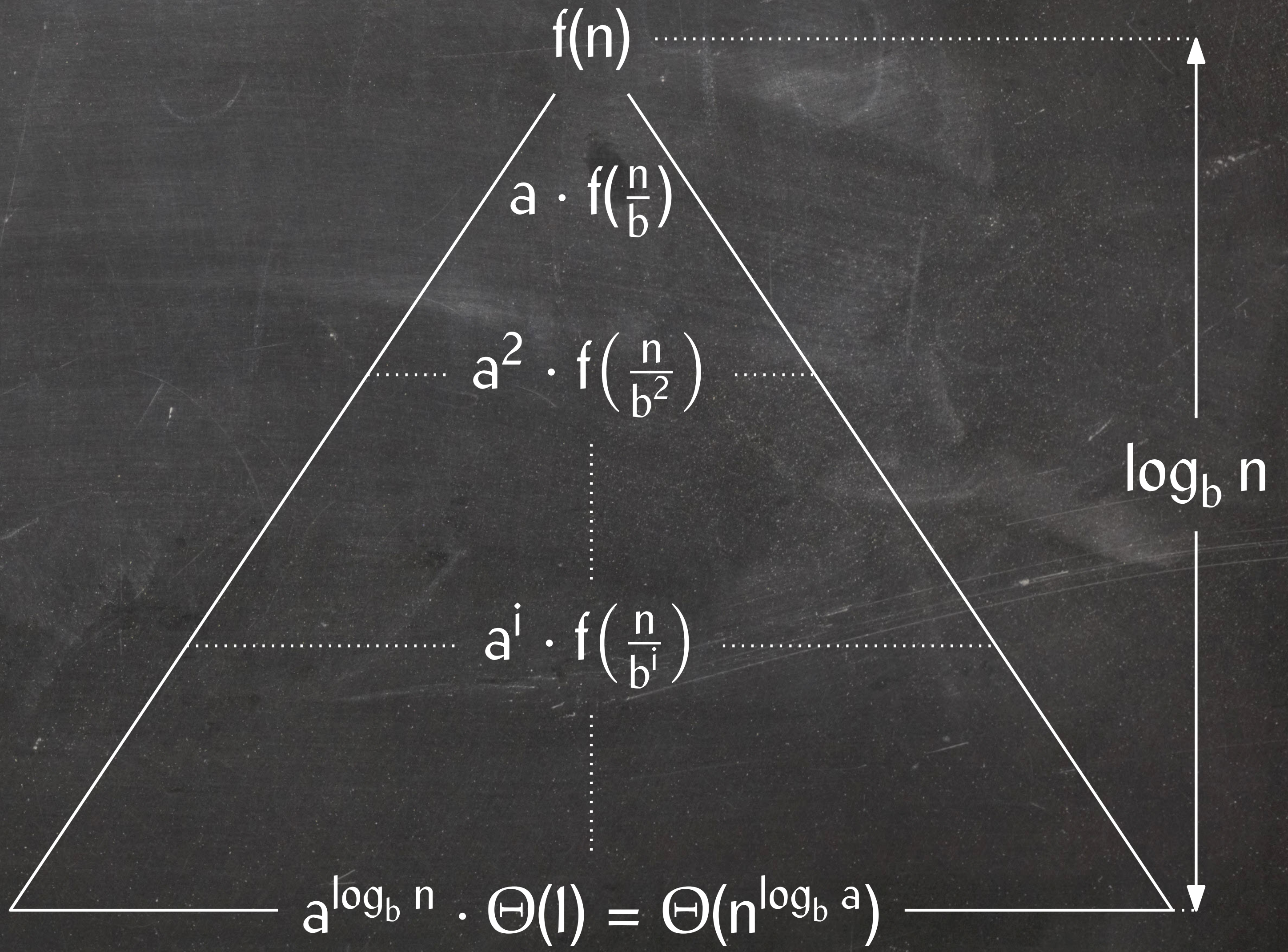

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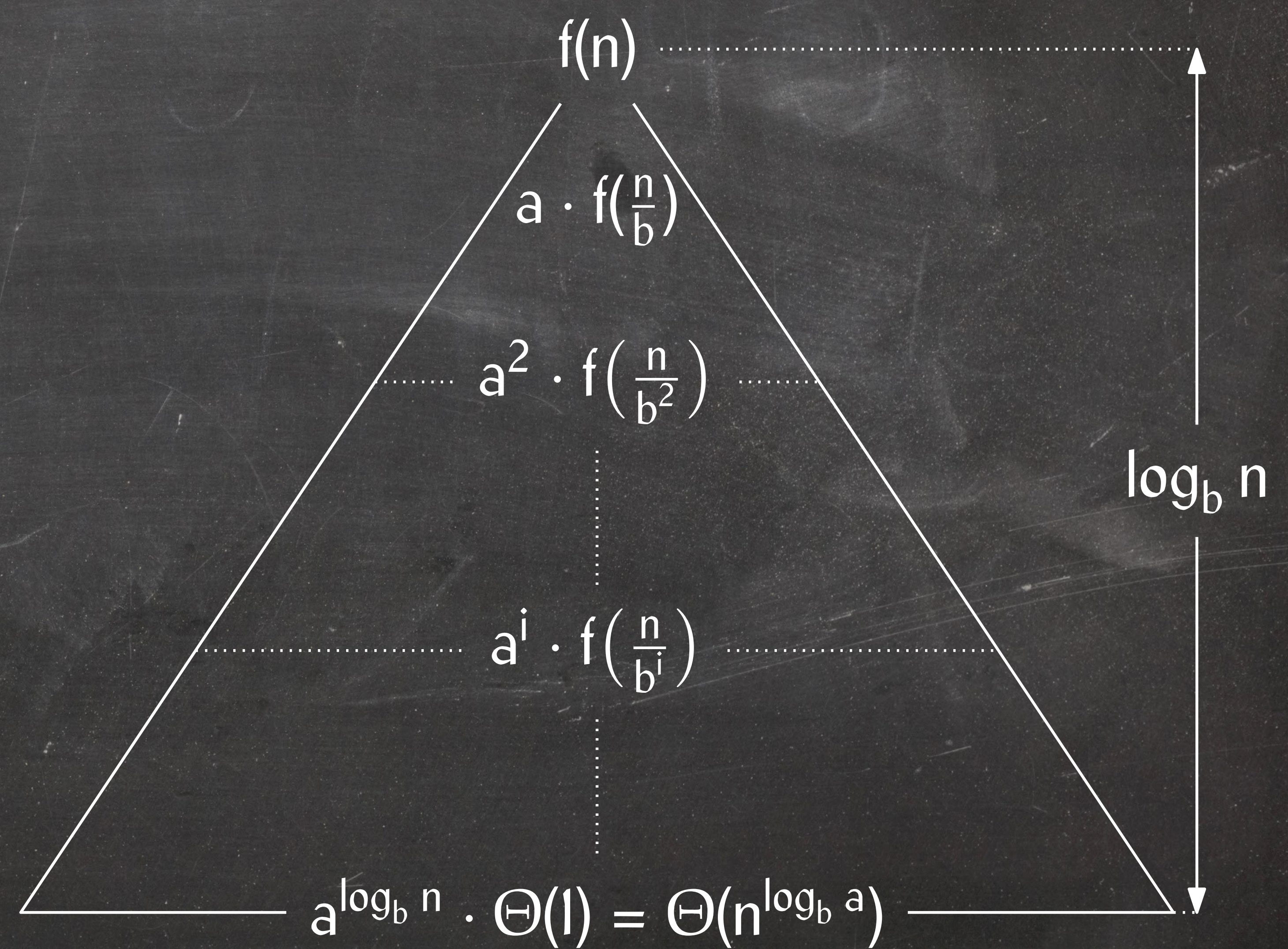

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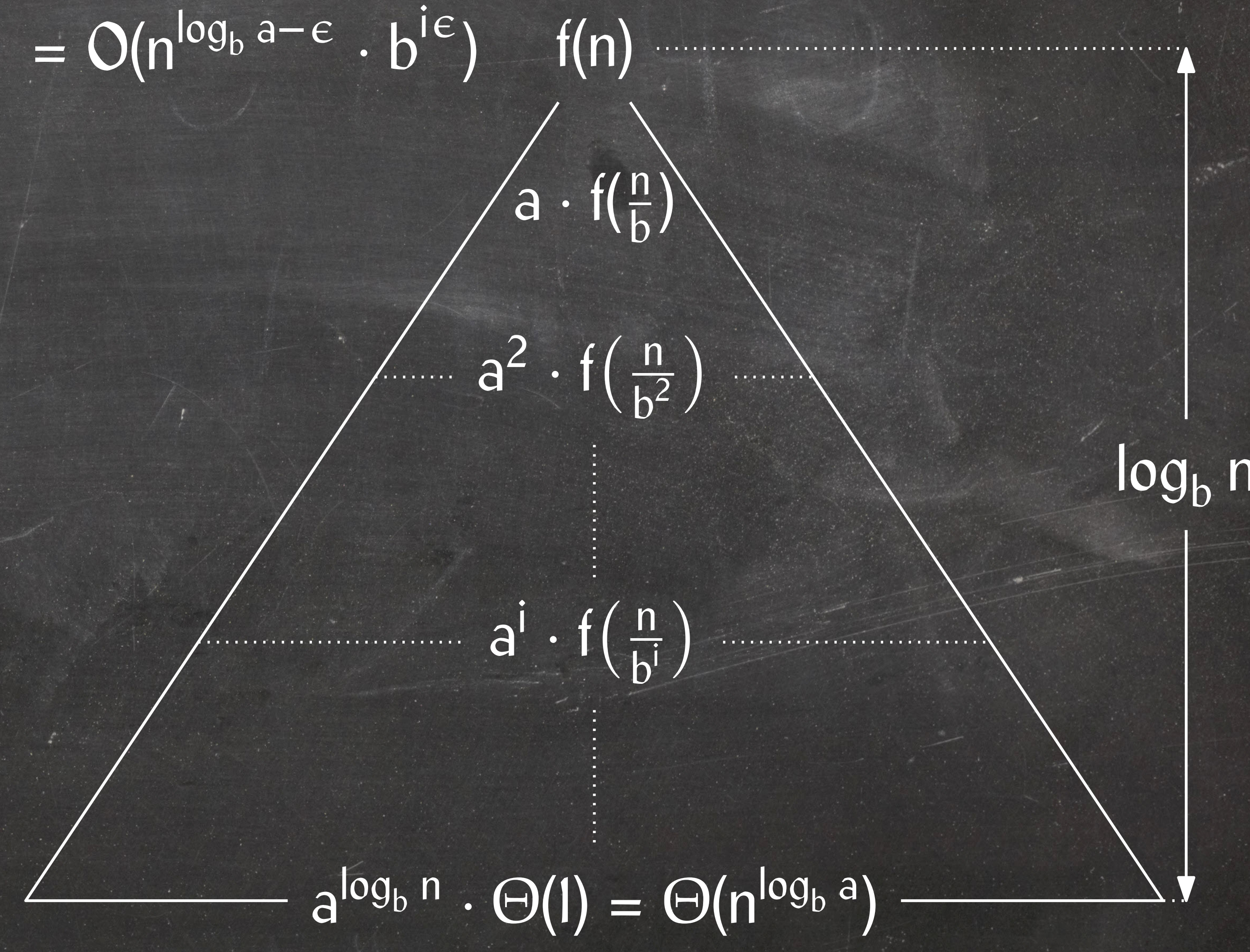

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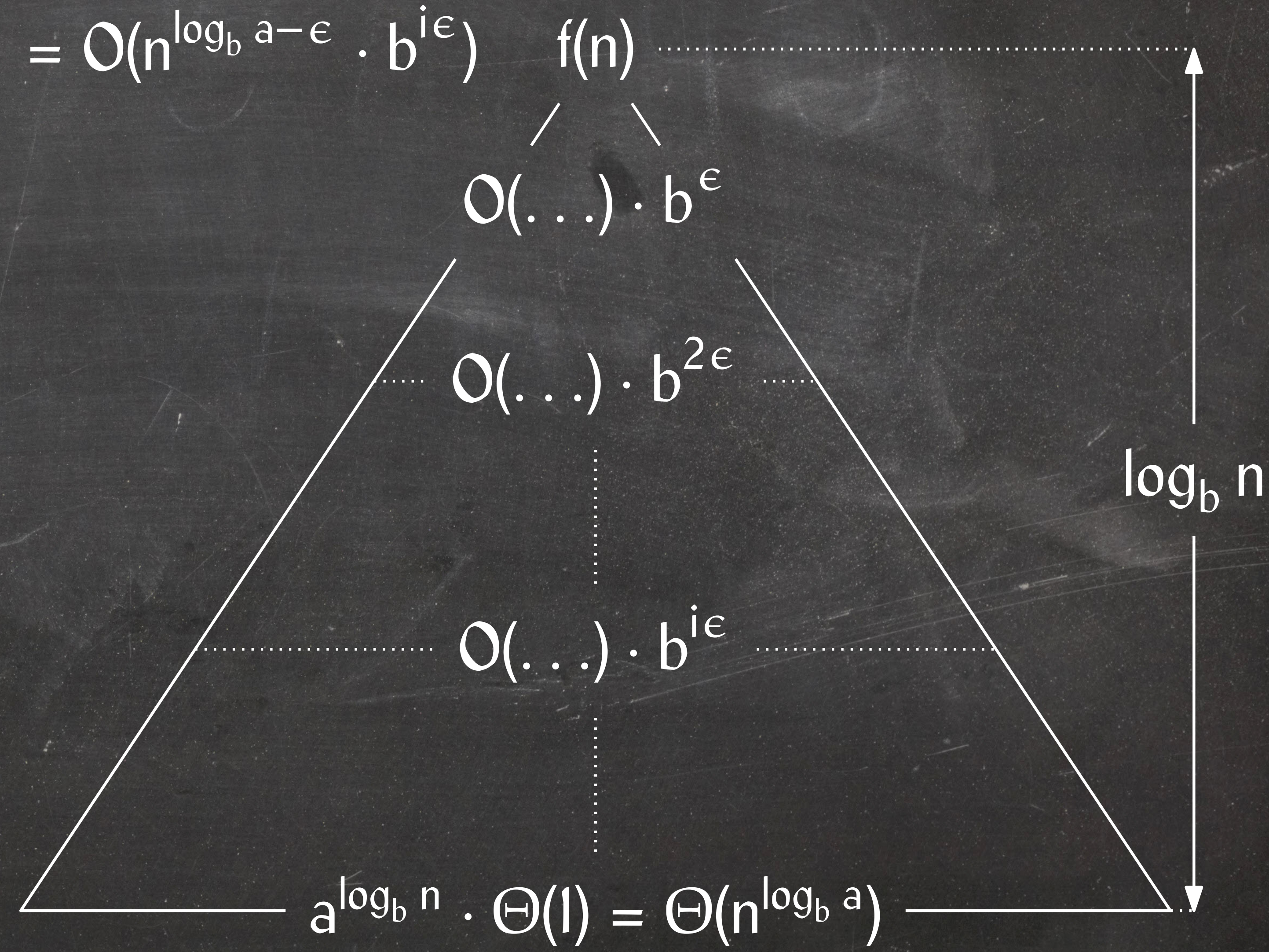

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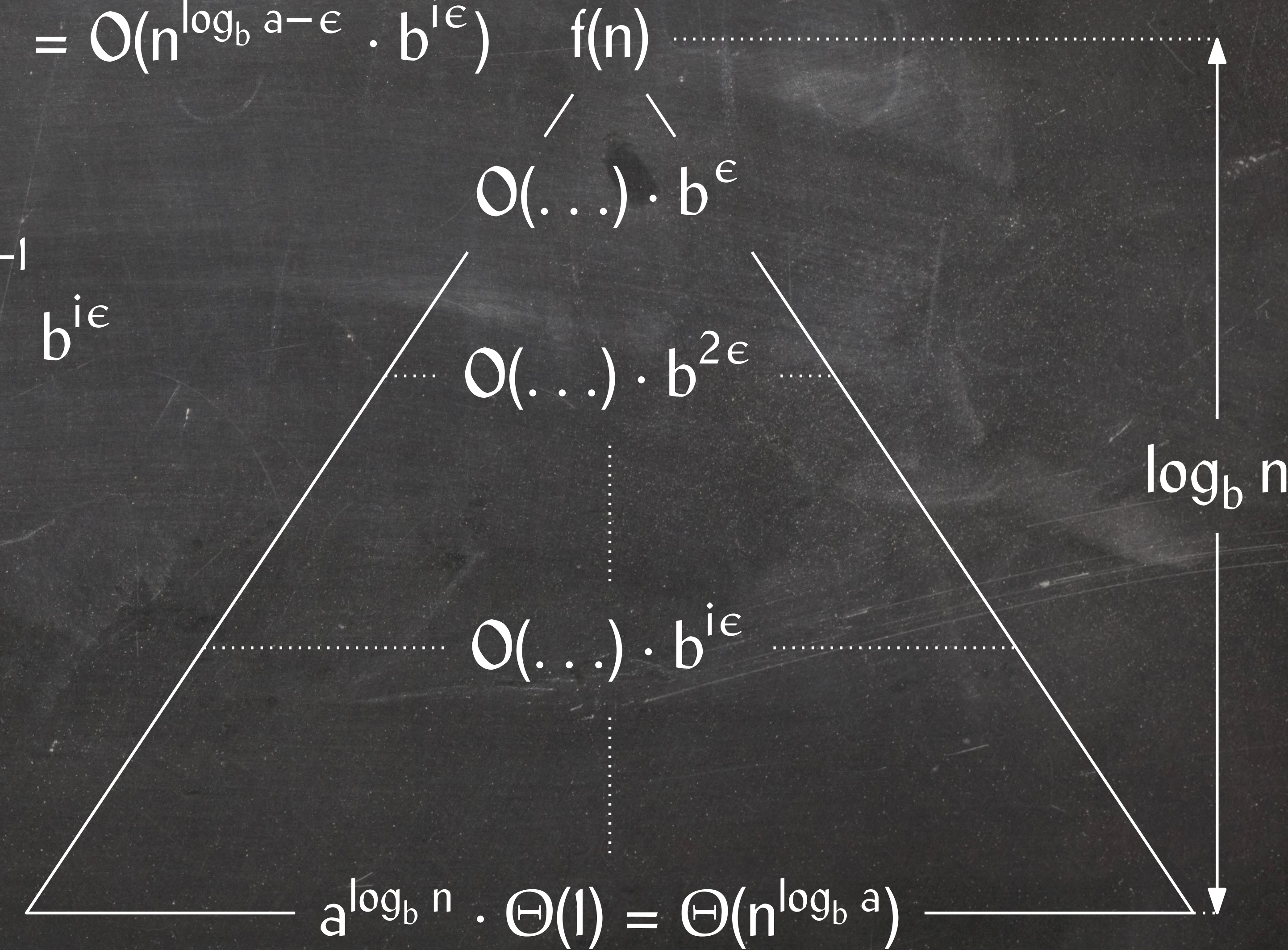

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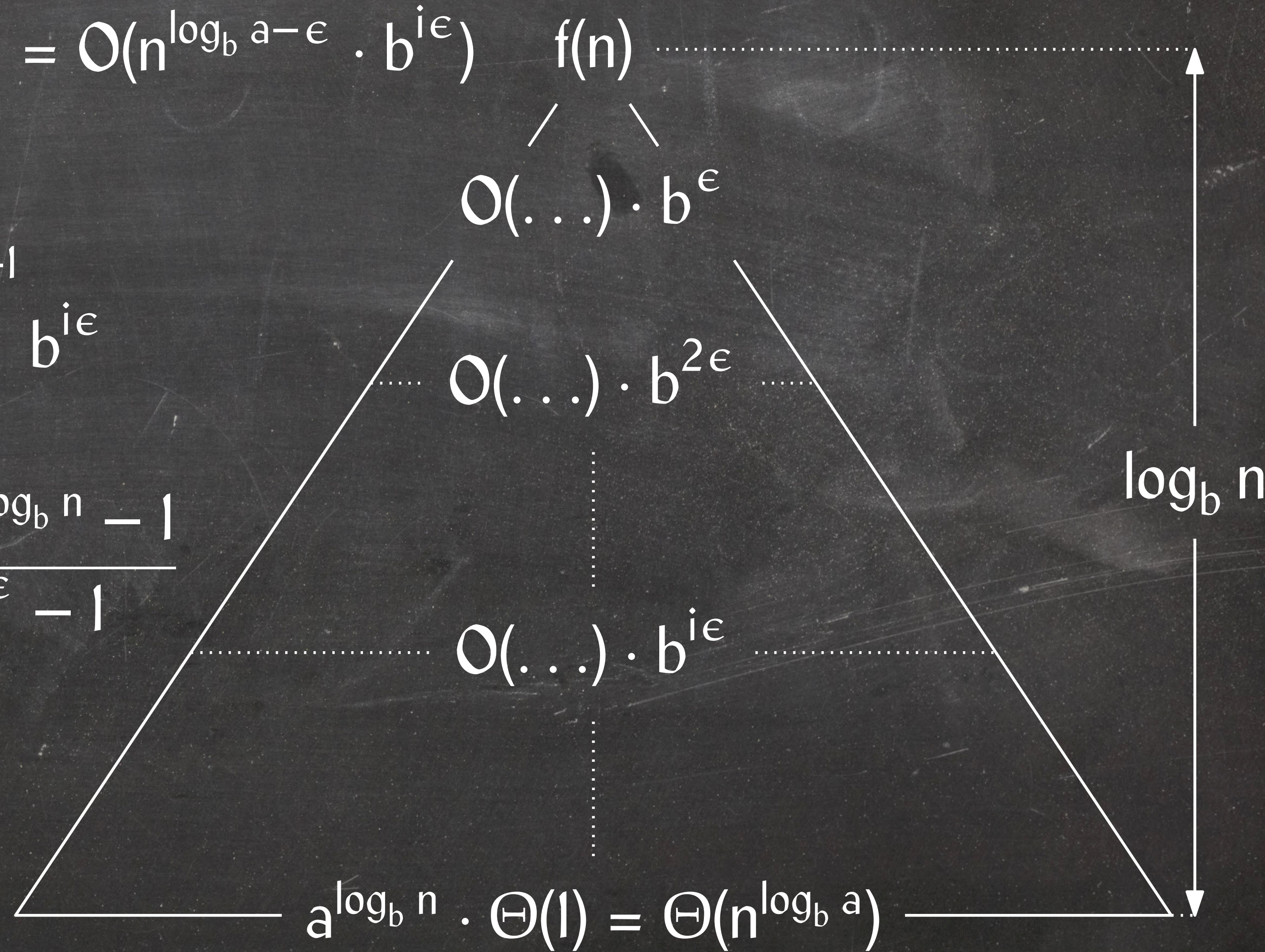

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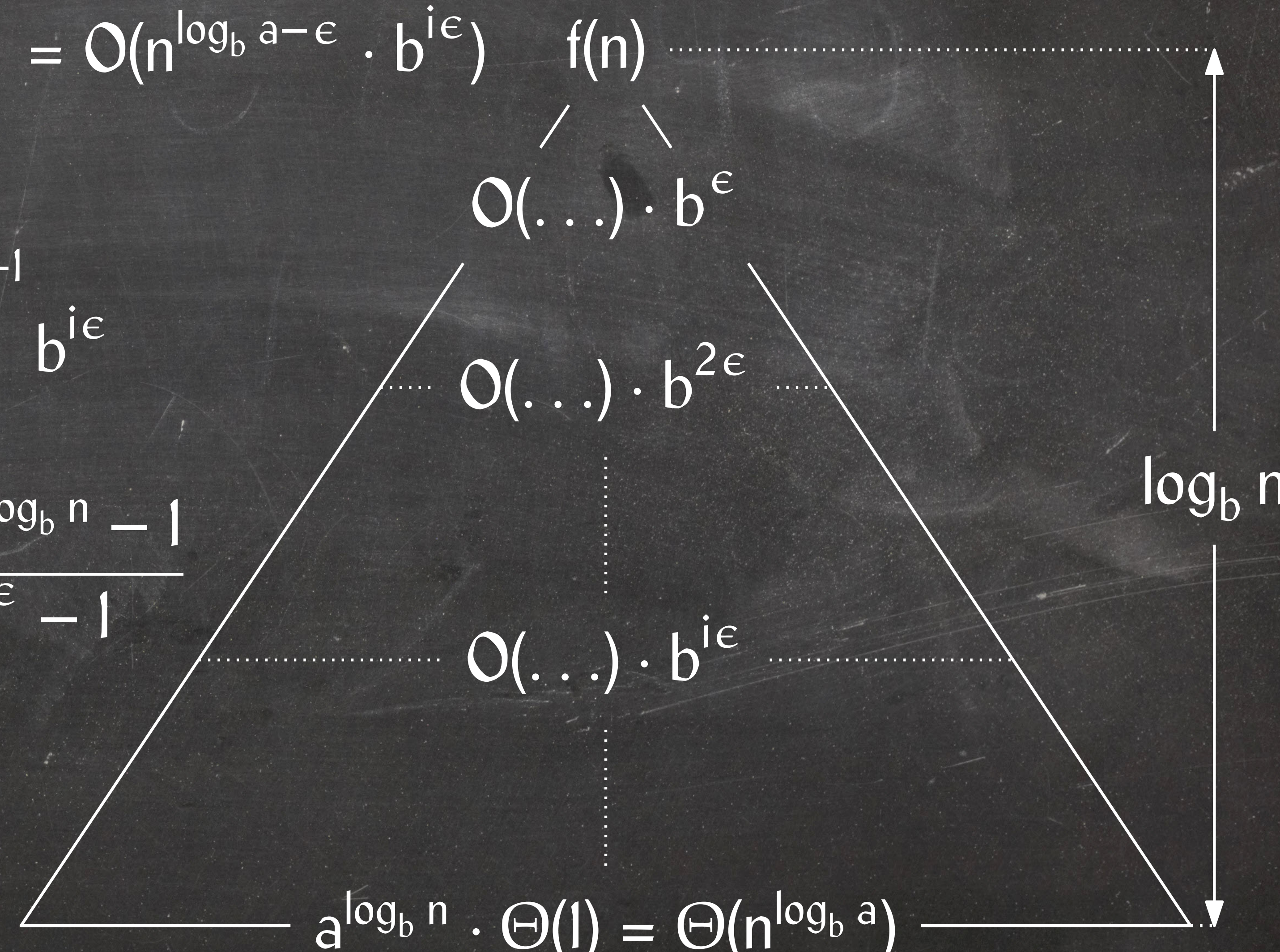

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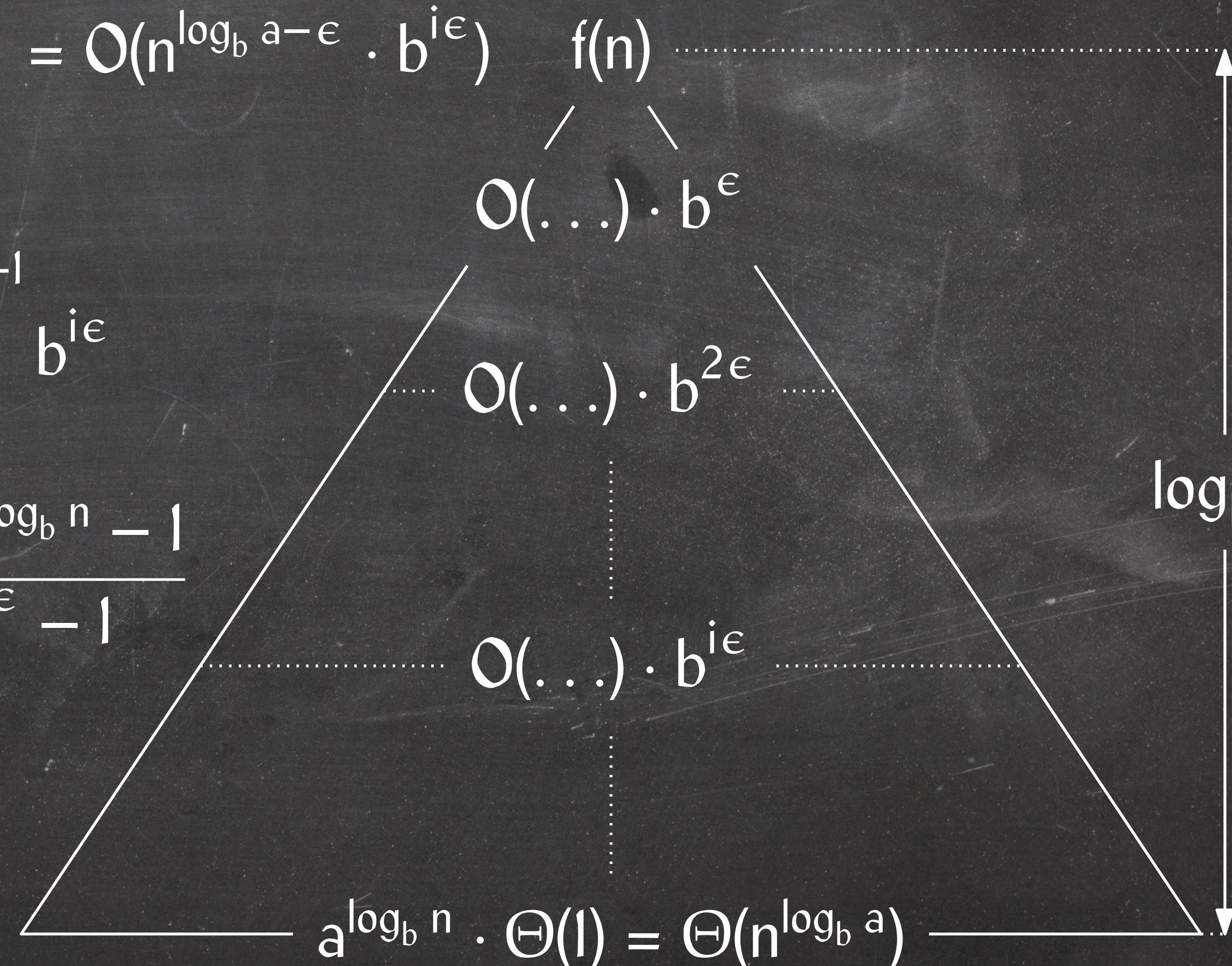

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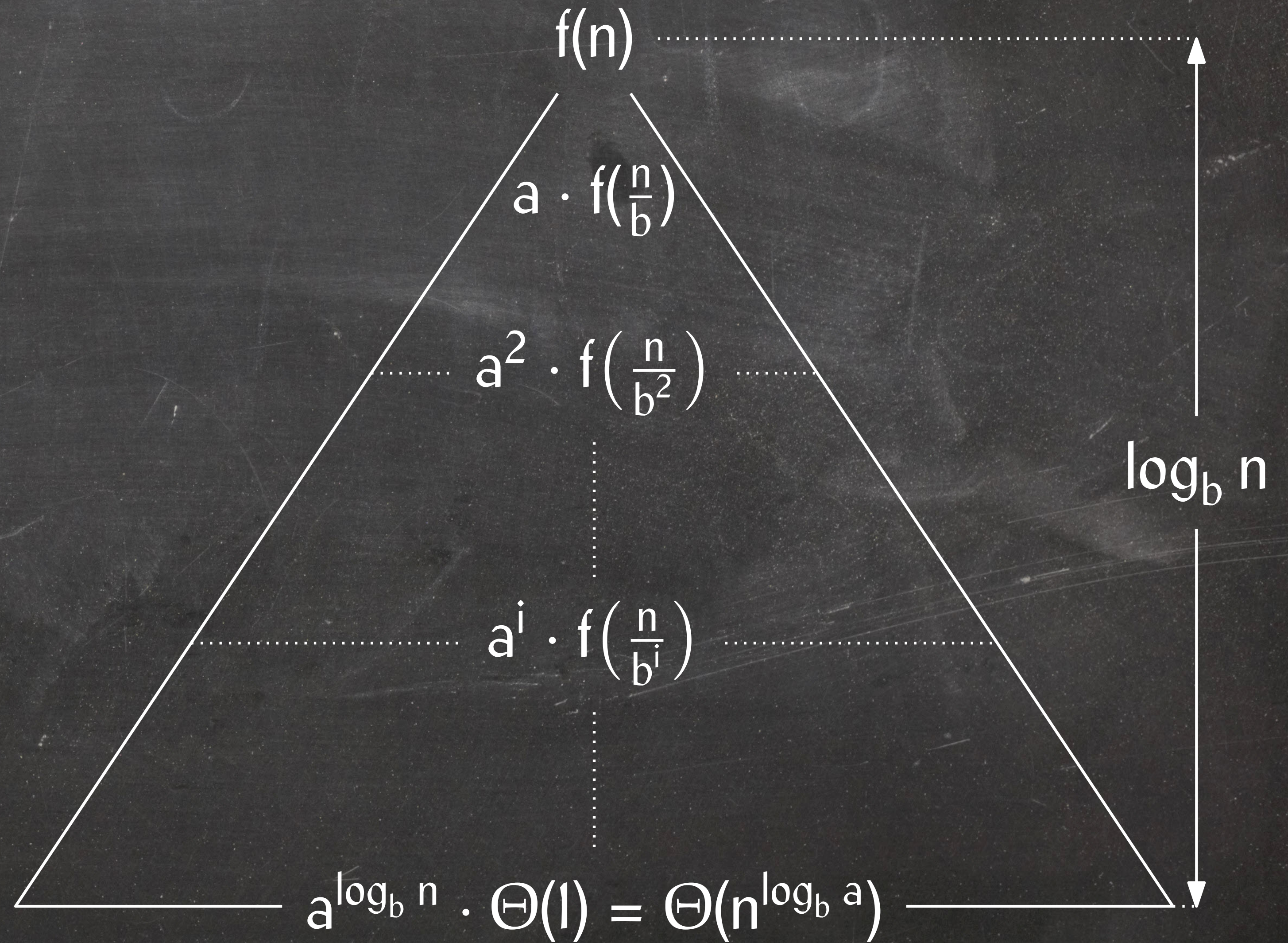
Case 2: $f(n) \in \Theta(n^{\log_b a})$



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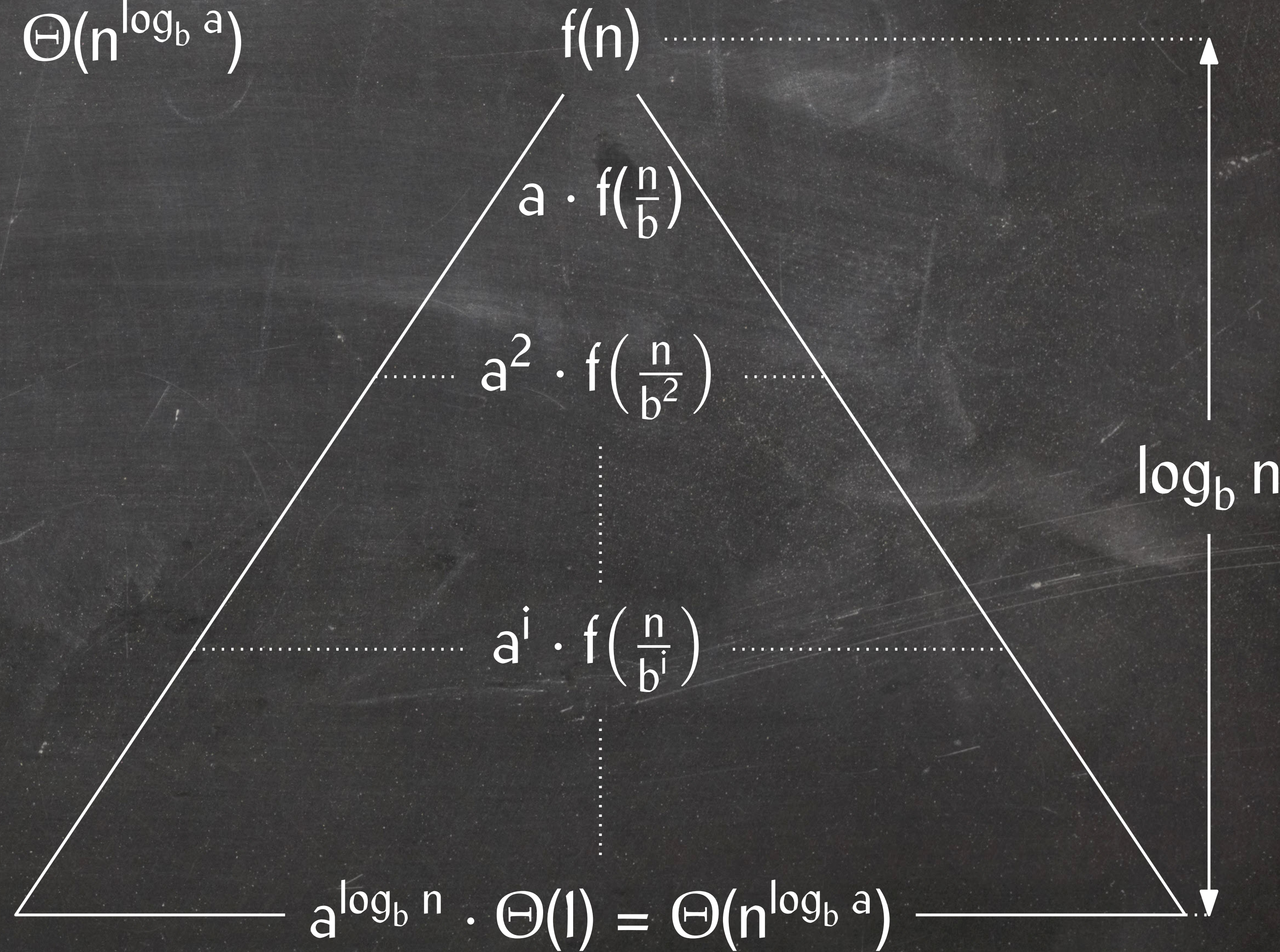

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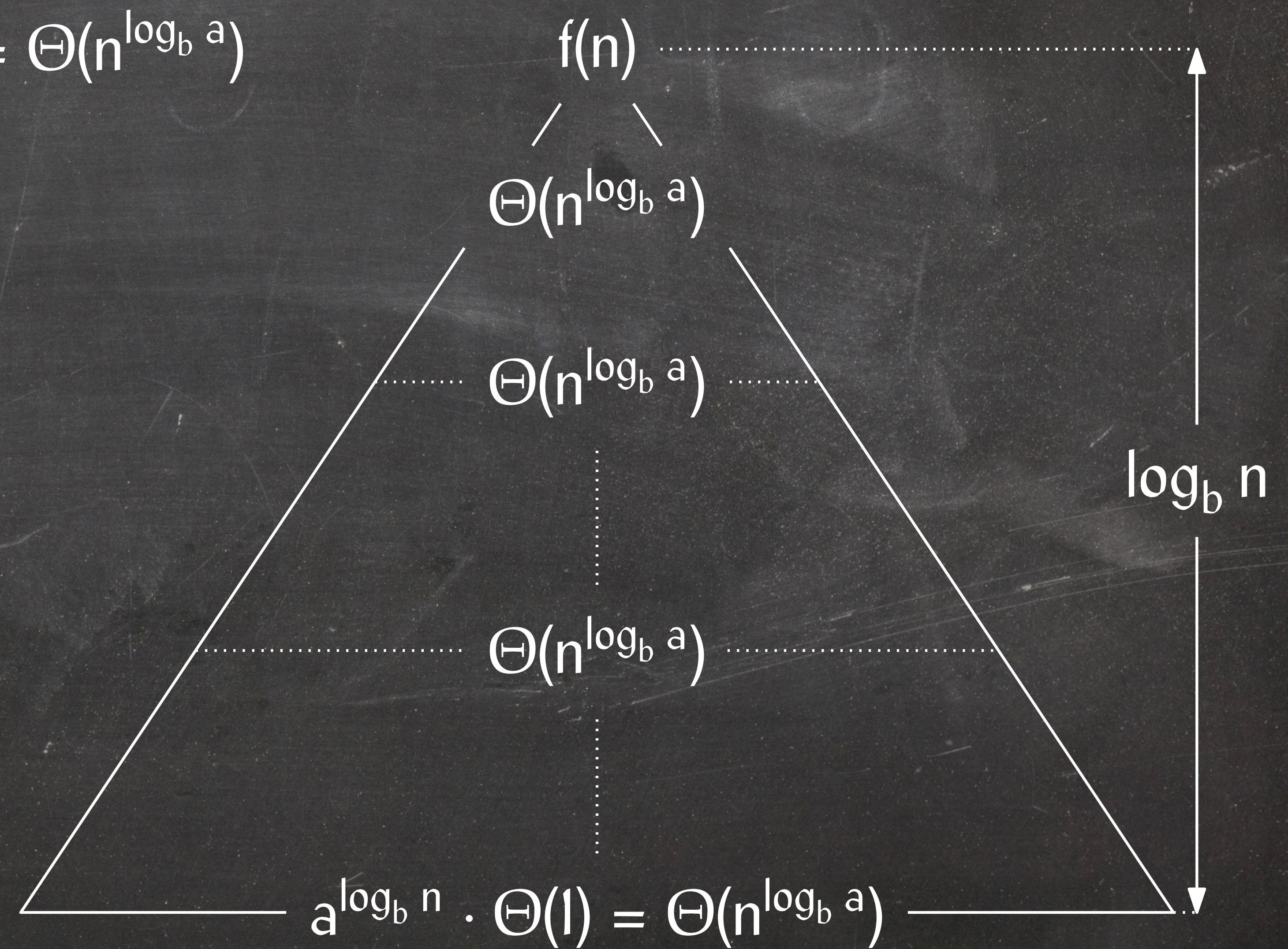

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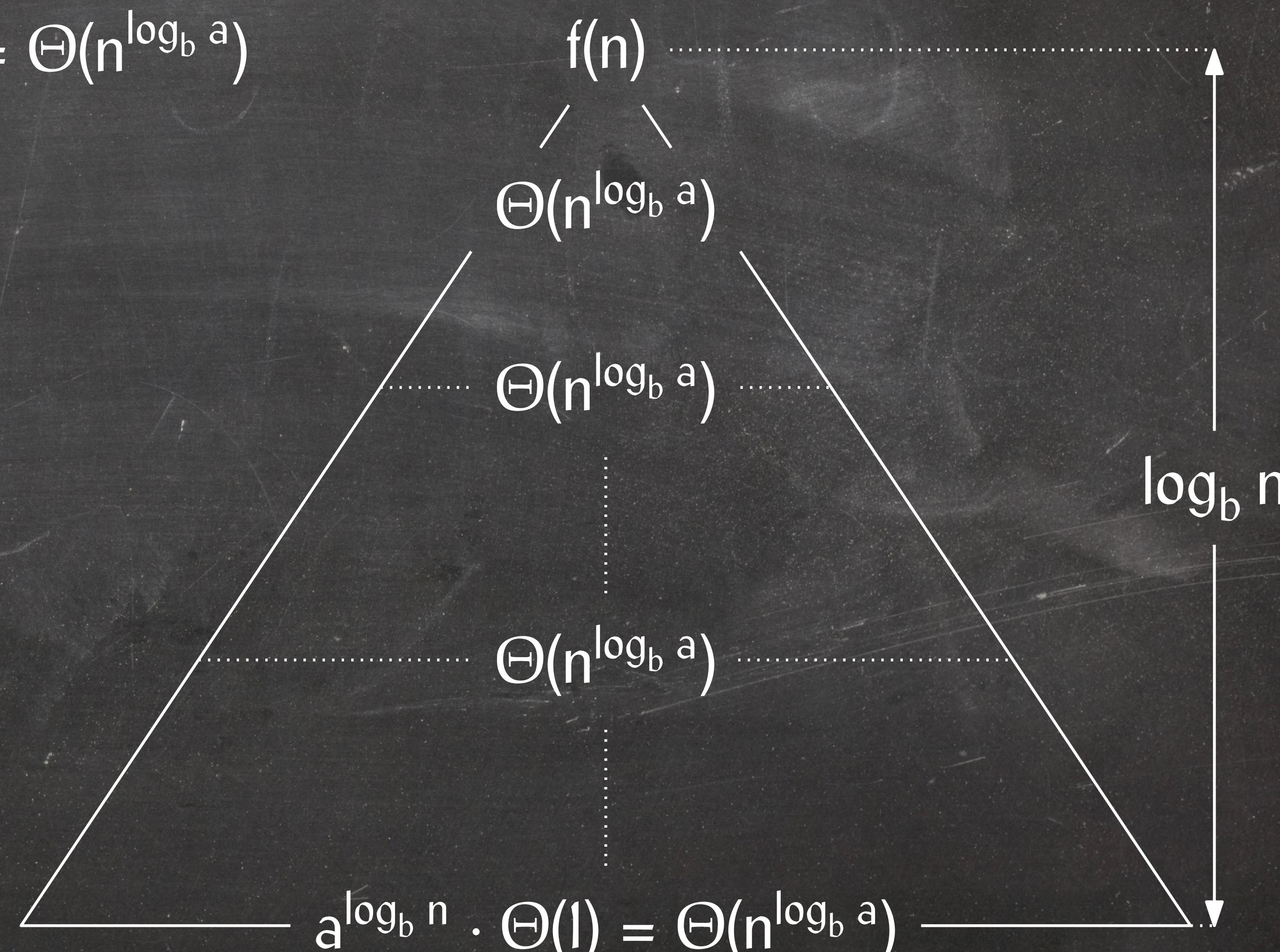
$$T(n) = \Theta(n^{\log_b a}) \cdot \log_b n$$



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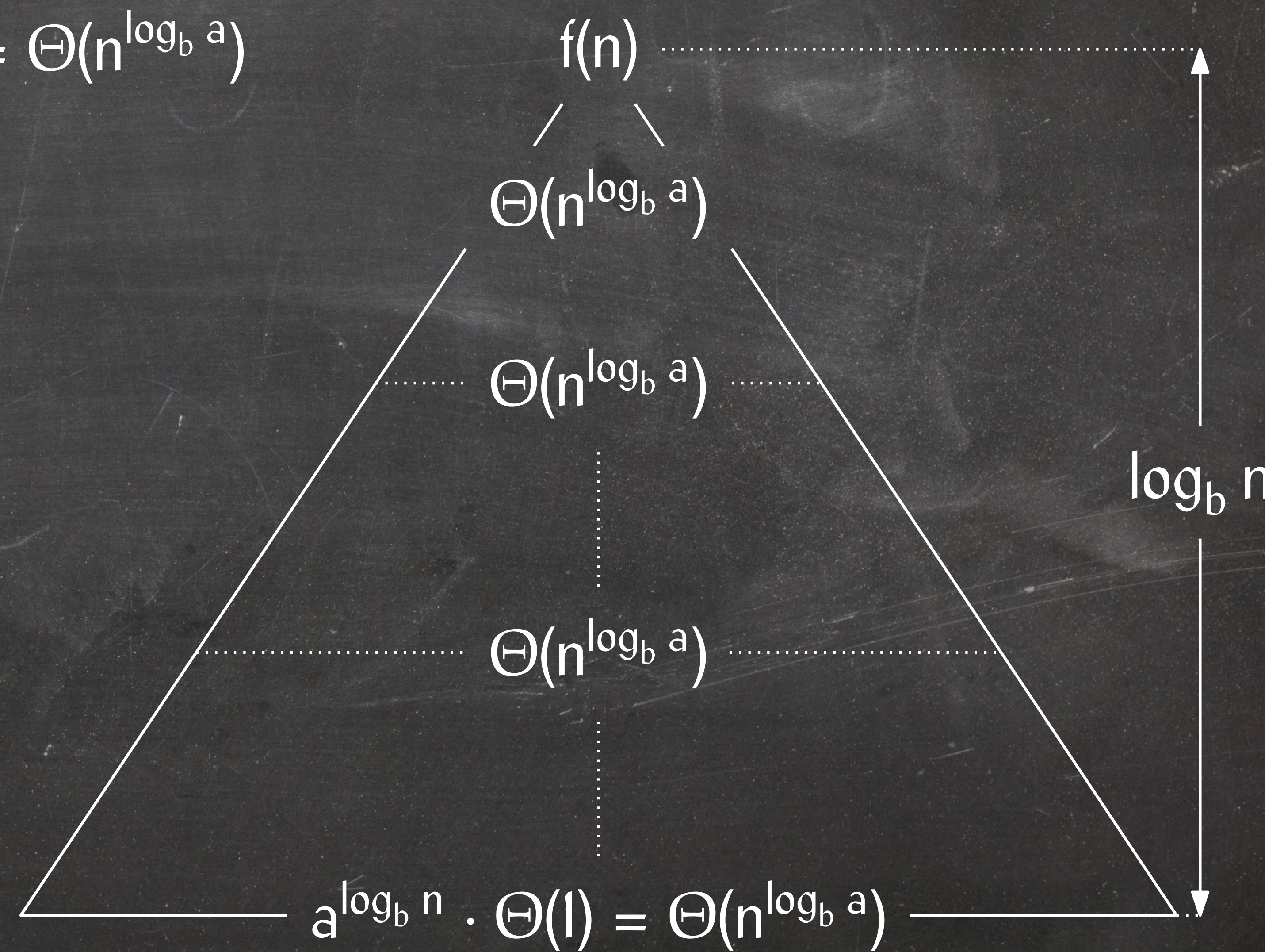

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$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

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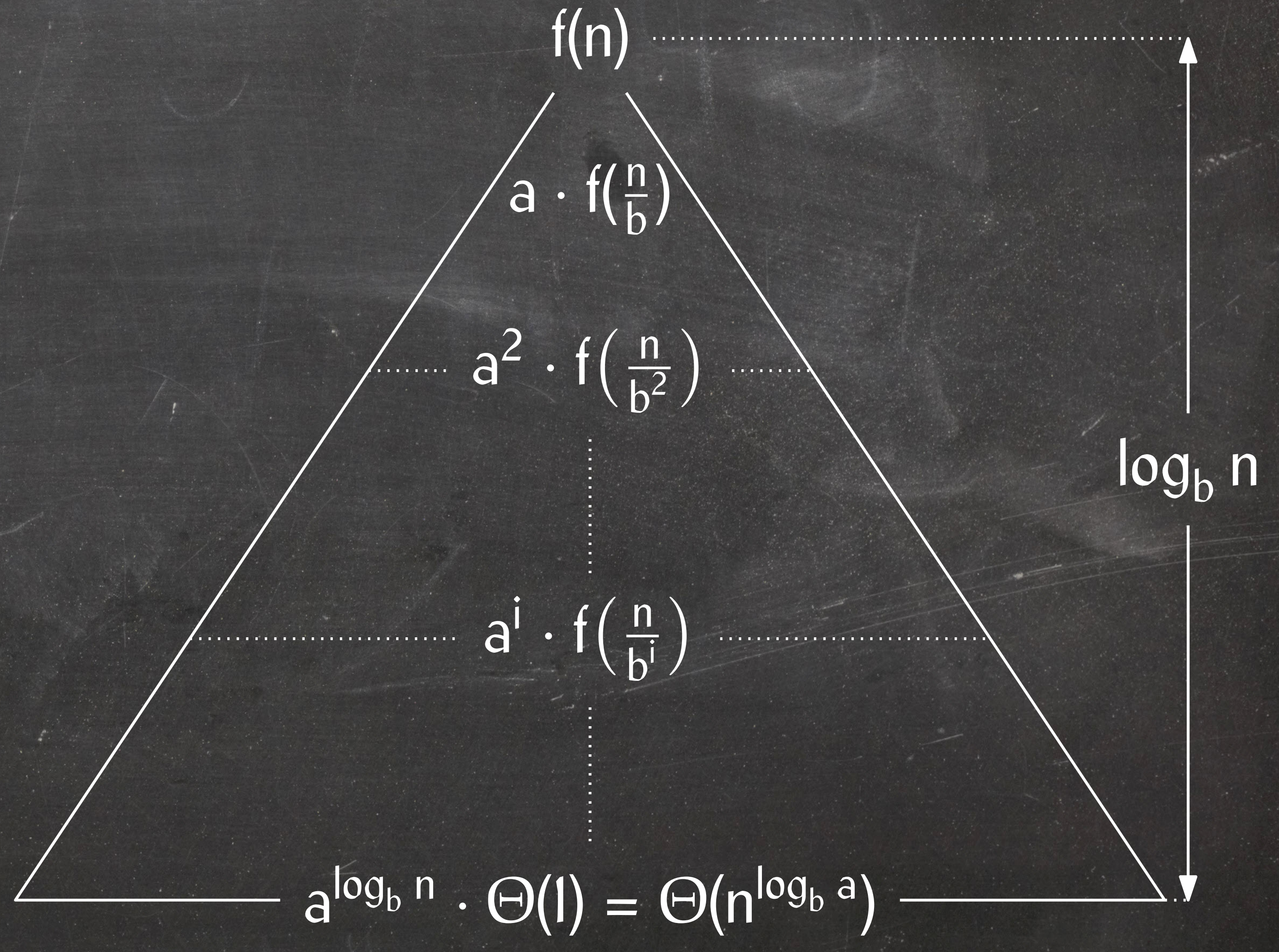

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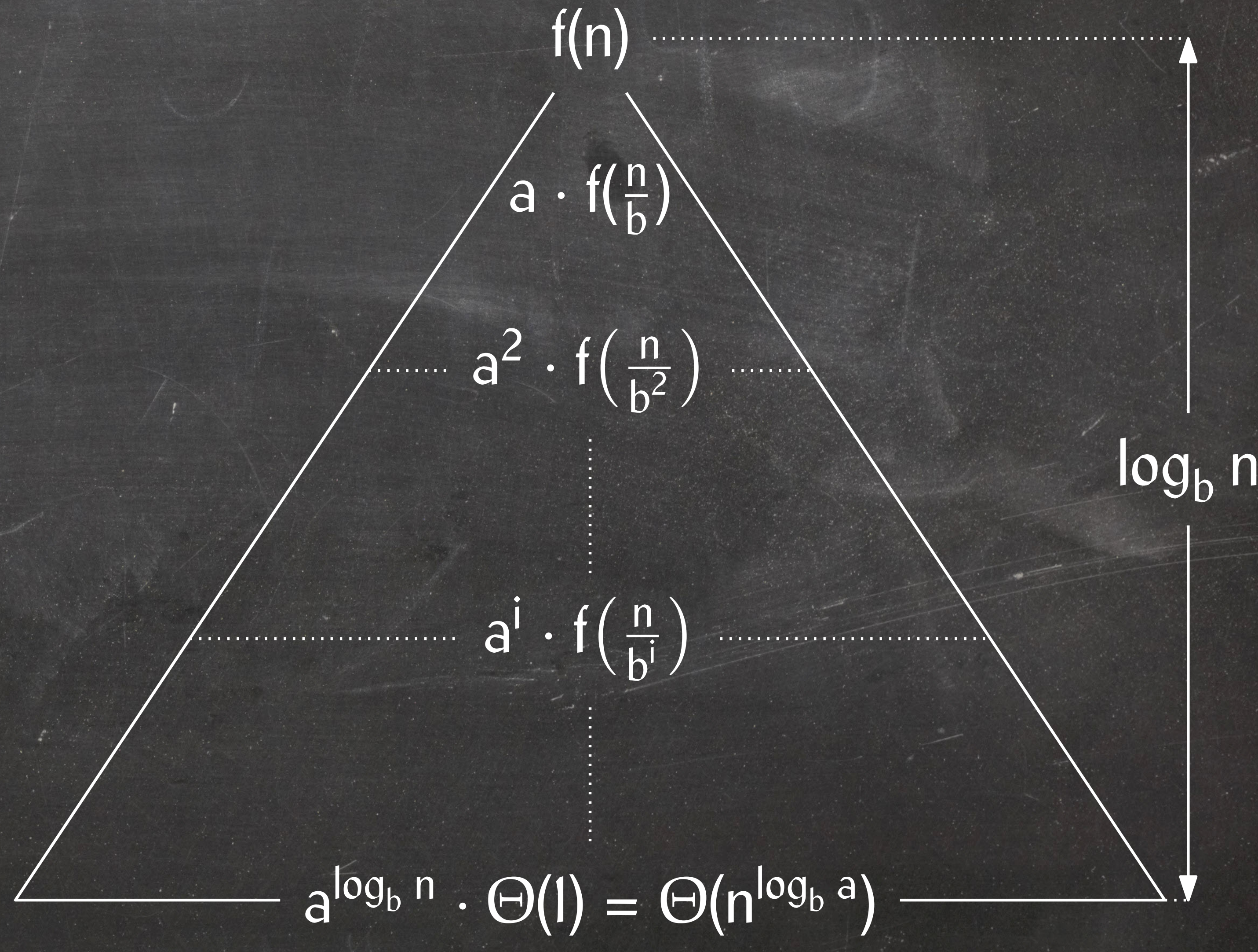

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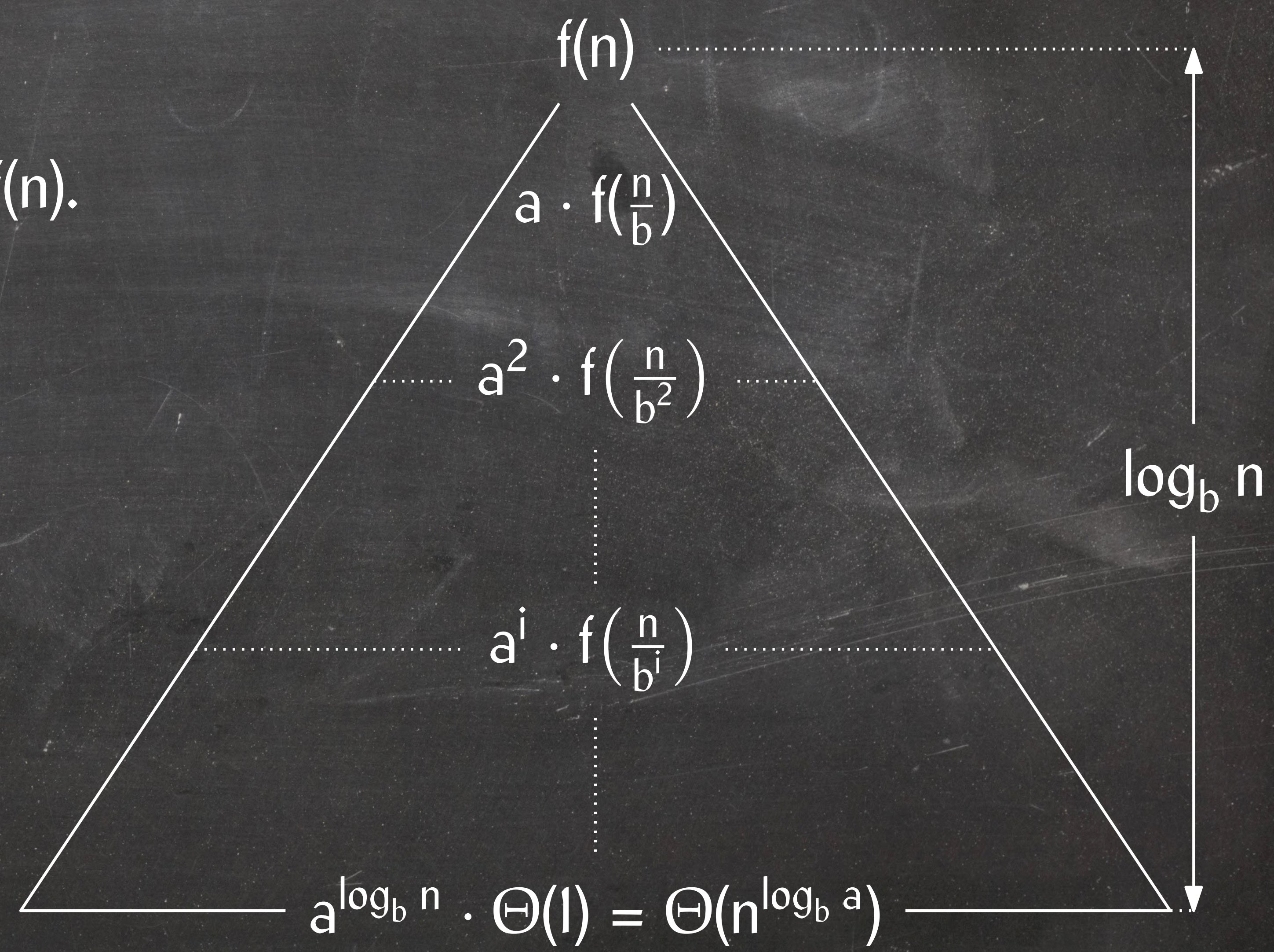
For $i = 0$, $a^0 \cdot f\left(\frac{n}{b^0}\right) = f(n) = c^0 \cdot f(n)$.



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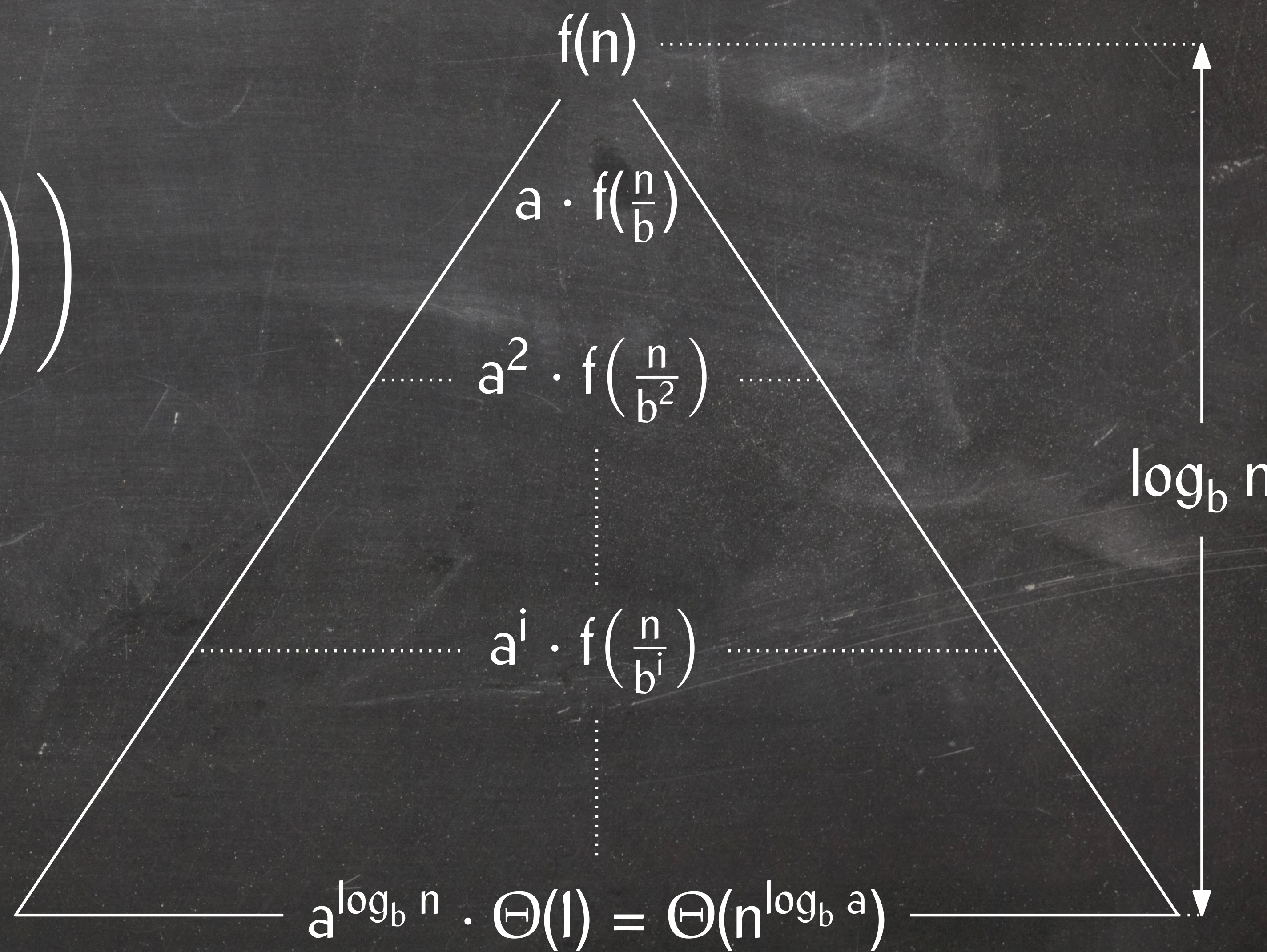

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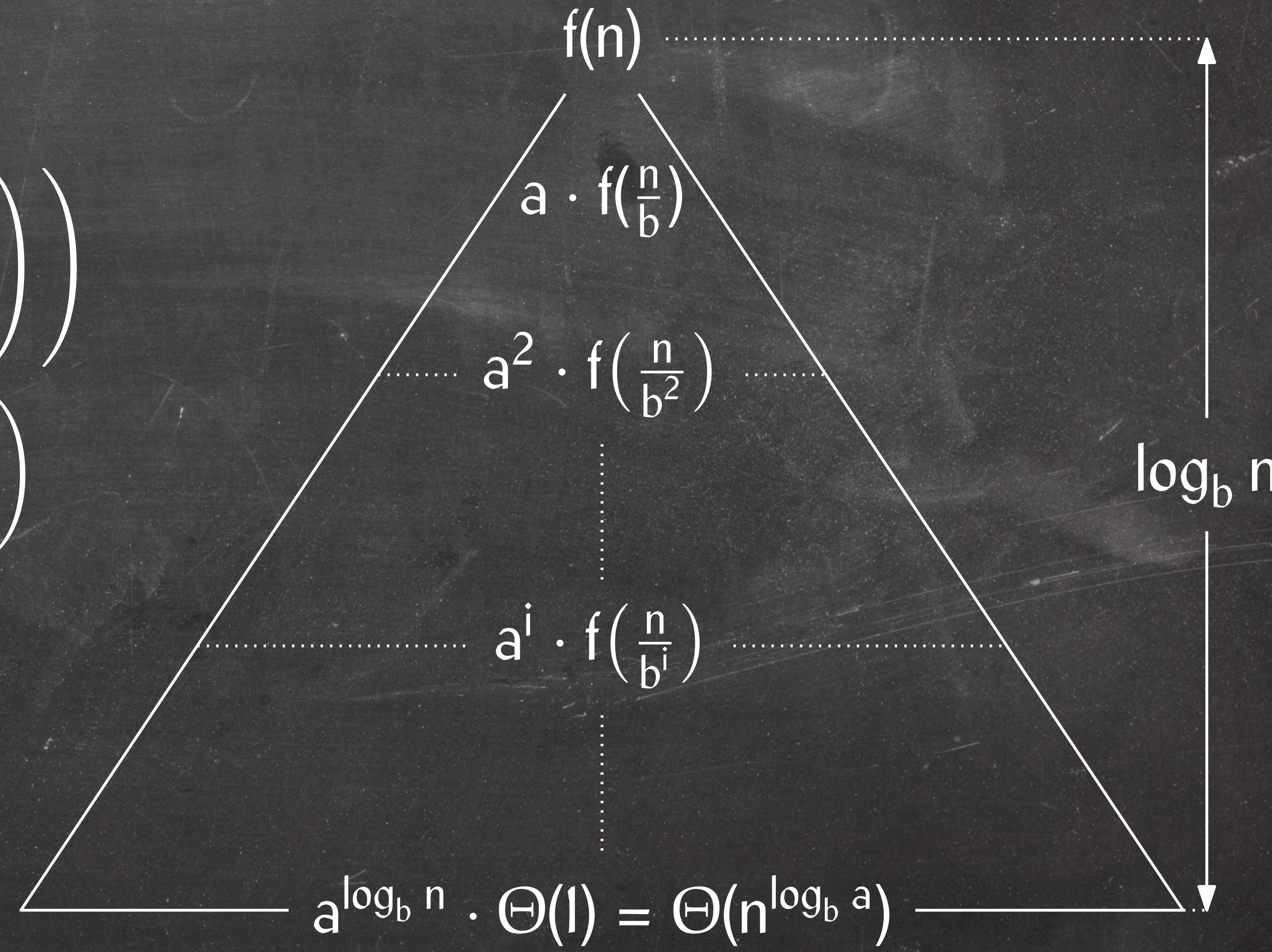

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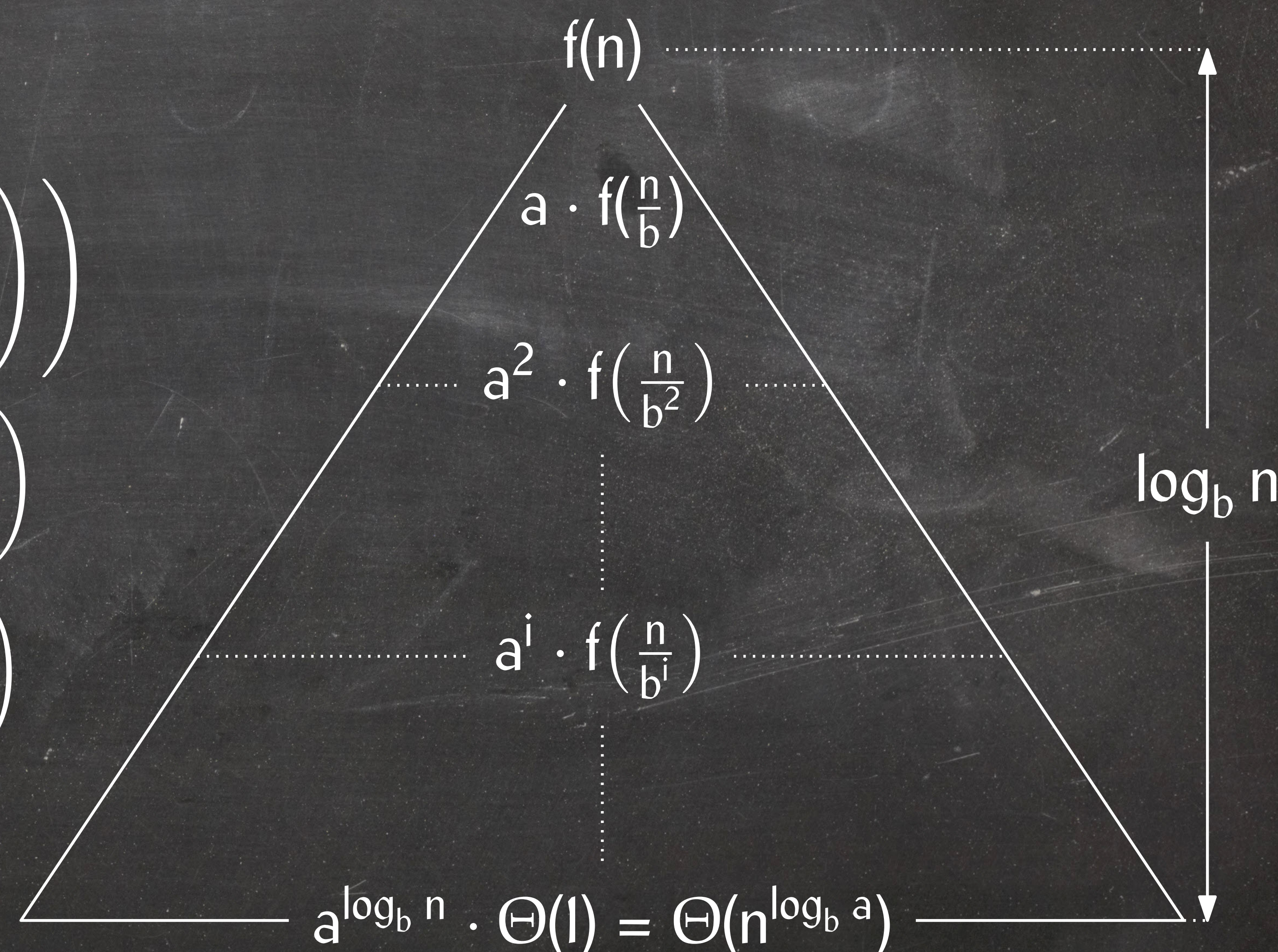

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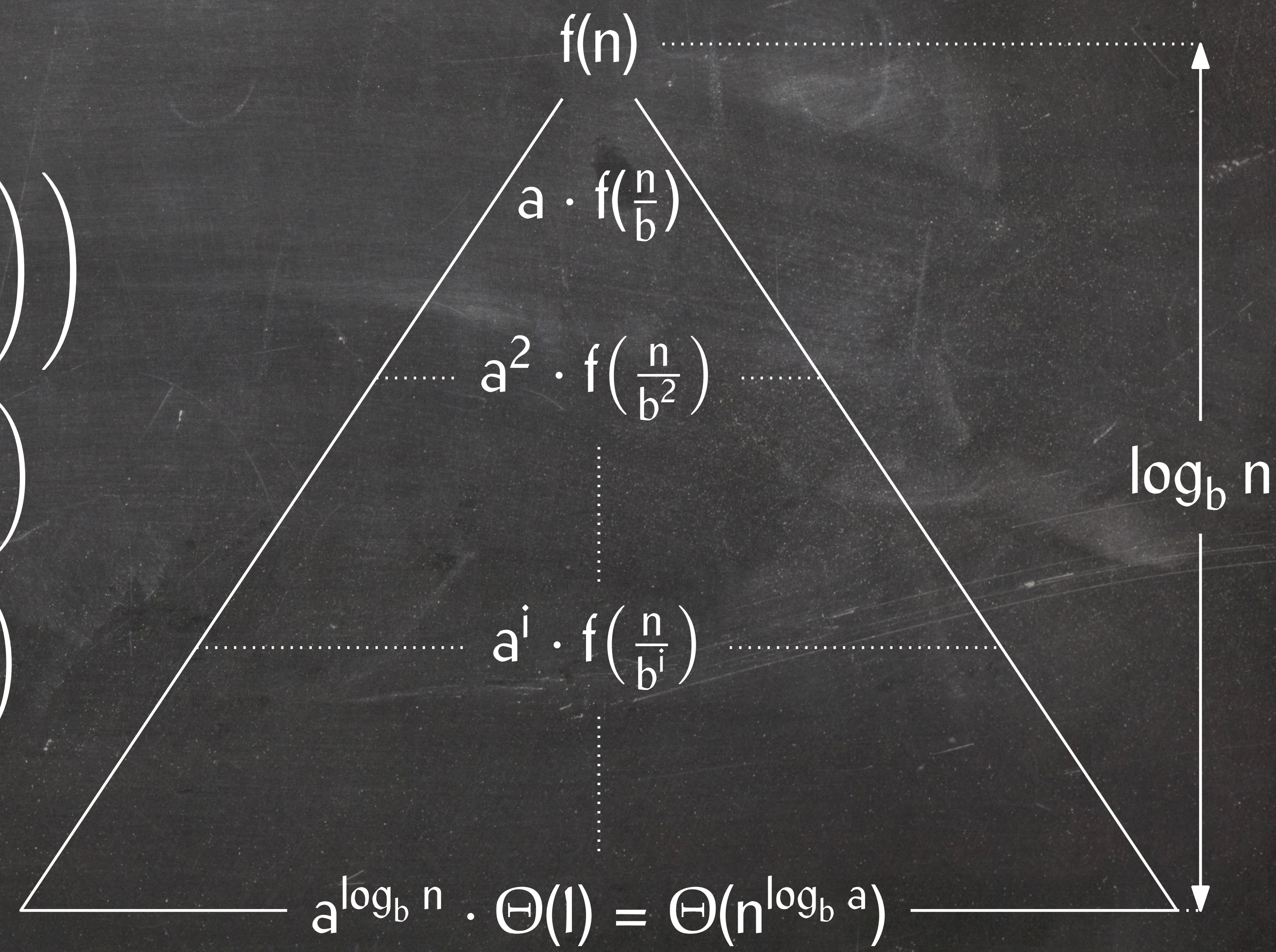

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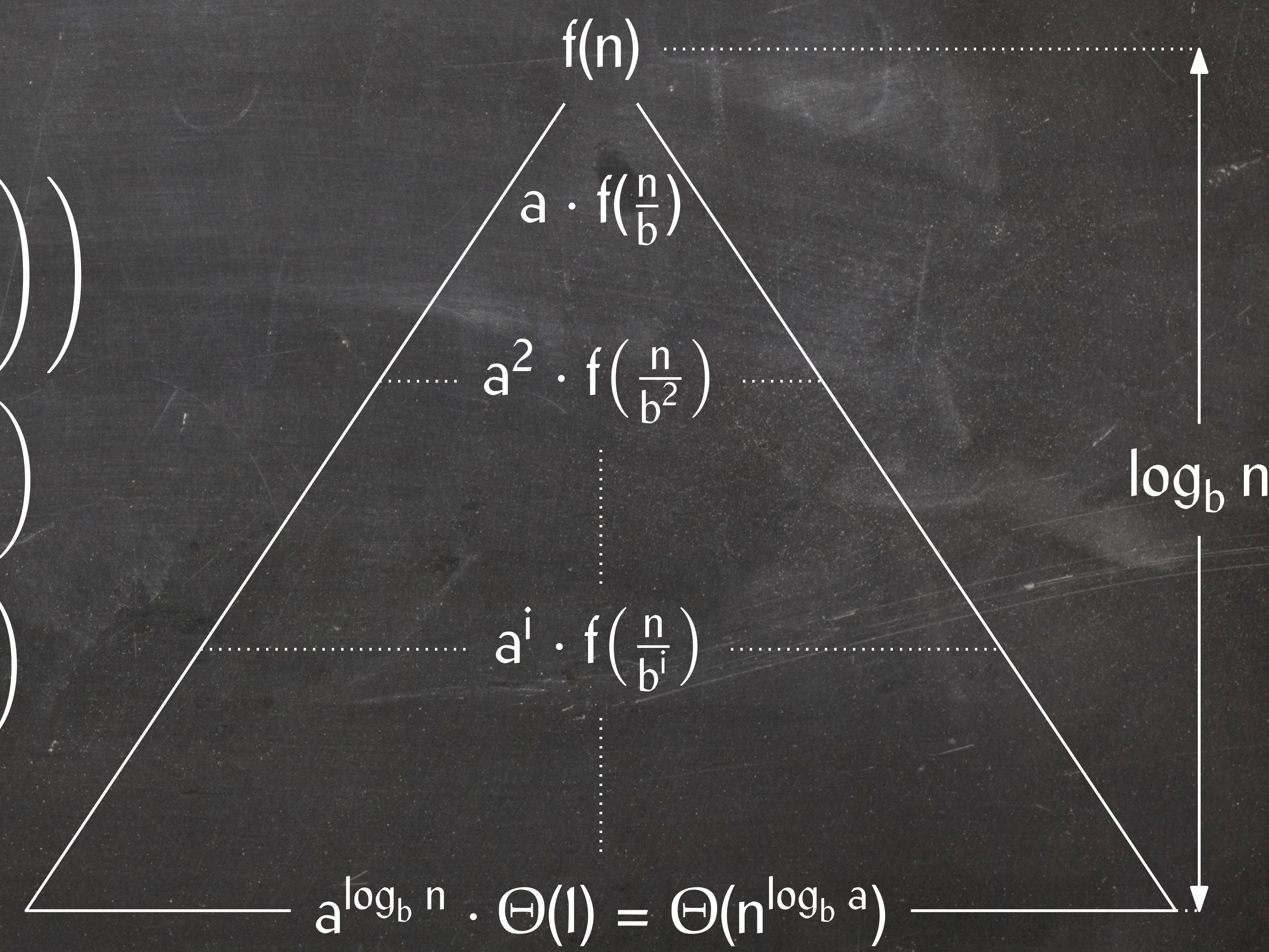

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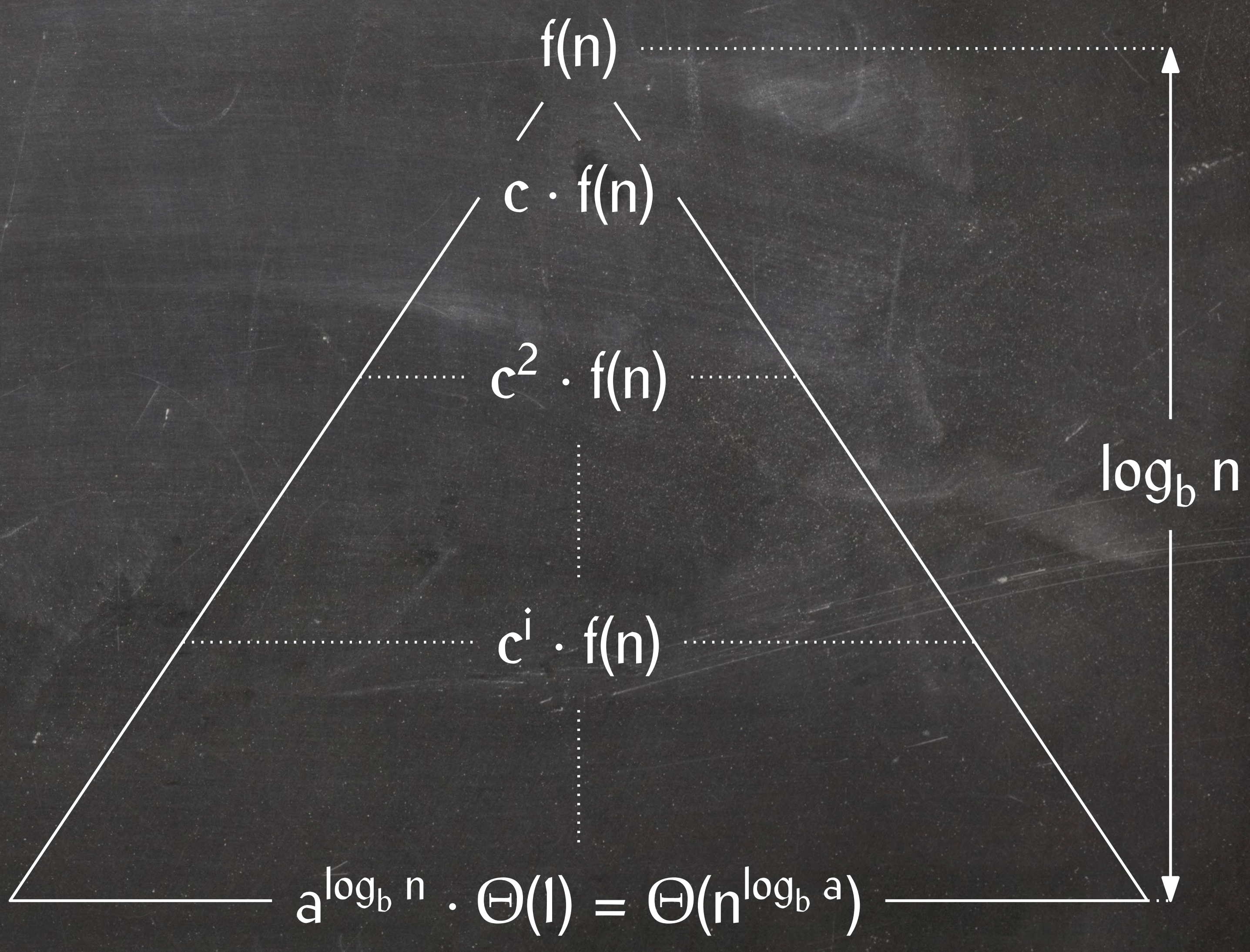

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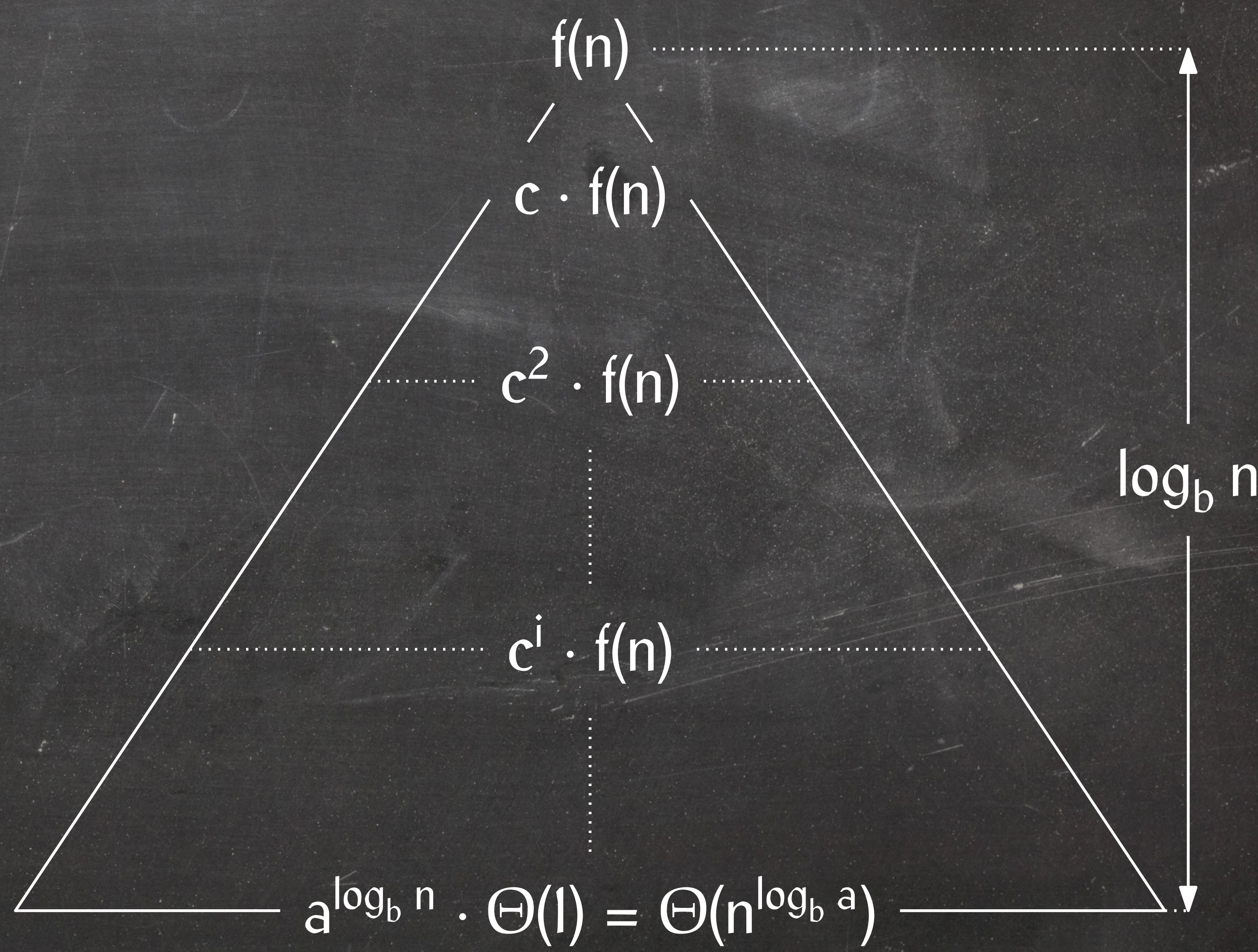

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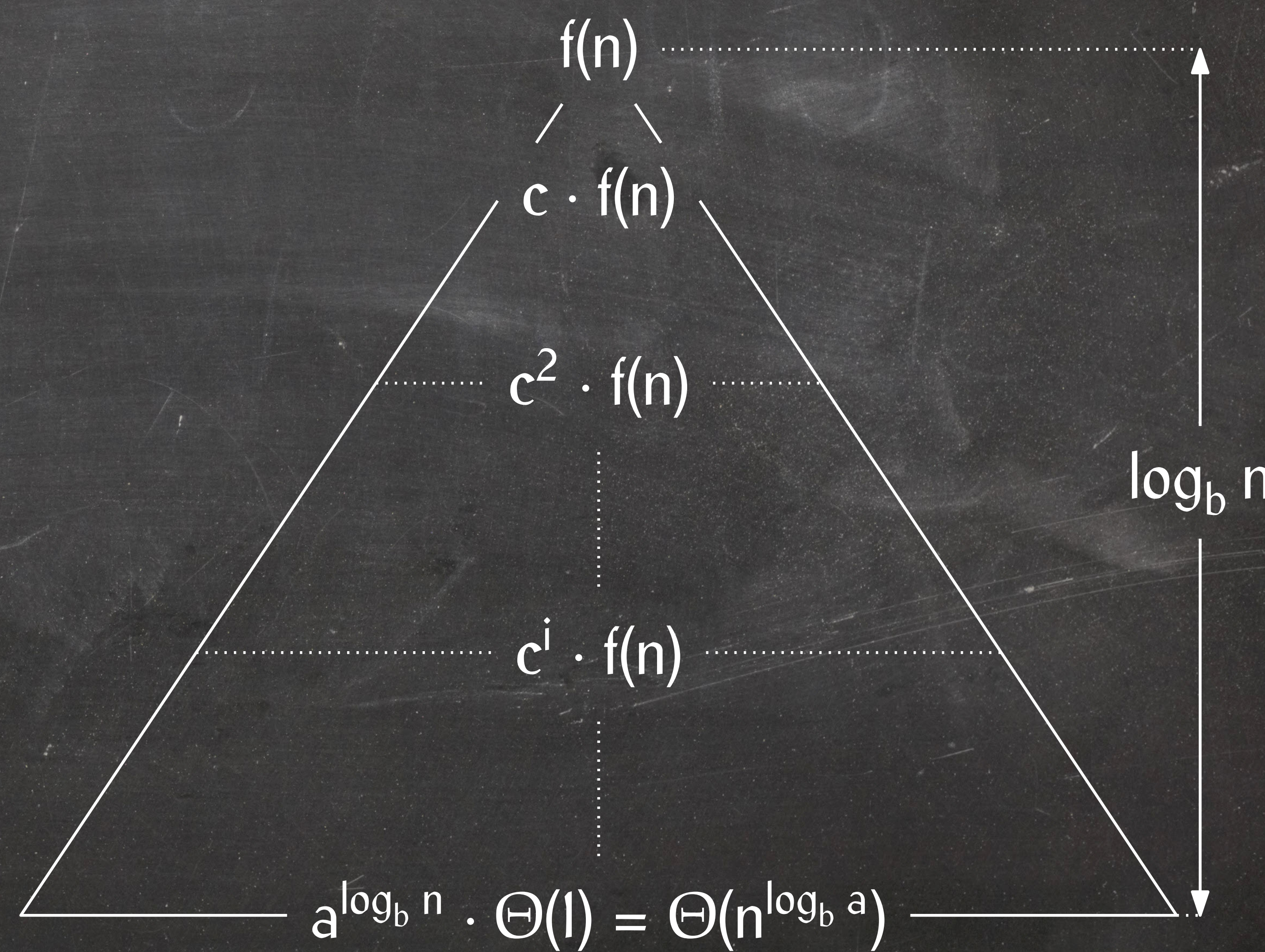

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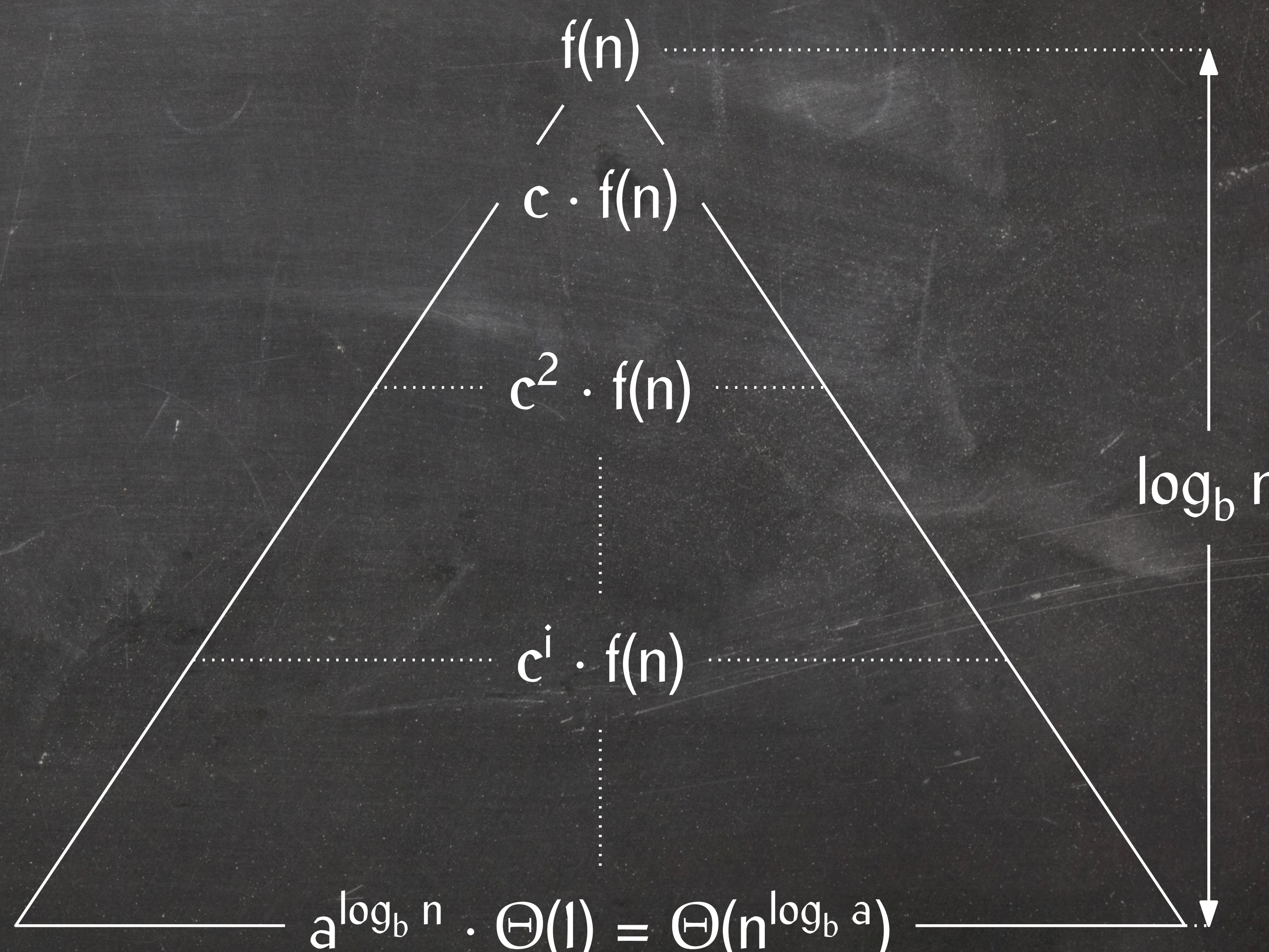

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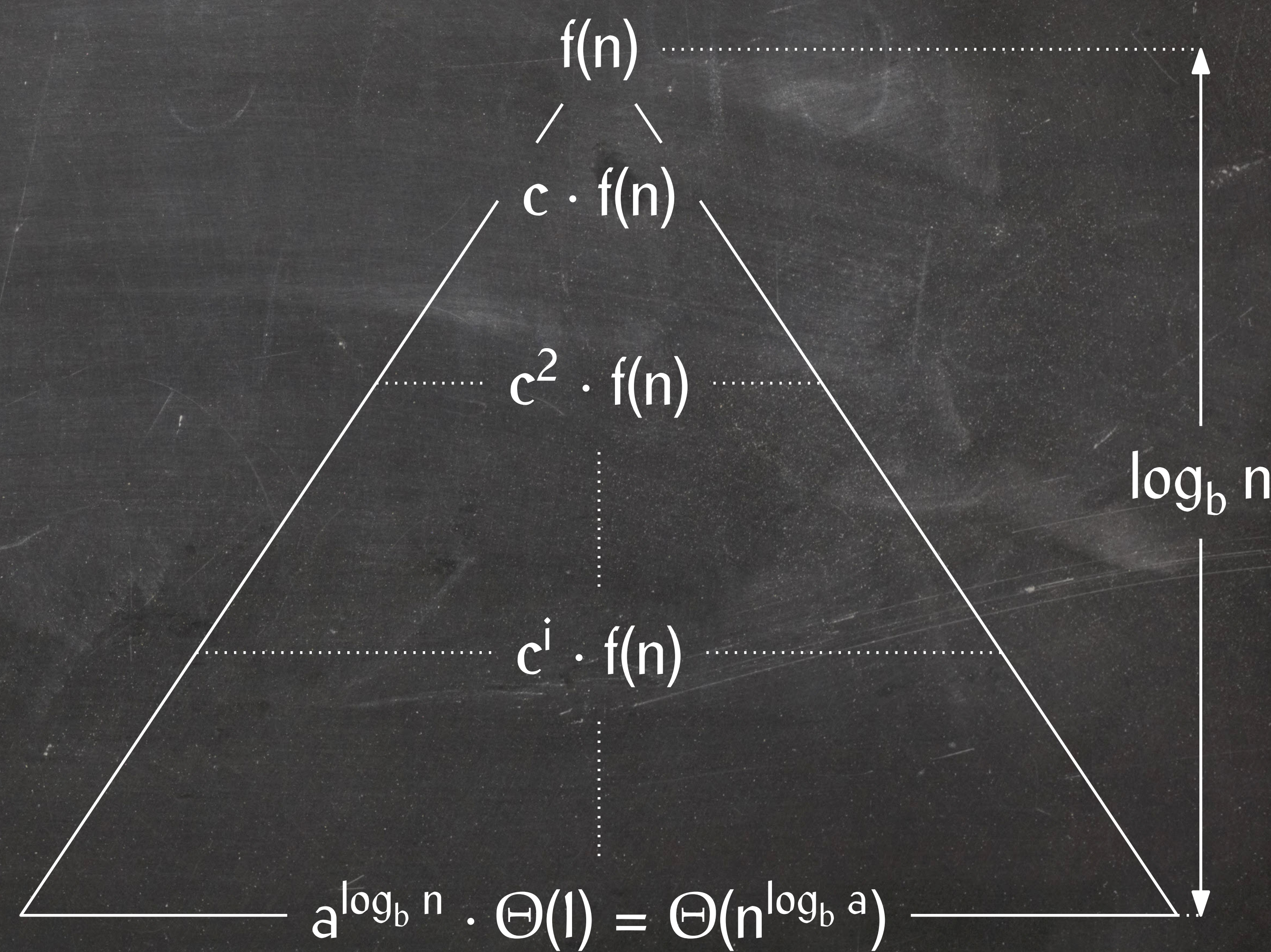

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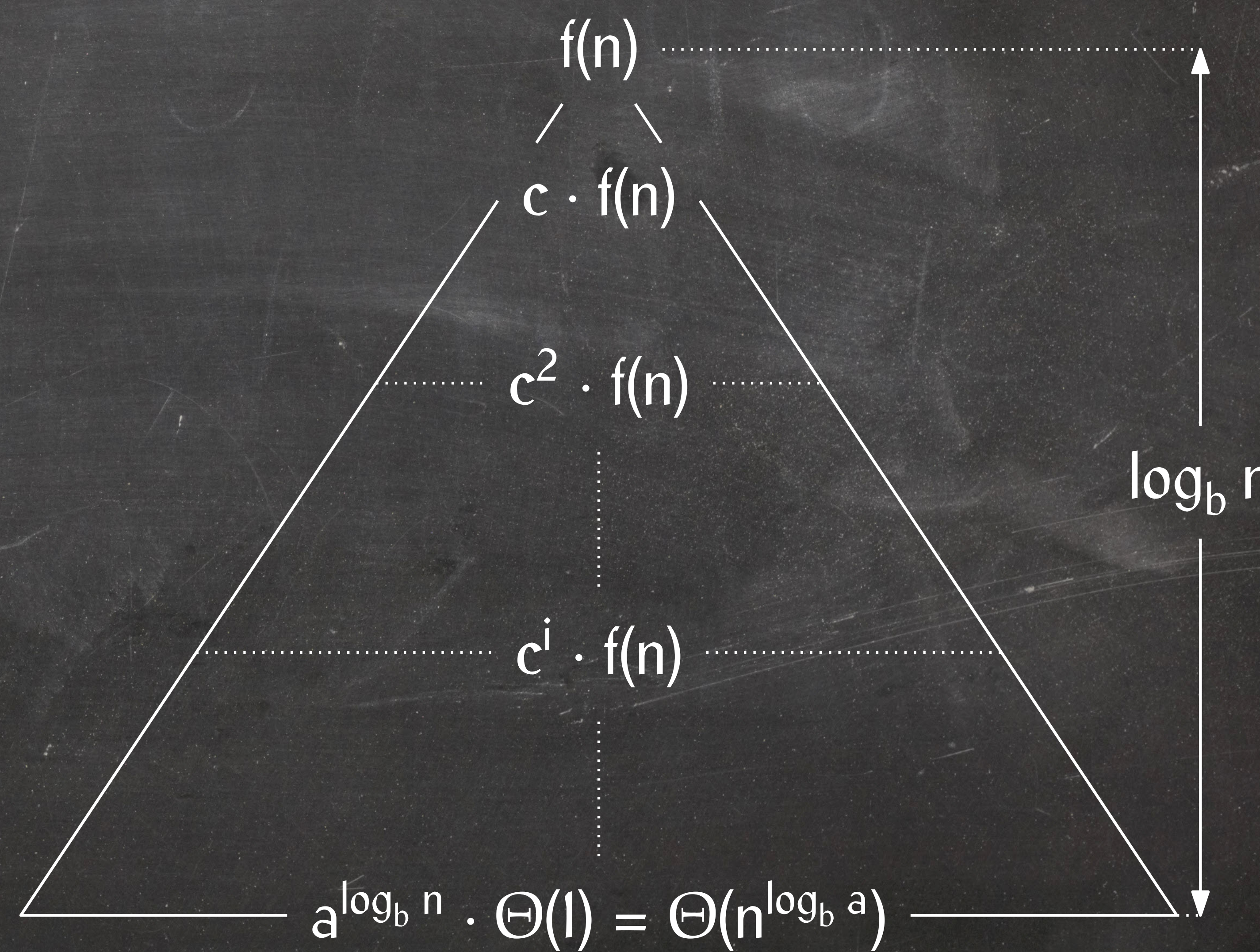

Quick Sort

QuickSort(A, ℓ, r)

```
if r ≤ ℓ
    then return
m = Partition(A, ℓ, r)
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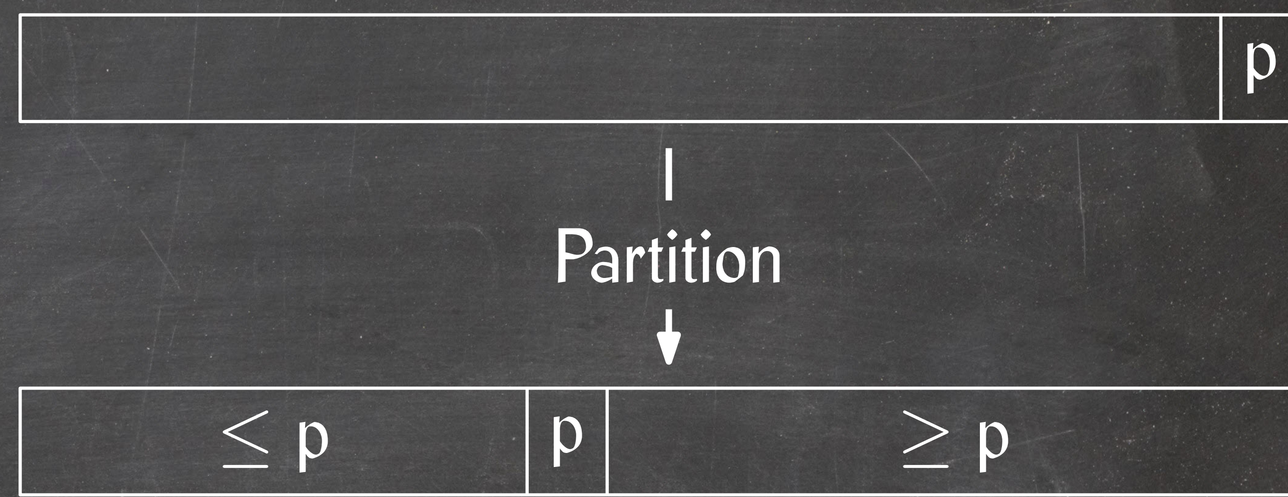

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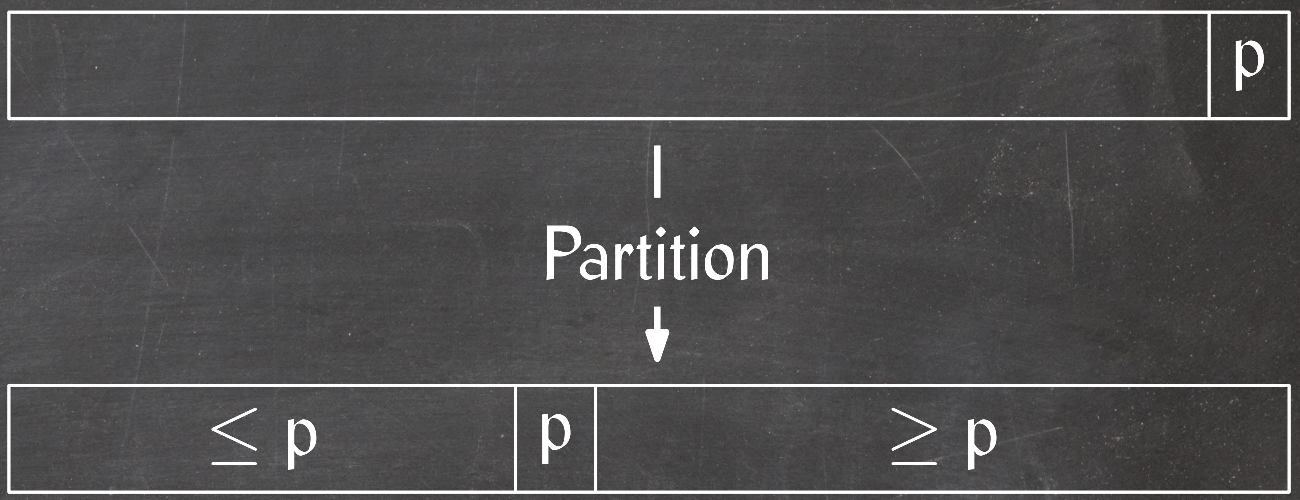

Worst case:

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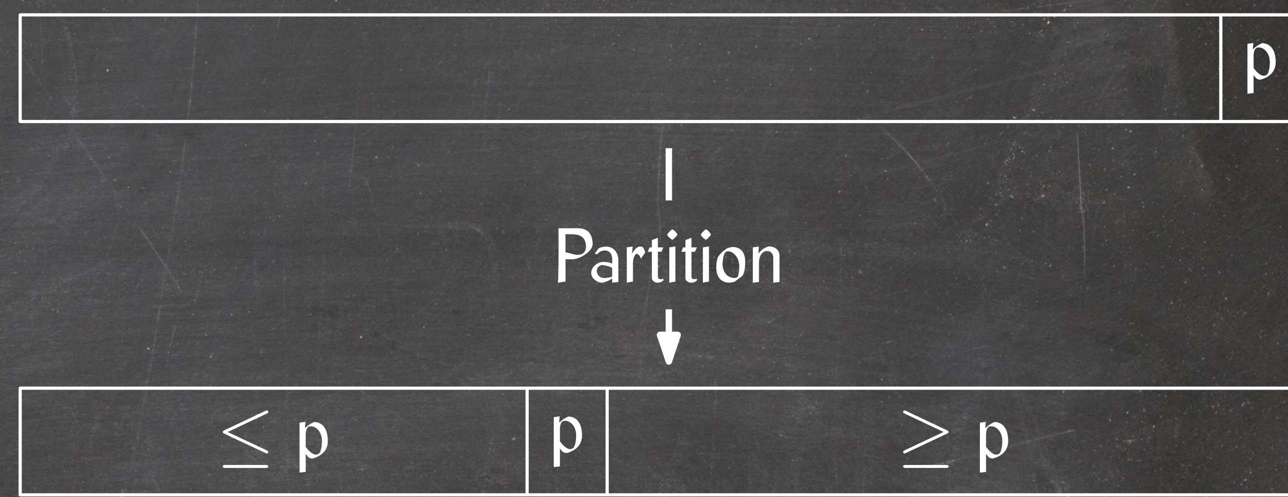

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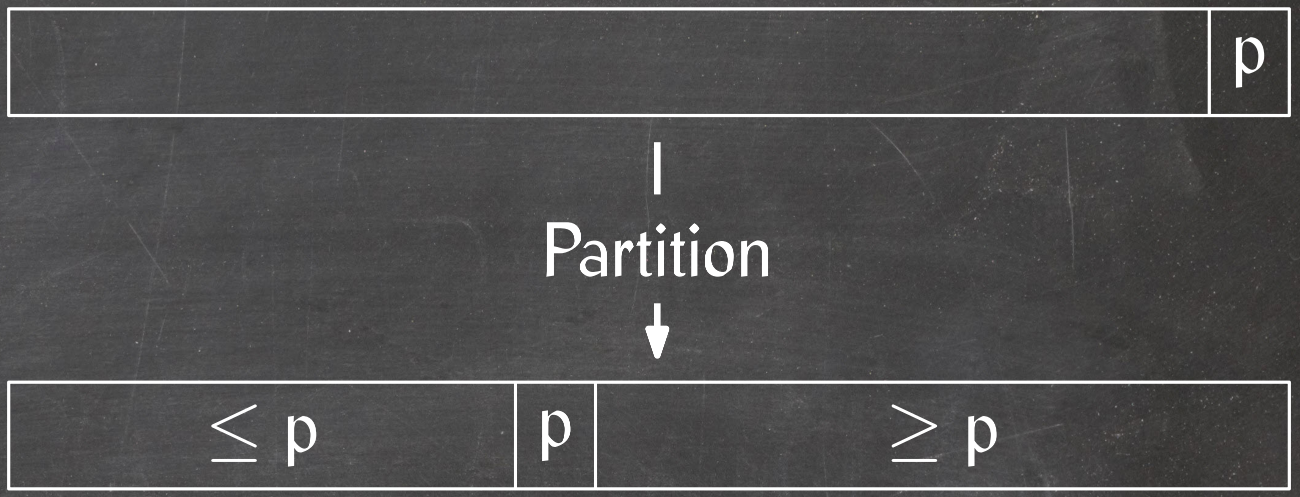

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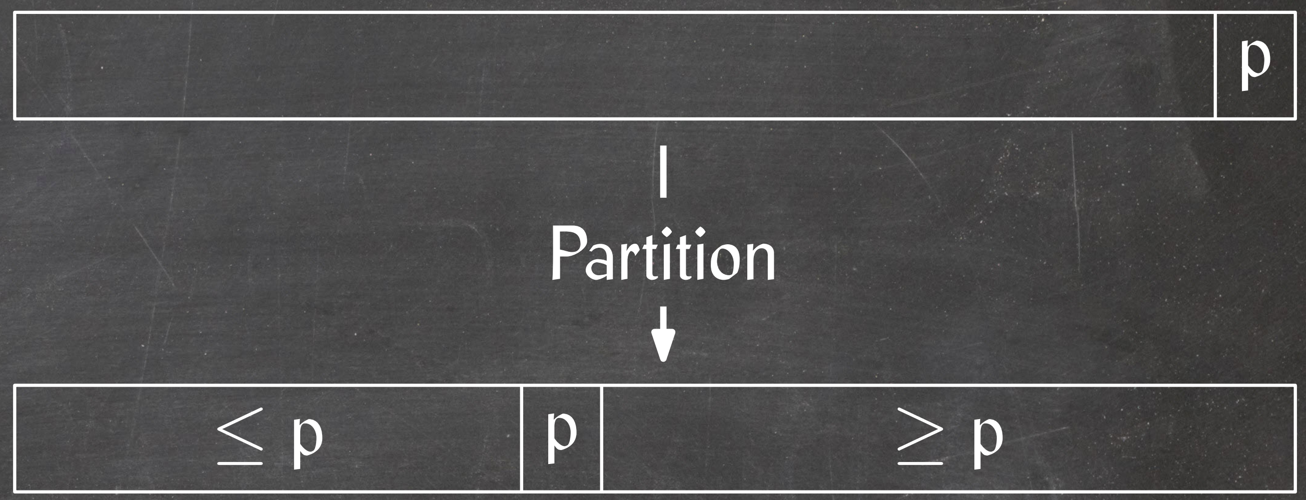

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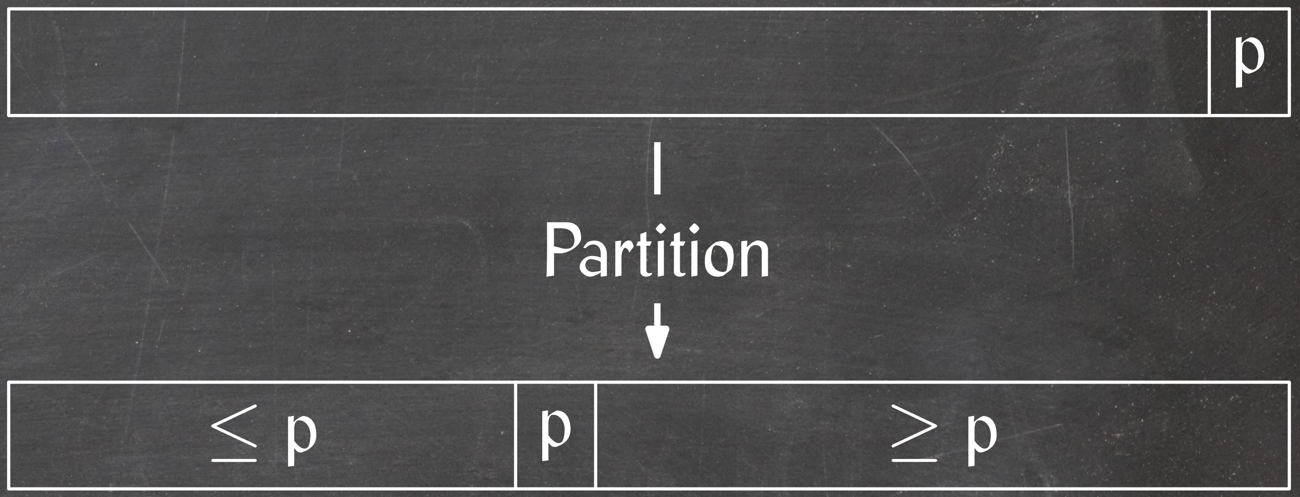

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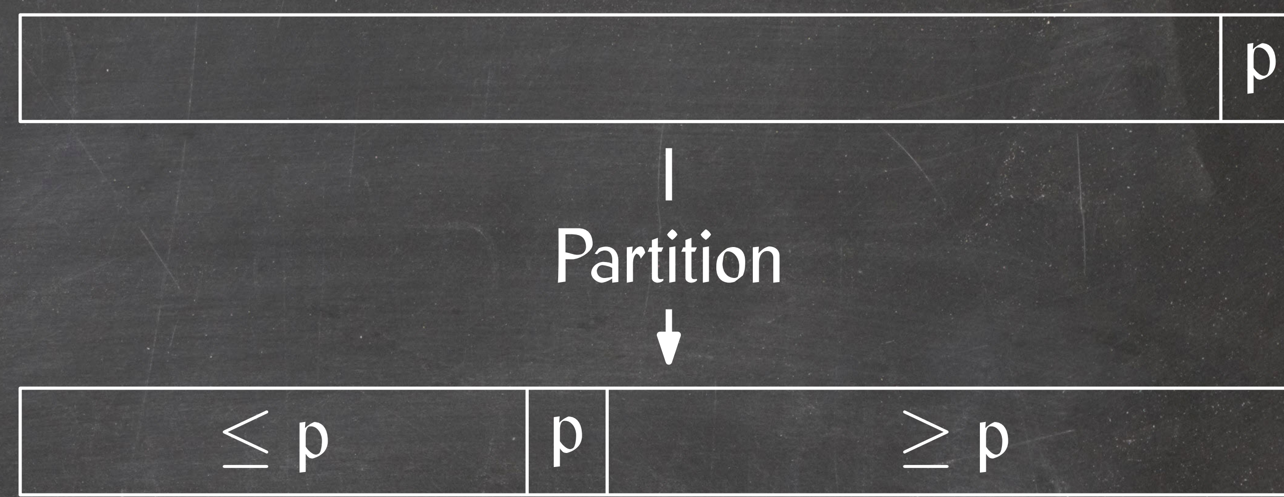

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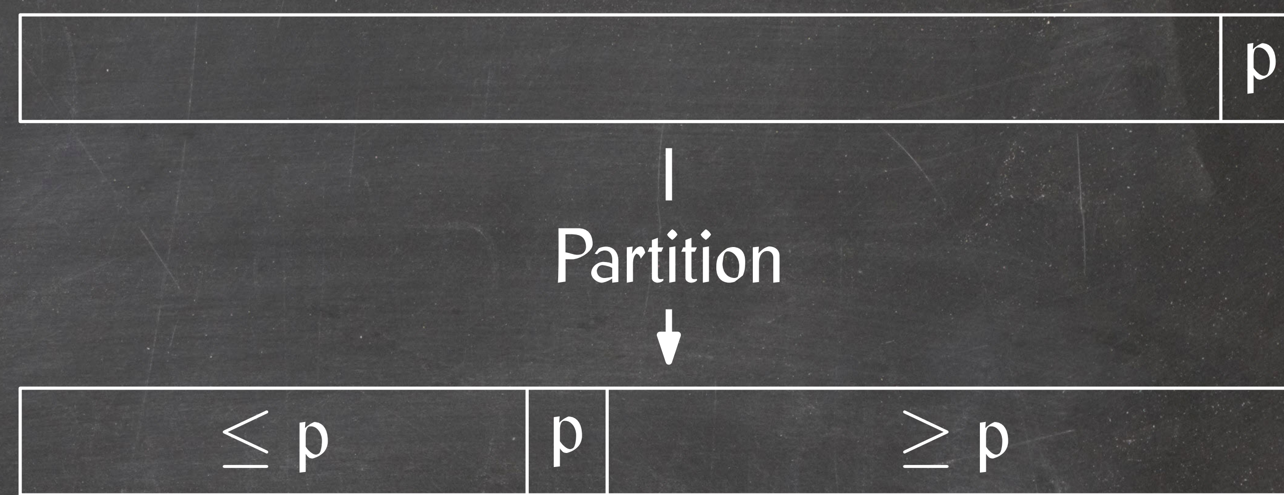

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Average case:

$$T(n) = \Theta(n \lg n)$$

Two Partitioning Algorithms

HoarePartition(A, l, r)

```
x = A[r]
i = l − 1
j = r + 1
while True
    do repeat i = i + 1
           until A[i] ≥ x
       repeat j = j − 1
           until A[j] ≤ x
       if i < j
           then swap A[i] and A[j]
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Loop invariants:

Two Partitioning Algorithms

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LomutoPartition(A, l, r)

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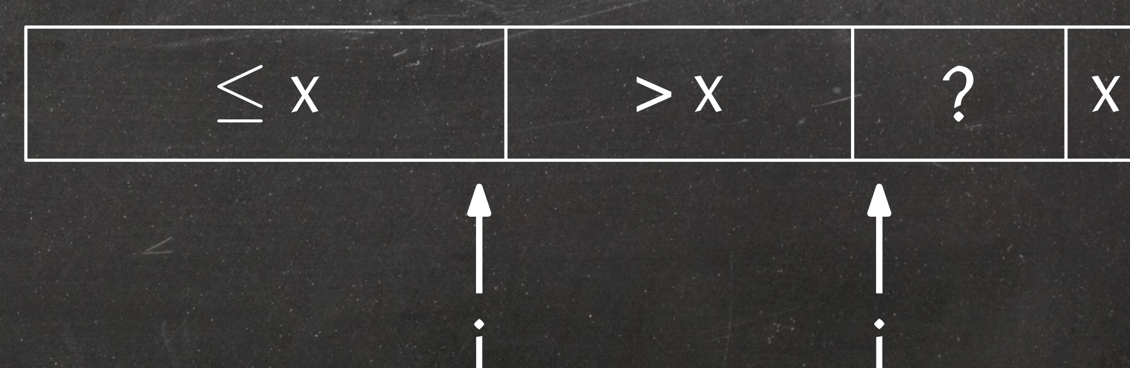
Loop invariants:



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HoarePartition is more efficient in practice.

LomutoPartition has some properties that make average-case analysis easier.

HoarePartition is more convenient for worst-case Quick Sort.

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Selection

Goal: Given an unsorted array A and an integer $1 \leq k \leq n$, find the kth smallest element in A.

8	17	5	43	3	12	64	21

$\uparrow$

4th smallest element

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It would be nice if we were able to find the kth smallest element, for any k, in $O(n)$ time.

The Sorting Idea Isn't Half Bad, But ...

To find the kth smallest element, we don't need to sort the input completely.

We only need to verify that there are exactly $k - 1$ elements smaller than the element we return.

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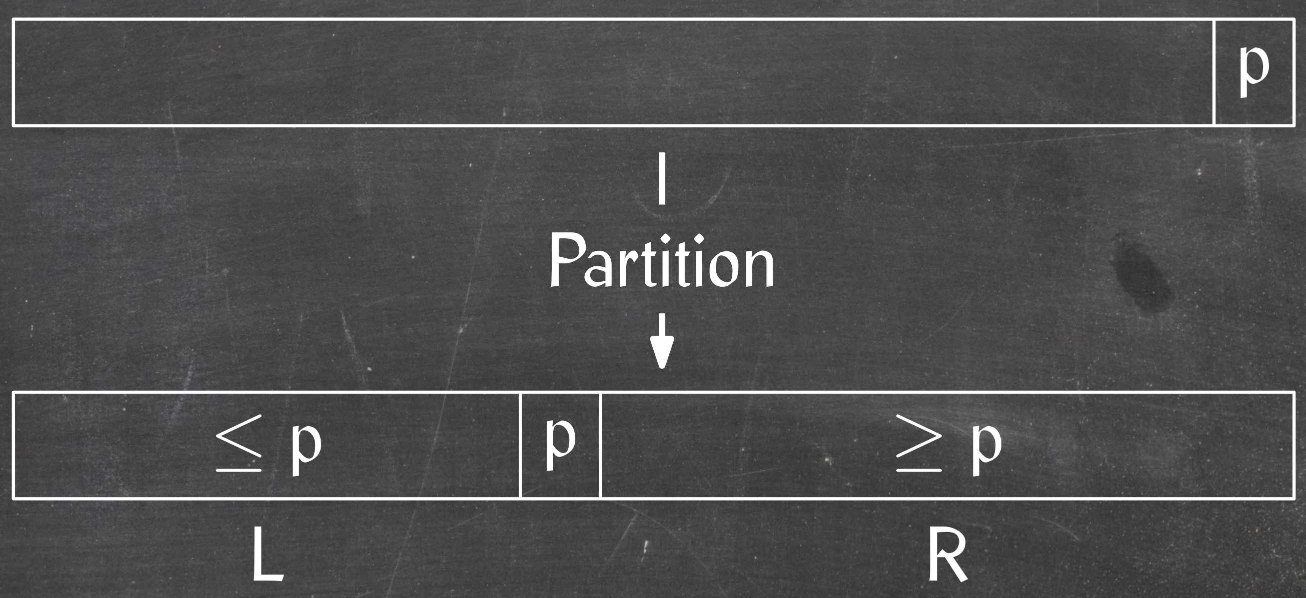

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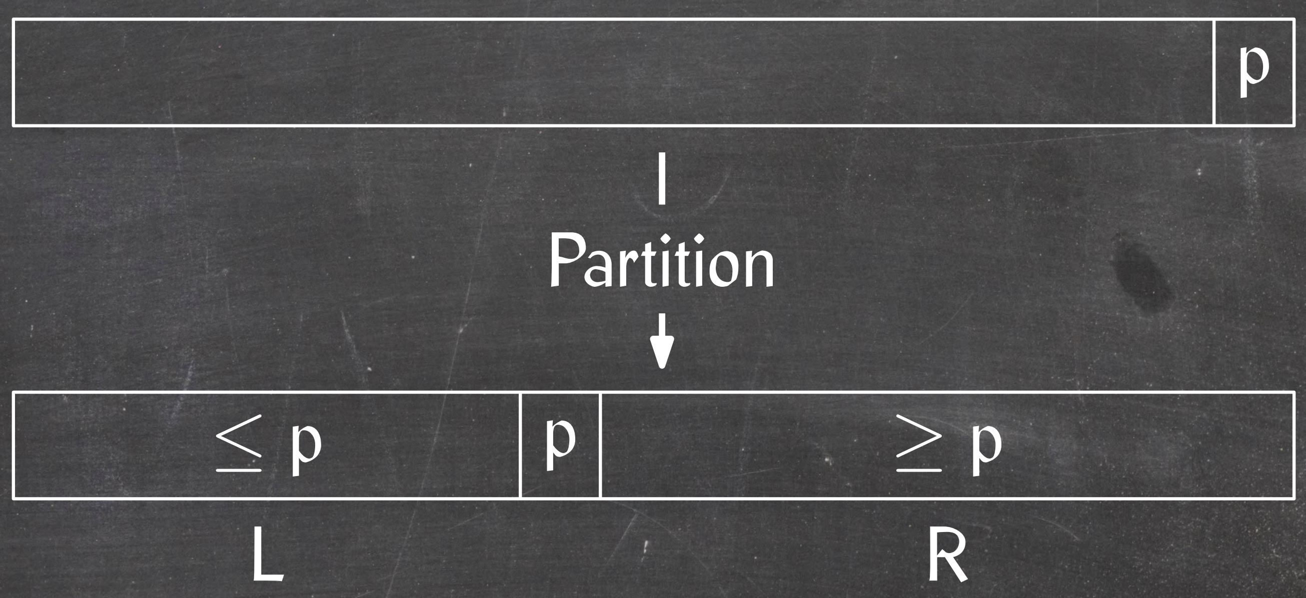

If $|L| = k - 1$, then p is the kth smallest element.

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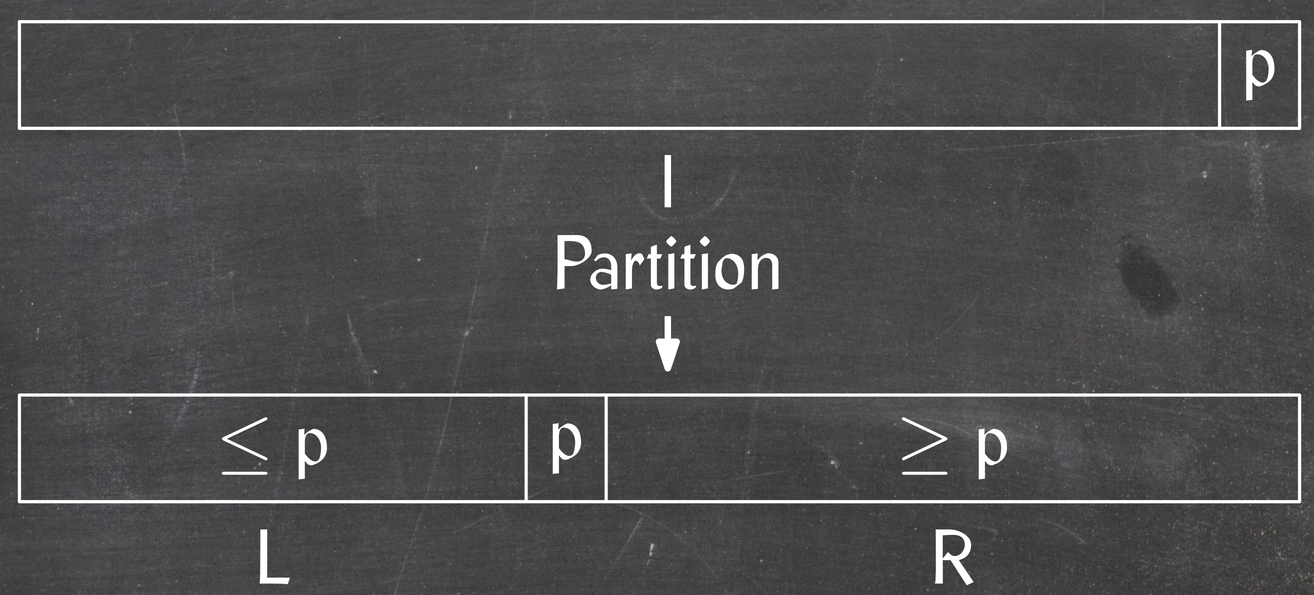
We only need to verify that there are exactly $k - 1$ elements smaller than the element we return.



If $|L| = k - 1$, then p is the kth smallest element.

If $|L| \geq k$, then the kth smallest element in L is the kth smallest element in A.

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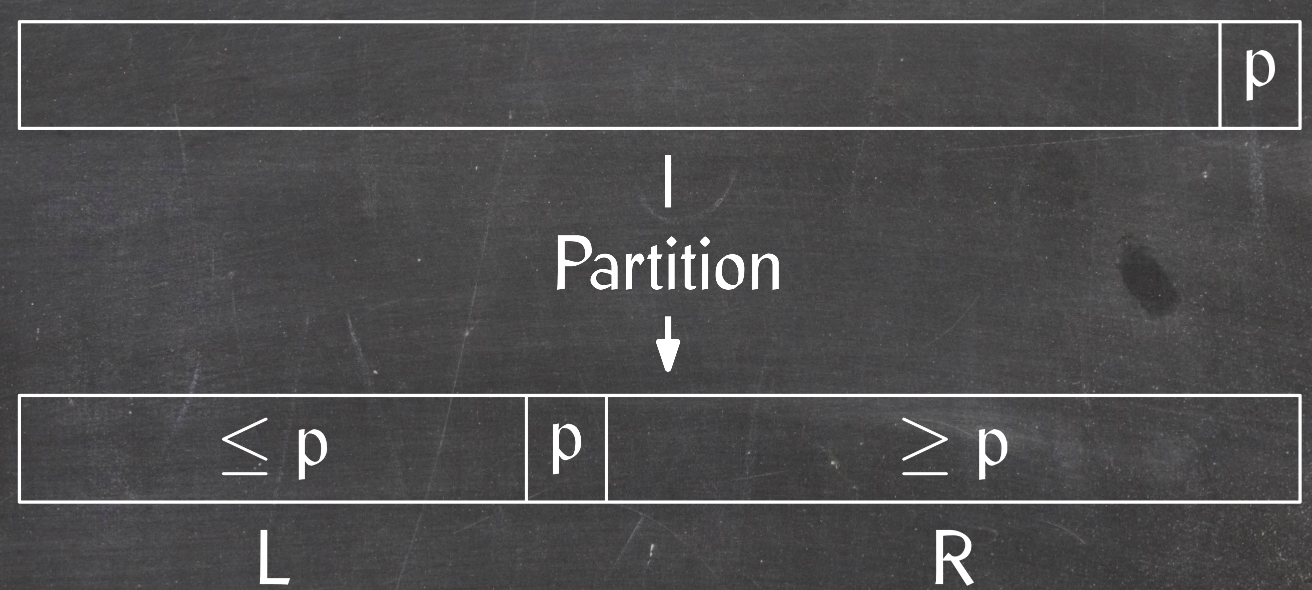


If $|L| = k - 1$, then p is the kth smallest element.

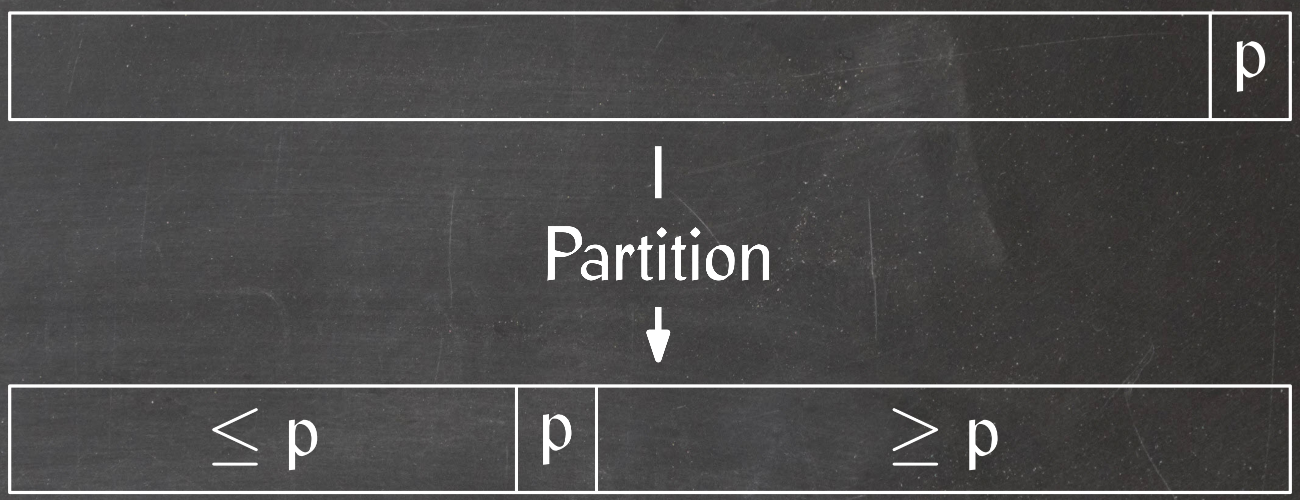
If $|L| \geq k$, then the kth smallest element in L is the kth smallest element in A.

If $|L| < k - 1$, then the $(k - |L| + 1)$st element in R is the kth smallest element in A.

Quick Select

QuickSelect(A, ℓ, r, k)

```
if r ≤ ℓ
    then return A[ℓ]
m = Partition(A, ℓ, r)
if m − ℓ = k − 1
    then return A[m]
    else if m − ℓ ≥ k
            then return QuickSelect(A, ℓ, m − 1, k)
            else return QuickSelect(A, m + 1, r, k − (m + 1 − ℓ))
```



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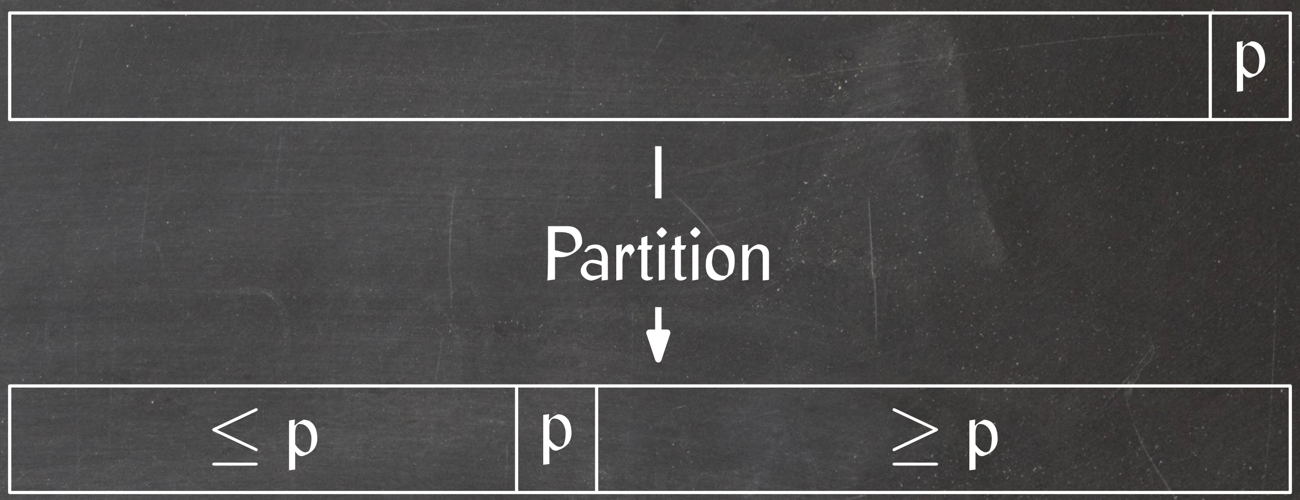

Worst case:

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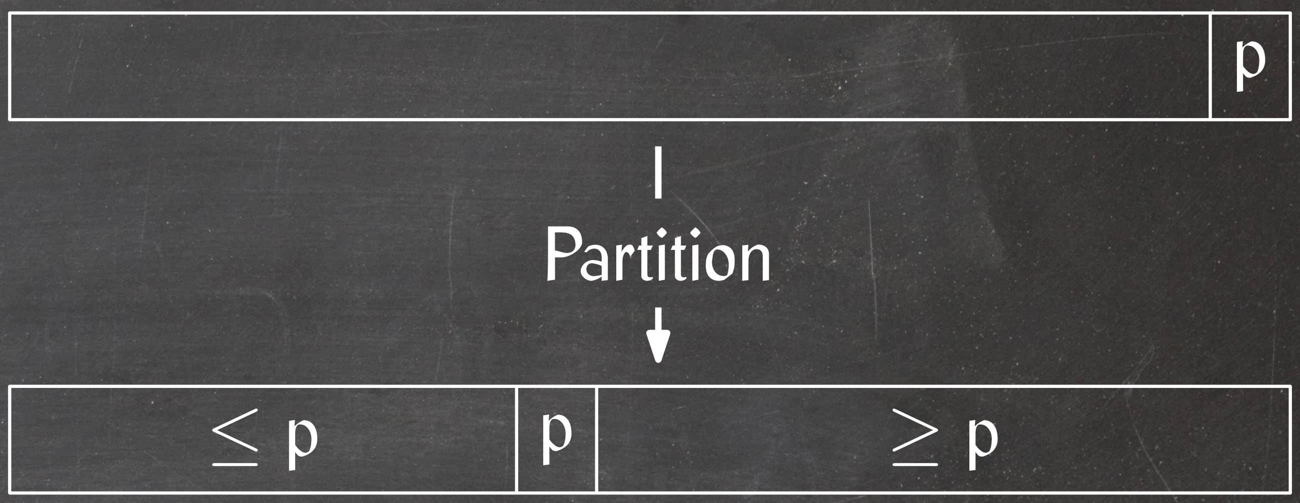

Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

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```

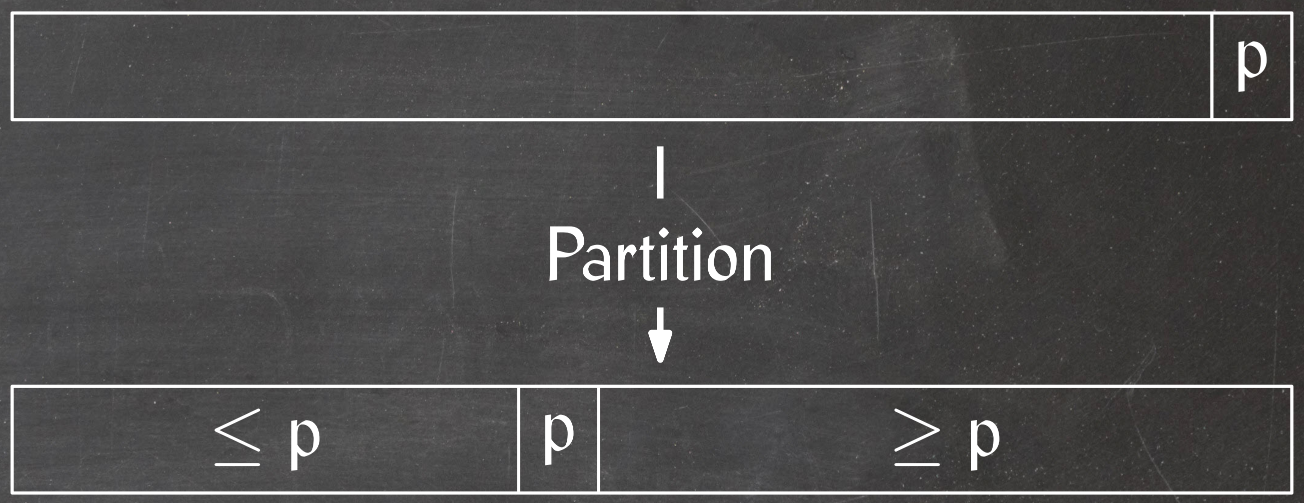

Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

Best case:

Quick Select

QuickSelect(A, ℓ, r, k)

```
if r ≤ ℓ
   then return A[ℓ]
m = Partition(A, ℓ, r)
if m − ℓ = k − 1
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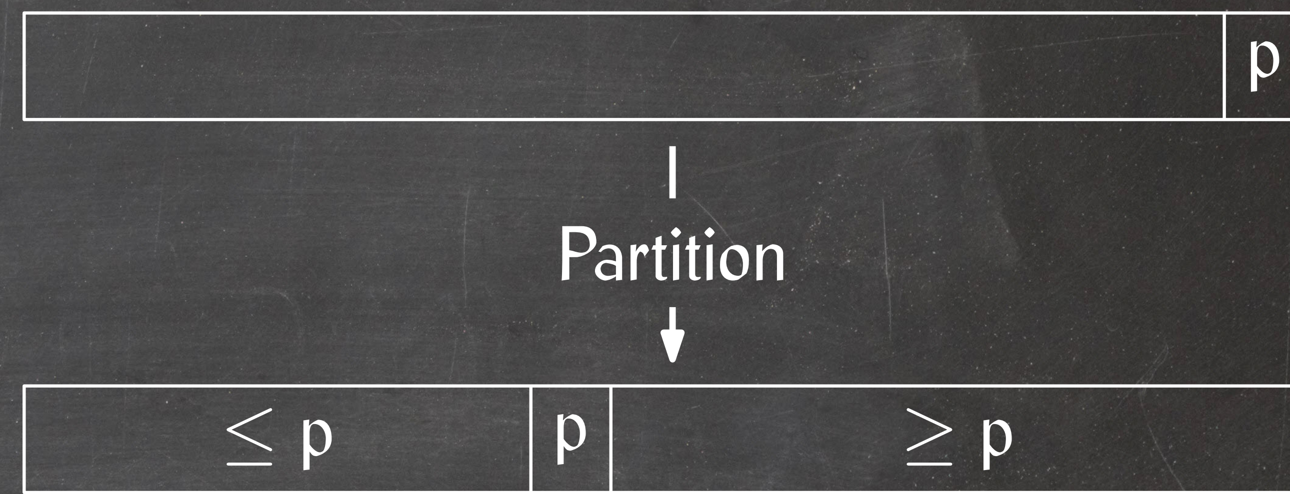

Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

Best case:

$$T(n) = \Theta(n)$$

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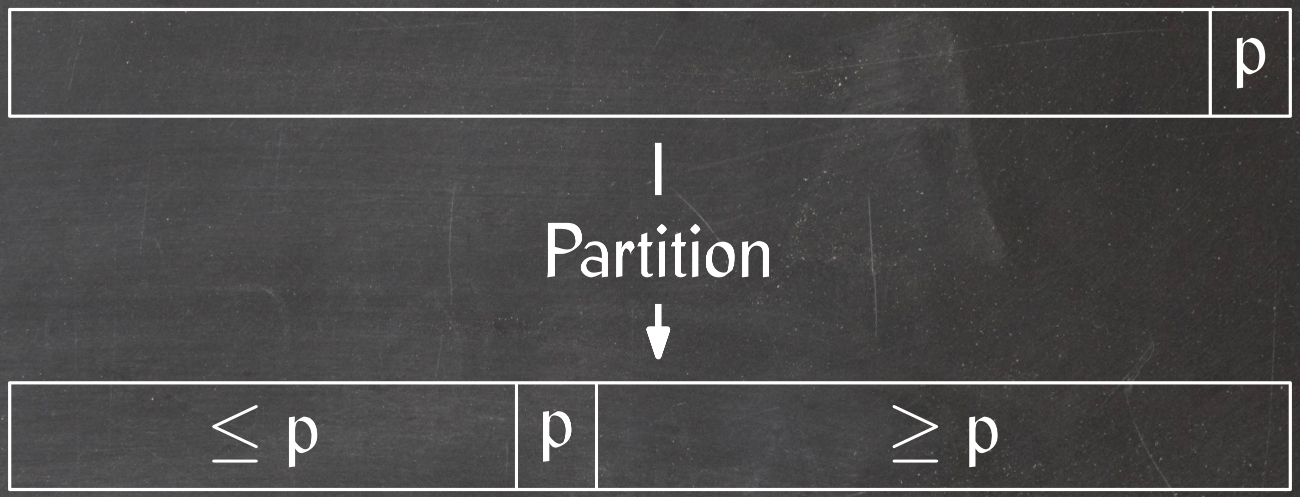

Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

Best case:

$$T(n) = \Theta(n) + T(n/2) = \Theta(n)$$

Quick Select

QuickSelect(A, ℓ, r, k)

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if r ≤ ℓ
    then return A[ℓ]
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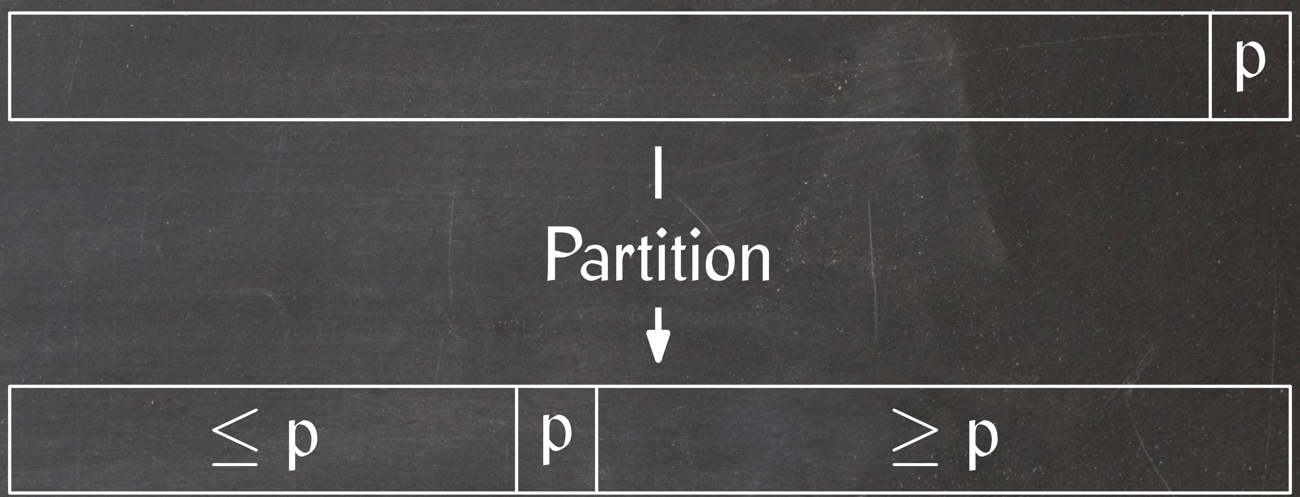

Worst case:

$$T(n) = \Theta(n) + T(n-1) = \Theta(n^2)$$

Best case:

$$T(n) = \Theta(n) + T(n/2) = \Theta(n)$$

Average case:

$$T(n) = \Theta(n)$$

Worst-Case Selection

QuickSelect(A, ℓ, r, k)

```
if r ≤ ℓ
    then return A[ℓ]
p = FindPivot(A, ℓ, r)
m = HoarePartition(A, ℓ, r, p)
if m − ℓ + 1 ≥ k
    then return QuickSelect(A, ℓ, m, k)
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Worst-Case Selection

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If we could guarantee that p is the median of A[ℓ . . r], then we'd recurse on at most n/2 elements.

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If we could guarantee that p is the median of $A[\ell \, .. \, r]$, then we'd recurse on at most n/2 elements.

$\Rightarrow \; T(n) = \Theta(n) + T(n/2) = \Theta(n)$.

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If we could guarantee that p is the median of $A[\ell \,..\, r]$, then we'd recurse on at most $n/2$ elements.

$\Rightarrow \ T(n) = \Theta(n) + T(n/2) = \Theta(n).$

Alas, finding the median is selection!

Making Do With An Approximate Median

$$\underbrace{\boxed{\quad \leq p \quad}}_{\geq \epsilon n}\boxed{p}\underbrace{\boxed{\quad \geq p \quad}}_{\geq \epsilon n}$$

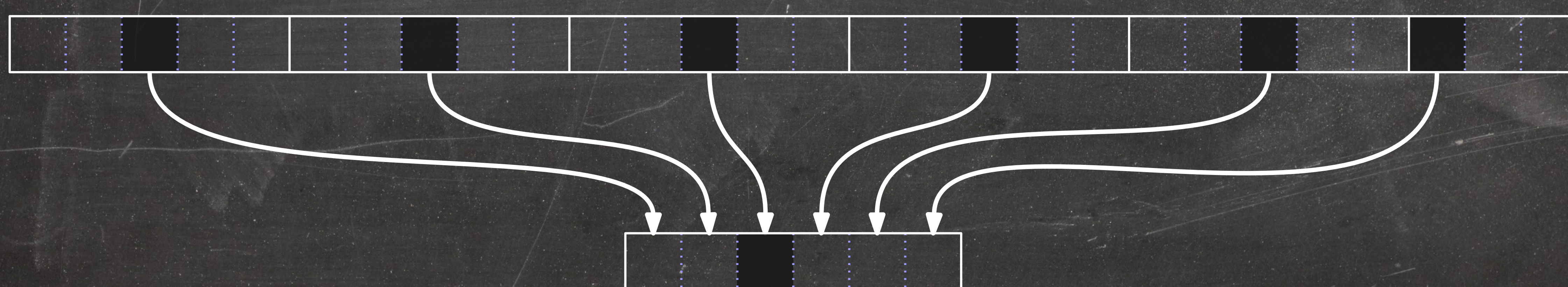
If there are at least ϵn elements smaller than p and at least ϵn elements greater than p, then

$$T(n) \leq \Theta(n) + T((1-\epsilon)n) = \Theta(n).$$

Finding An Approximate Median

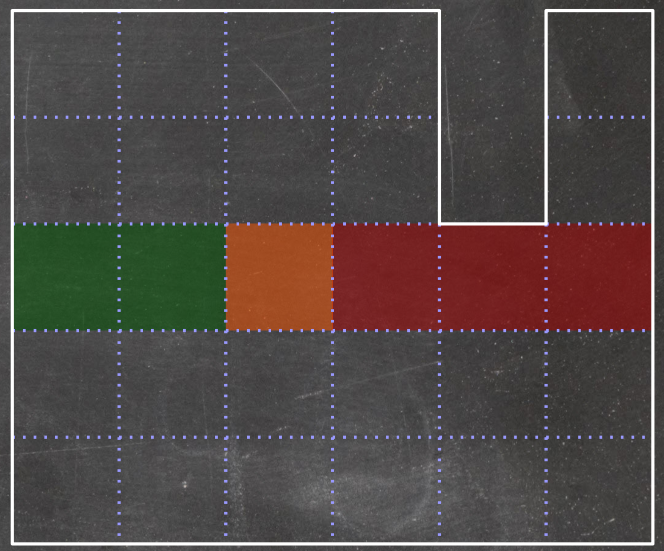
FindPivot(A, ℓ, r)

```
n′ = ⌊(r − ℓ)/5⌋ + 1
for i = 0 to n′ − 1
    do InsertionSort(A, ℓ + 5 · i, min(ℓ + 5 · i + 4, r))
       if ℓ + 5i + 4 ≤ r
          then B[i + 1] = A[ℓ + 5 · i + 2]
          else B[i + 1] = A[ℓ + 5 · i]
return QuickSelect(B, 1, n′, ⌈n′/2⌉)
```



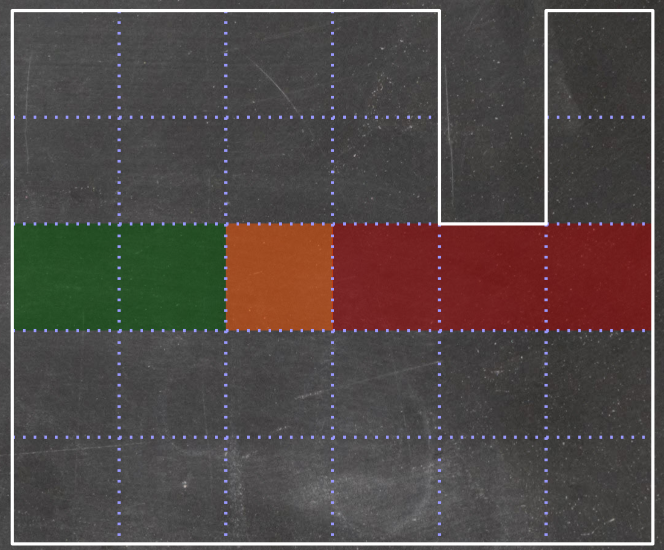
Approximate median

Finding An Approximate Median



Finding An Approximate Median

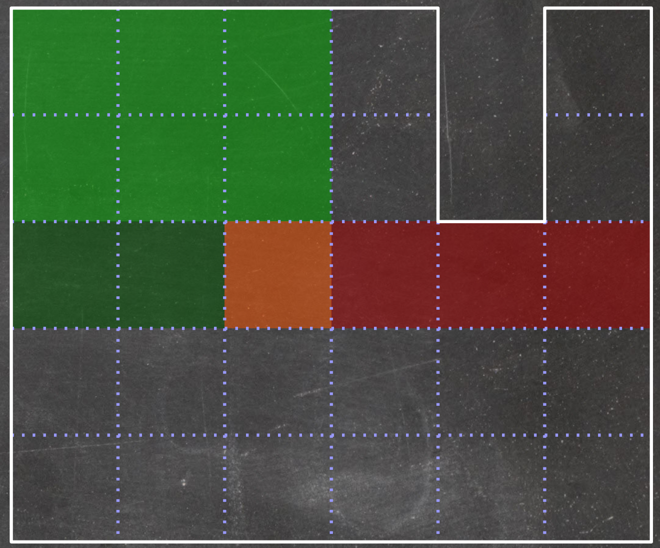
There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.



Finding An Approximate Median

There are at least $\lceil n'/2 \rceil - 1$ medians smaller than the median of medians.

For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.



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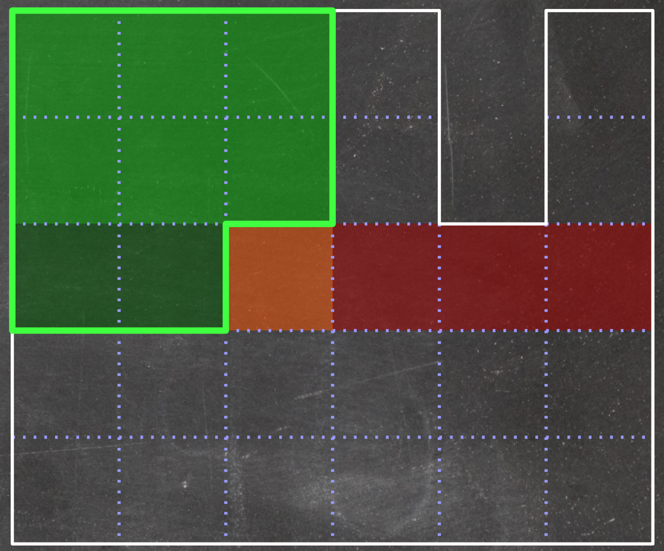

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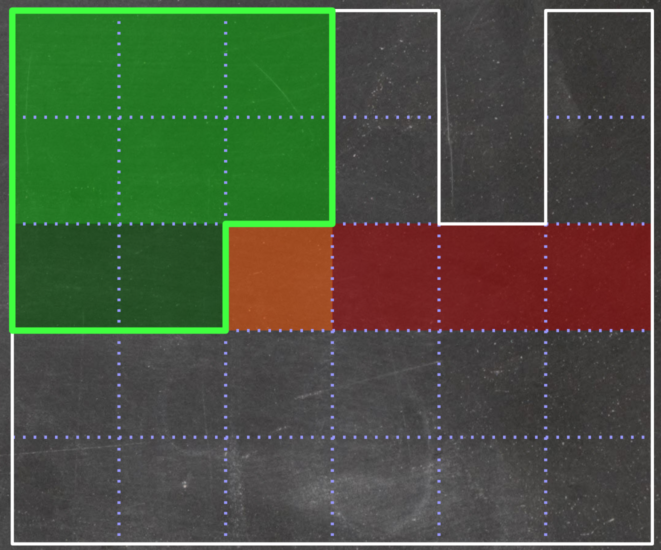
For at least $\lceil n'/2 \rceil - 1$ of the medians, there are two elements in each of their groups that are smaller than the median of medians.

$n' = \lceil n/5 \rceil$



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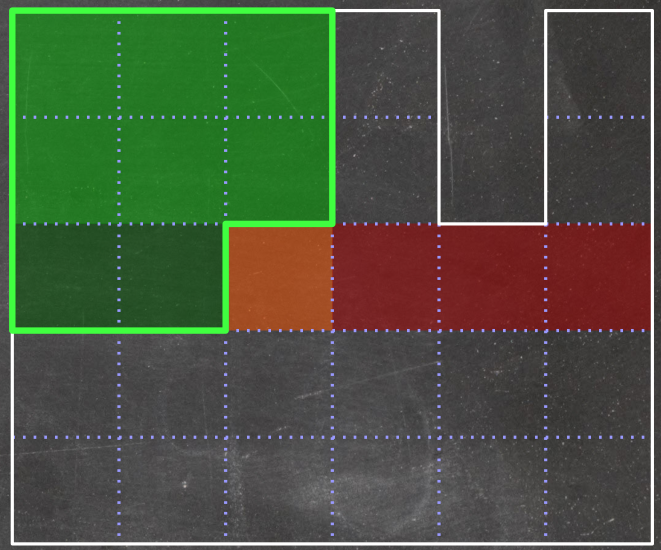

Total number of elements smaller than the median of medians:

$$3\left(\left\lceil \frac{n'}{2} \right\rceil - 1\right) = 3\left\lceil \frac{\lceil n/5 \rceil}{2} \right\rceil - 3 \geq \frac{3n}{10} - 3$$

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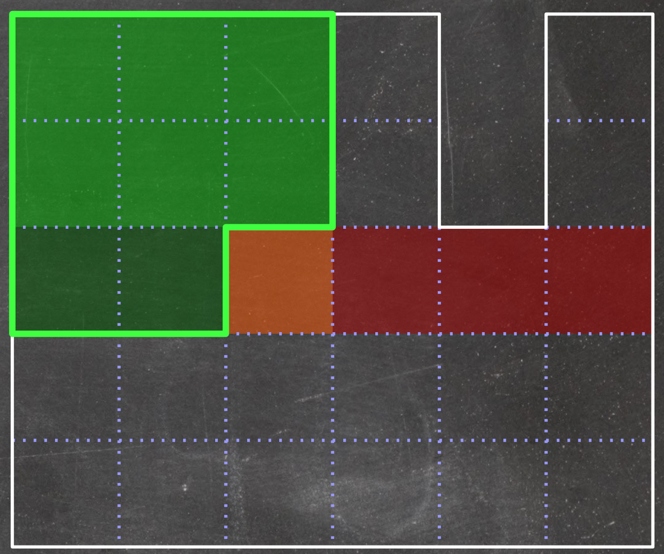

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The same analysis holds for counting the number of elements greater than the median of medians.

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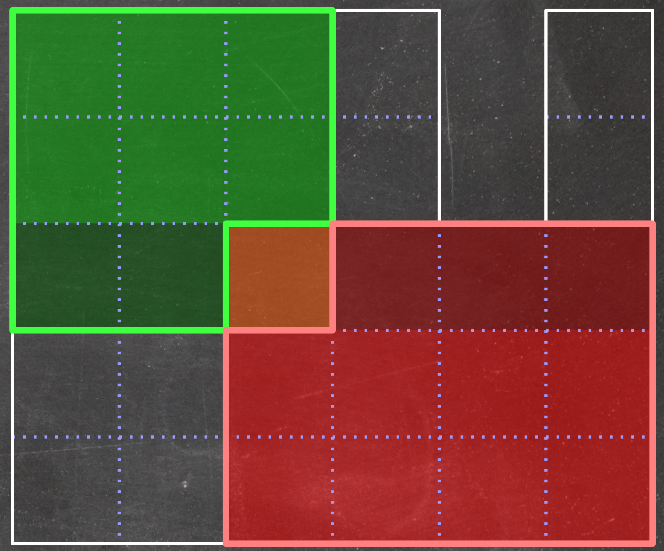

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Worst-Case Selection: Analysis

FindPivot (excluding the recursive call to QuickSelect) and Partition take $O(n)$ time.

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QuickSelect itself recurses on at most $\frac{7n}{10} + 3$ elements.

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QuickSelect itself recurses on at most $\frac{7n}{10} + 3$ elements.

$$T(n) \leq T(\tfrac{n}{5} + 1) + T(\tfrac{7n}{10} + 3) + O(n)$$

Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

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Base case: $(n < 80)$

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We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.

$\Rightarrow$ $T(n) \leq c' \leq cn$ for c sufficiently large.

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Inductive Step: $(n \geq 80)$

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Inductive Step: $(n \geq 80)$

$$T(n) \leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an$$

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$$\begin{aligned} T(n) &\leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an \\ &\leq c\left(\frac{n}{5} + 1\right) + c\left(\frac{7n}{10} + 3\right) + an \end{aligned}$$

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Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

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$$
\begin{aligned}
T(n) &\leq T\left(\frac{n}{5}+1\right) + T\left(\frac{7n}{10}+3\right) + an \\
&\leq c\left(\frac{n}{5}+1\right) + c\left(\frac{7n}{10}+3\right) + an \\
&\leq c\left(\frac{4n}{20}+\frac{14n}{20}+\frac{n}{20}\right) + an
\end{aligned}
$$

Worst-Case Selection: Analysis

Claim: $T(n) \in O(n)$, that is, $T(n) \le cn$, for some $c > 0$ and all $n \ge 1$.

Base case: $(n < 80)$

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.
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Inductive Step: $(n \ge 80)$

$$\begin{aligned} T(n) &\le T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an \\ &\le c\left(\frac{n}{5} + 1\right) + c\left(\frac{7n}{10} + 3\right) + an \\ &\le c\left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an \\ &= \left(\frac{19c}{20} + a\right) n \end{aligned}$$

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Claim: $T(n) \in O(n)$, that is, $T(n) \leq cn$, for some $c > 0$ and all $n \geq 1$.

Base case: $(n < 80)$

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\begin{aligned}
T(n) &\leq T\left(\frac{n}{5} + 1\right) + T\left(\frac{7n}{10} + 3\right) + an \\
&\leq c\left(\frac{n}{5} + 1\right) + c\left(\frac{7n}{10} + 3\right) + an \\
&\leq c\left(\frac{4n}{20} + \frac{14n}{20} + \frac{n}{20}\right) + an \\
&= \left(\frac{19c}{20} + a\right) n \\
&\leq cn \quad \forall c \geq 20a
\end{aligned}
$$

Worst-Case Quick Sort

QuickSort(A, ℓ, r)

```
if r ≤ ℓ
    then return
p = FindPivot(A, ℓ, r)
m = HoarePartition(A, ℓ, r, p)
return QuickSort(A, ℓ, m)
return QuickSort(A, m + 1, r)
```

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$T(n) = \Theta(n) + T(n_1) + T(n_2)$, where $n_1 + n_2 = n$ and $n_1, n_2 \le \frac{7n}{10} + 3$

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Claim: $T(n) \in O(n \lg n)$, that is, $T(n) \le cn \lg n$, for some $c > 0$ and all $n \ge 2$.

Worst-Case Quick Sort

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if r ≤ ℓ
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p = FindPivot(A, ℓ, r)
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return QuickSort(A, m + 1, r)
```

$$T(n) = \Theta(n) + T(n_l) + T(n_2), \text{ where } n_l + n_2 = n \text{ and } n_l, n_2 \le \frac{7n}{10} + 3$$

Claim: $T(n) \in O(n \lg n)$, that is, $T(n) \le cn \lg n$, for some $c > 0$ and all $n \ge 2$.

Base case: $(n < 30)$

We already observed that the running time is in $O(n^2)$ in the worst case. Since $n \in O(1)$, $n^2 \in O(1)$.

$\Rightarrow$ $T(n) \le c' \le cn$ for c sufficiently large.

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

Worst-Case Quick Sort

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Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$\begin{aligned} T(n) &\leq T(n_1) + T(n_2) + an \\ &\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \end{aligned}$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an
\end{aligned}
$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg\left(\frac{7n}{10} + 3\right) + cn_2 \lg\left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg\left(\frac{7n}{10} + 3\right) + an
\end{aligned}
$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg \left(\frac{7n}{10} + 3\right) + an \\
&\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an
\end{aligned}$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg \left(\frac{7n}{10} + 3\right) + an \\
&\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an \\
&= cn \lg \frac{4n}{5} + an
\end{aligned}
$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg \left(\frac{7n}{10} + 3\right) + an \\
&\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an \\
&= cn \lg \frac{4n}{5} + an \\
&= cn \left(\lg n - \lg \frac{5}{4}\right) + an
\end{aligned}
$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg \left(\frac{7n}{10} + 3\right) + cn_2 \lg \left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg \left(\frac{7n}{10} + 3\right) + an \\
&\leq cn \lg \left(\frac{7n}{10} + \frac{n}{10}\right) + an \\
&= cn \lg \frac{4n}{5} + an \\
&= cn \left(\lg n - \lg \frac{5}{4}\right) + an \\
&= cn \lg n + \left(a - c \lg \frac{5}{4}\right) n
\end{aligned}
$$

Worst-Case Quick Sort

Inductive Step: $(n \geq 30)$

$$
\begin{aligned}
T(n) &\leq T(n_1) + T(n_2) + an \\
&\leq cn_1 \lg n_1 + cn_2 \lg n_2 + an \\
&\leq cn_1 \lg\left(\frac{7n}{10} + 3\right) + cn_2 \lg\left(\frac{7n}{10} + 3\right) + an \\
&= cn \lg\left(\frac{7n}{10} + 3\right) + an \\
&\leq cn \lg\left(\frac{7n}{10} + \frac{n}{10}\right) + an \\
&= cn \lg \frac{4n}{5} + an \\
&= cn\left(\lg n - \lg \frac{5}{4}\right) + an \\
&= cn \lg n + \left(a - c \lg \frac{5}{4}\right) n \\
&\leq cn \lg n \quad \forall c \geq \frac{a}{\lg(5/4)} \approx 3.1a
\end{aligned}
$$

Matrix Multiplication

We want to compute $C = A \times B$, where $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices and hence $C = (c_{ij})$ is too.

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Definition:

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The naïve algorithm implementing the definition:

MatrixProduct(A, B)

```
C = an n × n array
for i = 1 to n
    do for k = 1 to n
            do C[i, k] = 0
                for j = 1 to n
                    do C[i, k] = C[i, k] + A[i, j] · B[j, k]
return C
```

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return C
```

Cost: $\Theta(n^3)$

Matrix Multiplication: Divide and Conquer

For simplicity, assume $n = 2^t$ for some integer t.

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Base case: $t = 0$

$$c_{11} = a_{11} \cdot b_{11}$$

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Base case: $t = 0$

$$c_{11} = a_{11} \cdot b_{11}$$

Inductive step: $t > 0$

$$\begin{array}{|c|c|}\hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline\end{array} = \begin{array}{|c|c|}\hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline\end{array} \times \begin{array}{|c|c|}\hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline\end{array}$$

For $1 \leq i, k \leq 2$,

$$C_{ik} = \sum_{j=1}^{2} (A_{ij} \times B_{jk}).$$

Matrix Multiplication: Divide and Conquer

MatrixProductDNC$(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)$

if $i_l = i_u$
    then $C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] \times B[j_l, k_l]$
    else $i_m = (i_l + i_u - 1)/2$
          $j_m = (j_l + j_u - 1)/2$
          $k_m = (k_l + k_u - 1)/2$
          MatrixProductDNC$(A, B, C, i_l, i_m, j_l, j_m, k_l, k_m)$
          MatrixProductDNC$(A, B, C, i_l, i_m, j_m + 1, j_u, k_l, k_m)$
          MatrixProductDNC$(A, B, C, i_l, i_m, j_l, j_m, k_m + 1, k_u)$
          MatrixProductDNC$(A, B, C, i_l, i_m, j_m + 1, j_u, k_m + 1, k_u)$
          MatrixProductDNC$(A, B, C, i_m + 1, i_u, j_l, j_m, k_l, k_m)$
          MatrixProductDNC$(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l, k_m)$
          MatrixProductDNC$(A, B, C, i_m + 1, i_u, j_l, j_m, k_m + 1, k_u)$
          MatrixProductDNC$(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_m + 1, k_u)$

Matrix Multiplication: Divide and Conquer

MatrixProductDNC($A, B, C, i_l, i_u, j_l, j_u, k_l, k_u$)

if $i_l = i_u$
    then $C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] \times B[j_l, k_l]$
    else $i_m = (i_l + i_u - 1)/2$
        $j_m = (j_l + j_u - 1)/2$
        $k_m = (k_l + k_u - 1)/2$
        MatrixProductDNC($A, B, C, i_l, i_m, j_l, j_m, k_l, k_m$)
        MatrixProductDNC($A, B, C, i_l, i_m, j_m + 1, j_u, k_l, k_m$)
        MatrixProductDNC($A, B, C, i_l, i_m, j_l, j_m, k_m + 1, k_u$)
        MatrixProductDNC($A, B, C, i_l, i_m, j_m + 1, j_u, k_m + 1, k_u$)
        MatrixProductDNC($A, B, C, i_m + 1, i_u, j_l, j_m, k_l, k_m$)
        MatrixProductDNC($A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l, k_m$)
        MatrixProductDNC($A, B, C, i_m + 1, i_u, j_l, j_m, k_m + 1, k_u$)
        MatrixProductDNC($A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_m + 1, k_u$)

Cost: $T(n) = 8T(n/2) + \Theta(1)$

Matrix Multiplication: Divide and Conquer

MatrixProductDNC(A, B, C, i_l, i_u, j_l, j_u, k_l, k_u)

```
if i_l = i_u
    then C[i_l, k_l] = C[i_l, k_l] + A[i_l, j_l] × B[j_l, k_l]
    else i_m = (i_l + i_u − 1)/2
         j_m = (j_l + j_u − 1)/2
         k_m = (k_l + k_u − 1)/2
         MatrixProductDNC(A, B, C, i_l,     i_m, j_l,     j_m, k_l,     k_m)
         MatrixProductDNC(A, B, C, i_l,     i_m, j_m + 1, j_u, k_l,     k_m)
         MatrixProductDNC(A, B, C, i_l,     i_m, j_l,     j_m, k_m + 1, k_u)
         MatrixProductDNC(A, B, C, i_l,     i_m, j_m + 1, j_u, k_m + 1, k_u)
         MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l,     j_m, k_l,     k_m)
         MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_l,     k_m)
         MatrixProductDNC(A, B, C, i_m + 1, i_u, j_l,     j_m, k_m + 1, k_u)
         MatrixProductDNC(A, B, C, i_m + 1, i_u, j_m + 1, j_u, k_m + 1, k_u)
```

Cost: $T(n) = 8T(n/2) + \Theta(1) \in \Theta\left(n^3\right)$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta\left(n^2\right) \in \Theta\left(n^{\lg 7}\right) \approx \Theta\left(n^{2.81}\right)$$

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Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21}$$

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Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{A_{11}\ A_{12}\ A_{21}\ A_{22}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

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$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

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Goal:

$$T(n) = 7T(n/2) + \Theta\left(n^2\right) \in \Theta\left(n^{\lg 7}\right) \approx \Theta\left(n^{2.81}\right)$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right) \approx \Theta(n^{2.81})$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$\begin{aligned} M &= A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \\ &= \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix} \\ &= (A_{11} + A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta\left(n^2\right) \in \Theta\left(n^{\lg 7}\right) \approx \Theta\left(n^{2.81}\right)$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

2 inconveniently placed ones
= 2 multiplications

$$\begin{aligned} M &= A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12} \\ &= \begin{bmatrix} A_{11} & A_{12} & A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix} \\ &= (A_{11} + A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

Matrix Multiplication: Strassen's Algorithm

Goal:

$$T(n) = 7T(n/2) + \Theta\left(n^2\right) \in \Theta\left(n^{\lg 7}\right) \approx \Theta\left(n^{2.81}\right)$$

Idea:

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \boxed{\begin{matrix} A_{11} & A_{12} & A_{21} & A_{22} \end{matrix}} \times \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$M = A_{11} \times B_{11} + A_{11} \times B_{12} + A_{21} \times B_{11} + A_{21} \times B_{12}$$

$$= \boxed{\begin{matrix} A_{11} & A_{12} & A_{21} & A_{22} \end{matrix}} \times \begin{bmatrix} 1 & & 1 & \\ & & & \\ 1 & & 1 & \\ & & & \end{bmatrix} \times \begin{bmatrix} B_{11} \\ B_{21} \\ B_{12} \\ B_{22} \end{bmatrix}$$

$$= (A_{11} + A_{21}) \times (B_{11} + B_{12})$$

2 inconveniently placed ones
= 2 multiplications

4 conveniently placed ones
= 1 multiplication

Matrix Multiplication: Strassen's Alg.

$$M_1 = (A_{11} + A_{22}) \times (B_{11} + B_{22})$$
$$M_2 = A_{11} \times (B_{12} + B_{22})$$
$$M_3 = A_{22} \times (B_{11} + B_{21})$$
$$M_4 = (A_{11} - A_{12}) \times B_{22}$$
$$M_5 = (A_{22} - A_{21}) \times B_{11}$$
$$M_6 = (A_{11} + A_{21}) \times (B_{11} - B_{12})$$
$$M_7 = (A_{12} + A_{22}) \times (B_{21} - B_{22})$$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				1
A_{12}				-1
A_{21}				
A_{22}				

M_4

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	-1			
A_{22}	1			

M_5

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1		-1	
A_{12}				
A_{21}	1		-1	
A_{22}				

M_6

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}		1		-1
A_{21}				
A_{22}		1		-1

M_7

Matrix Multiplication: Strassen's Alg.

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				1
A_{12}				-1
A_{21}				
A_{22}				

M_4

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	-1			
A_{22}	1			

M_5

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1		-1	
A_{12}				
A_{21}	1		-1	
A_{22}				

M_6

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}		1		-1
A_{21}				
A_{22}		1		-1

M_7

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			
A_{12}		1		
A_{21}				
A_{22}				

C_{11}

Matrix Multiplication: Strassen's Alg.

$$C_{11} = M_1 + M_7 - M_4 - M_3$$

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & 1 & & & 1 \\ A_{12} & & & & \\ A_{21} & & & & \\ A_{22} & 1 & & & 1 \\ \hline \end{array}$$

M_1

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & 1 & 1 \\ A_{12} & & & & \\ A_{21} & & & & \\ A_{22} & & & & \\ \hline \end{array}$$

M_2

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & \\ A_{12} & & & & \\ A_{21} & & & & \\ A_{22} & 1 & 1 & & \\ \hline \end{array}$$

M_3

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & 1 \\ A_{12} & & & & -1 \\ A_{21} & & & & \\ A_{22} & & & & \\ \hline \end{array}$$

M_4

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & \\ A_{12} & & & & \\ A_{21} & -1 & & & \\ A_{22} & 1 & & & \\ \hline \end{array}$$

M_5

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & 1 & & -1 & \\ A_{12} & & & & \\ A_{21} & 1 & & -1 & \\ A_{22} & & & & \\ \hline \end{array}$$

M_6

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & \\ A_{12} & & 1 & & -1 \\ A_{21} & & & & \\ A_{22} & & 1 & & -1 \\ \hline \end{array}$$

M_7

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & 1 & & & \\ A_{12} & & 1 & & \\ A_{21} & & & & \\ A_{22} & & & & \\ \hline \end{array}$$

C_{11}

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & 1 & & & 1 \\ A_{12} & & & & \\ A_{21} & & & & \\ A_{22} & 1 & & & 1 \\ \hline \end{array}$$

M_1

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & \\ A_{12} & & 1 & & -1 \\ A_{21} & & & & \\ A_{22} & & 1 & & -1 \\ \hline \end{array}$$

M_7

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & -1 \\ A_{12} & & & & 1 \\ A_{21} & & & & \\ A_{22} & & & & \\ \hline \end{array}$$

$-M_4$

$$\begin{array}{c|cccc|} & B_{11} & B_{21} & B_{12} & B_{22} \\ \hline A_{11} & & & & \\ A_{12} & & & & \\ A_{21} & & & & \\ A_{22} & -1 & -1 & & \\ \hline \end{array}$$

$-M_3$

Matrix Multiplication: Strassen's Alg.

$$C_{11} = M_1 + M_7 - M_4 - M_3$$

$$C_{12} = M_2 - M_4$$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	
A_{12}				1
A_{21}				
A_{22}				

C_{12}

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				-1
A_{12}				1
A_{21}				
A_{22}				

$-M_4$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				1
A_{12}				-1
A_{21}				
A_{22}				

M_4

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	-1			
A_{22}	1			

M_5

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1		-1	
A_{12}				
A_{21}	1		-1	
A_{22}				

M_6

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}		1		-1
A_{21}				
A_{22}		1		-1

M_7

Matrix Multiplication: Strassen's Alg.

$$C_{11} = M_1 + M_7 - M_4 - M_3$$

$$C_{12} = M_2 - M_4$$

$$C_{21} = M_3 - M_5$$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	1			
A_{22}		1		

C_{12}

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	1			
A_{22}	-1			

$-M_5$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				1
A_{12}				-1
A_{21}				
A_{22}				

M_4

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	-1			
A_{22}	1			

M_5

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1		-1	
A_{12}				
A_{21}	1		-1	
A_{22}				

M_6

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}		1		-1
A_{21}				
A_{22}		1		-1

M_7

Matrix Multiplication: Strassen's Alg.

$$C_{11} = M_1 + M_7 - M_4 - M_3$$

$$C_{12} = M_2 - M_4$$

$$C_{21} = M_3 - M_5$$

$$C_{22} = M_1 - M_6 - M_2 - M_5$$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			1	1
A_{12}				
A_{21}				
A_{22}				

M_2

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}				
A_{22}	1	1		

M_3

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				1
A_{12}				-1
A_{21}				
A_{22}				

M_4

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	-1			
A_{22}	1			

M_5

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}			1	
A_{22}				1

C_{22}

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1			1
A_{12}				
A_{21}				
A_{22}	1			1

M_1

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	-1		1	
A_{12}				
A_{21}	-1		1	
A_{22}				

$-M_6$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}			-1	-1
A_{12}				
A_{21}				
A_{22}				

$-M_2$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}				
A_{21}	1			
A_{22}	-1			

$-M_5$

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}	1		-1	
A_{12}				
A_{21}	1		-1	
A_{22}				

M_6

	B_{11}	B_{21}	B_{12}	B_{22}
A_{11}				
A_{12}		1		-1
A_{21}				
A_{22}		1		-1

M_7

Matrix Multiplication: Strassen's Algorithm

Strassen(A, B)

let $n \times n$ be the dimension of A and B
if $n = 1$
    then return $A[1,1] \cdot B[1,1]$
    else partition A into submatrices $A_{11}, A_{12}, A_{21}, A_{22}$
        partition B into submatrices $B_{11}, B_{12}, B_{21}, B_{22}$
        $M_1 = \text{Strassen}(A_{11} + A_{22}, B_{11} + B_{22})$
        $M_2 = \text{Strassen}(A_{11}, B_{12} + B_{22})$
        $M_3 = \text{Strassen}(A_{22}, B_{11} + B_{21})$
        $M_4 = \text{Strassen}(A_{11} - A_{12}, B_{22})$
        $M_5 = \text{Strassen}(A_{22} - A_{21}, B_{11})$
        $M_6 = \text{Strassen}(A_{11} + A_{21}, B_{11} - B_{12})$
        $M_7 = \text{Strassen}(A_{12} + A_{22}, B_{21} - B_{22})$
        $C_{11} = M_1 + M_7 - M_4 - M_3$
        $C_{12} = M_2 - M_4$
        $C_{21} = M_3 - M_5$
        $C_{22} = M_1 - M_6 - M_2 - M_5$
        assemble C from $C_{11}, C_{12}, C_{21}, C_{22}$
        return C

Matrix Multiplication: Strassen's Algorithm

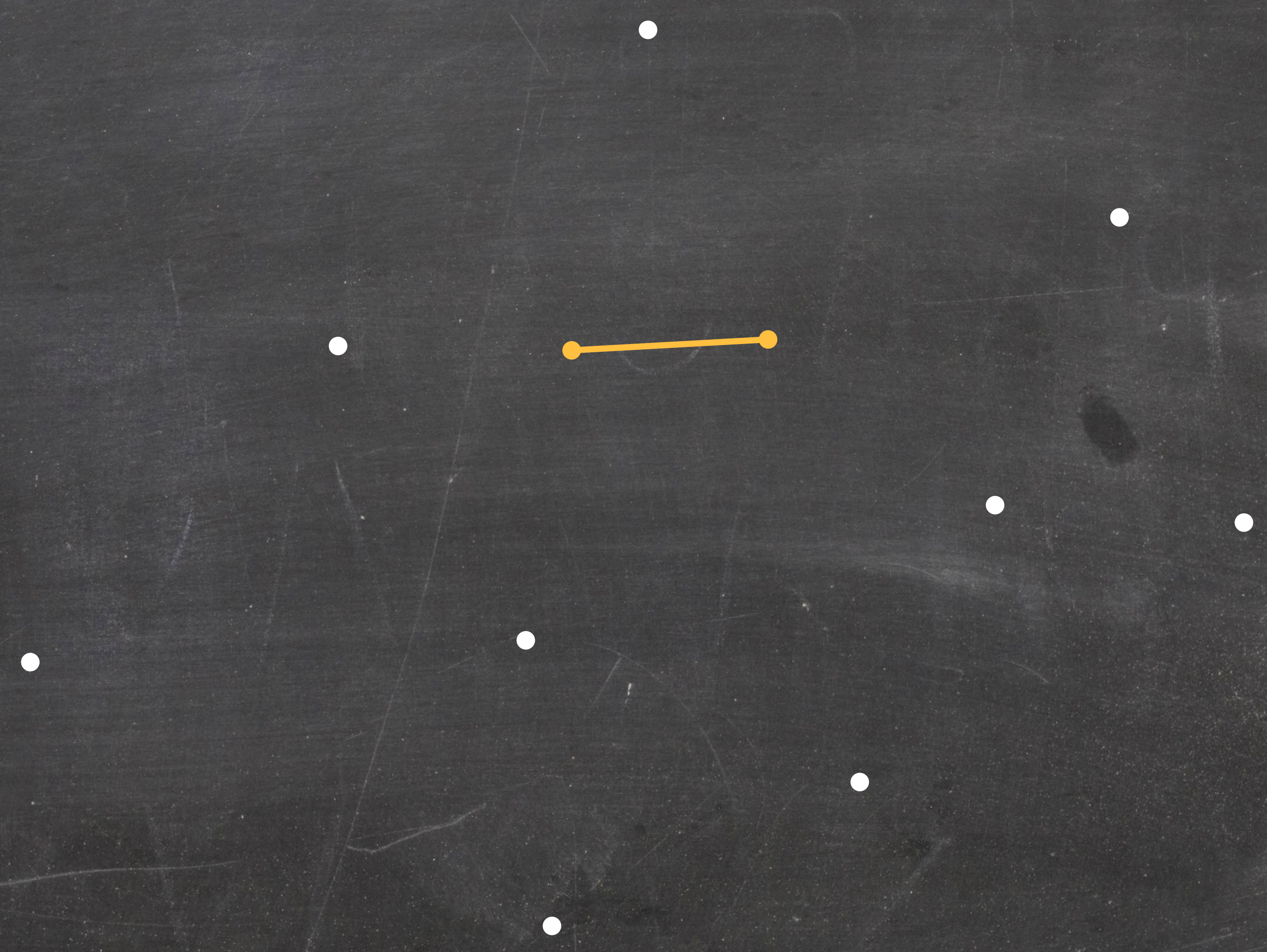
Strassen(A, B)

let $n \times n$ be the dimension of A and B
if $n = 1$
  then return $A[1,1] \cdot B[1,1]$
  else partition A into submatrices $A_{11}, A_{12}, A_{21}, A_{22}$
    partition B into submatrices $B_{11}, B_{12}, B_{21}, B_{22}$
    $M_1 = \text{Strassen}(A_{11} + A_{22}, B_{11} + B_{22})$
    $M_2 = \text{Strassen}(A_{11}, B_{12} + B_{22})$
    $M_3 = \text{Strassen}(A_{22}, B_{11} + B_{21})$
    $M_4 = \text{Strassen}(A_{11} - A_{12}, B_{22})$
    $M_5 = \text{Strassen}(A_{22} - A_{21}, B_{11})$
    $M_6 = \text{Strassen}(A_{11} + A_{21}, B_{11} - B_{12})$
    $M_7 = \text{Strassen}(A_{12} + A_{22}, B_{21} - B_{22})$
    $C_{11} = M_1 + M_7 - M_4 - M_3$
    $C_{12} = M_2 - M_4$
    $C_{21} = M_3 - M_5$
    $C_{22} = M_1 - M_6 - M_2 - M_5$
    assemble C from $C_{11}, C_{12}, C_{21}, C_{22}$
    return C

Cost: $T(n) = 7T(n/2) + \Theta(n^2) \in \Theta\left(n^{\lg 7}\right)$

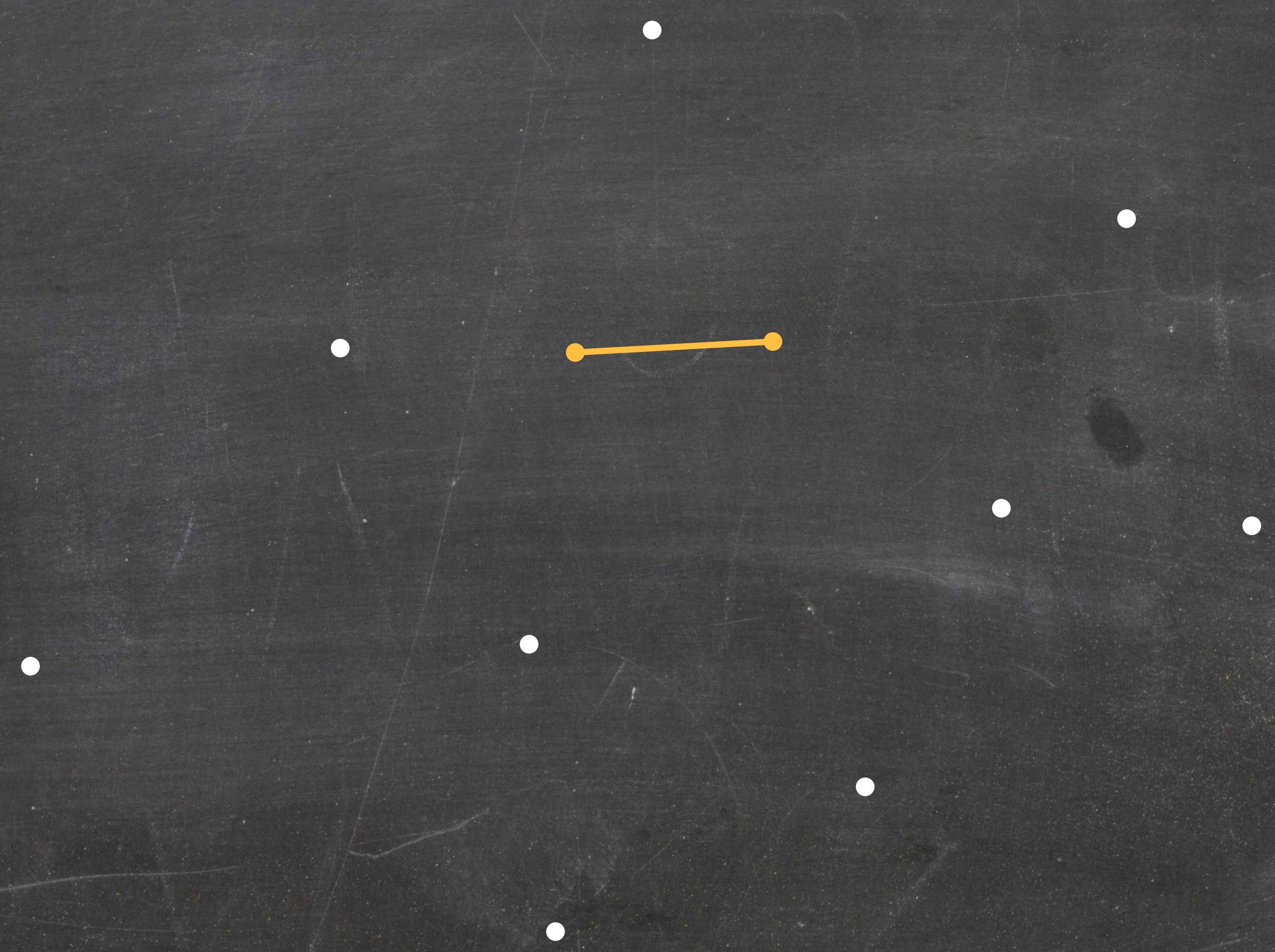
Closest Pair

Given a point set P in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from p to q).



Closest Pair

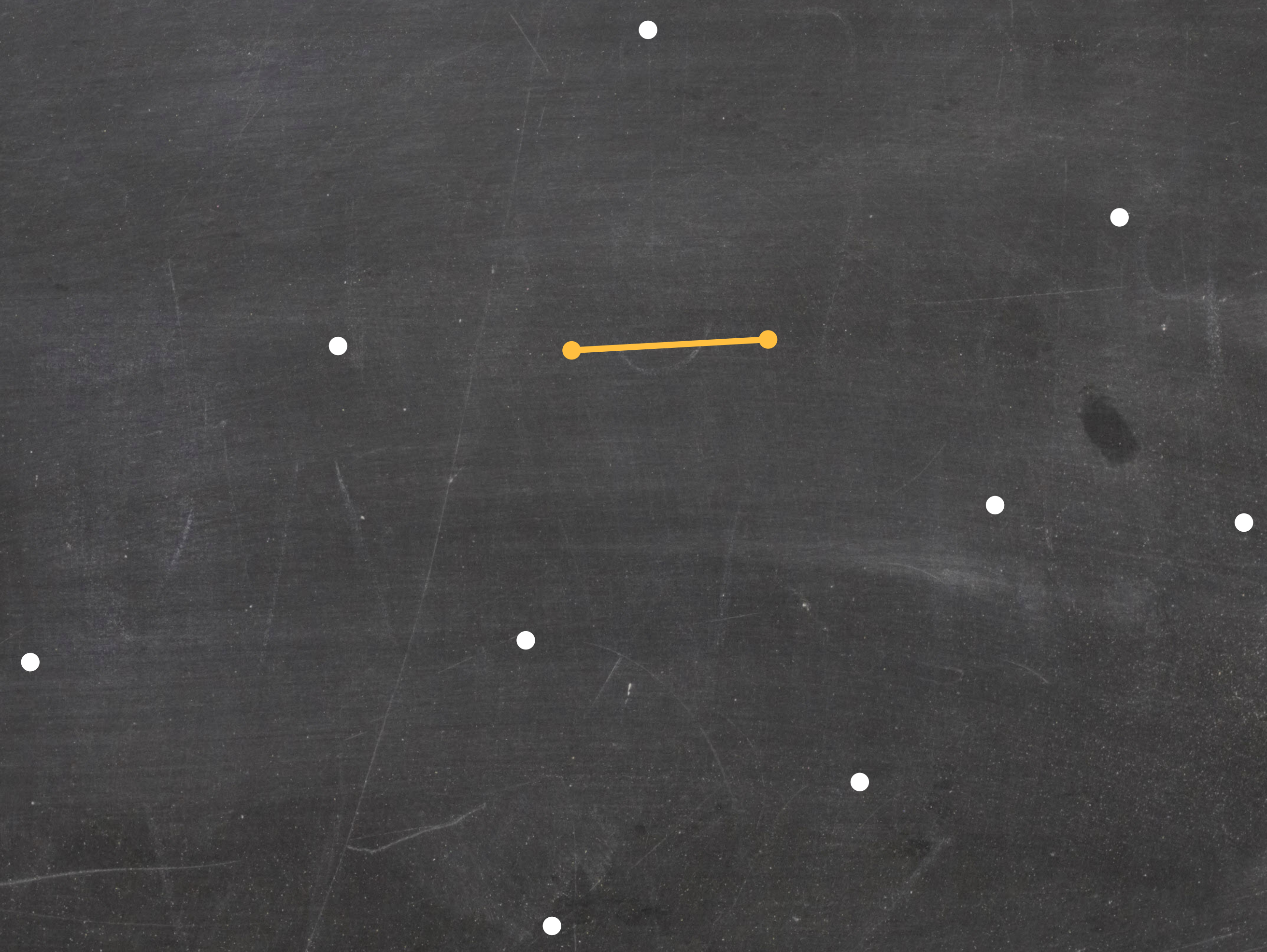
Given a point set P in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from p to q).



Can be computed in $O(n^2)$ time. How?

Closest Pair

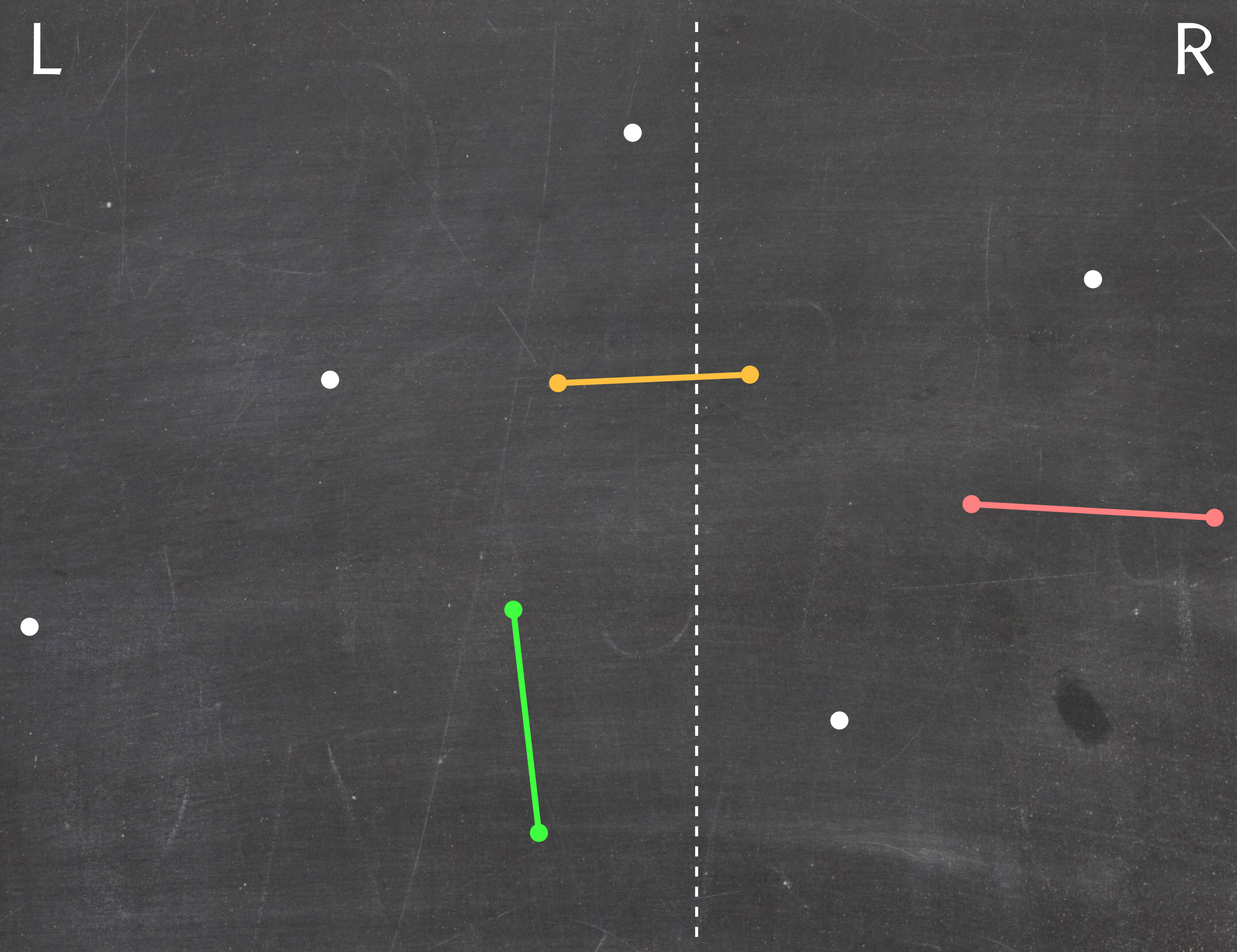
Given a point set P in the plane, the closest pair is the pair of points $p, q \in P$ that minimizes $\|p - q\|_2$ (the Euclidean distance from p to q).



Can be computed in $O(n^2)$ time. How?

Can we do better?

Closest Pair: Divide and Conquer



If we divide the point set into the leftmost $\lceil n/2 \rceil$ points (L) and the rightmost $\lfloor n/2 \rfloor$ points (R), then the closest pair has

- both points in L,
- both points in R or
- one point in L and one point in R.

Closest Pair: Divide and Conquer

ClosestPair(P)

if $|P| \leq 1$
  then return Nothing
Split P into two sets L and R containing the leftmost $\lceil n/2 \rceil$ and the rightmost $\lfloor n/2 \rfloor$ points, respectively.
(p_ℓ, q_ℓ) = ClosestPair(L)
(p_r, q_r) = ClosestPair(R)
(p_m, q_m) = ClosestPairLR(L, R)
return the pair (p_i, q_i), $i \in \{\ell, r, m\}$, that minimizes $\|p_i - q_i\|_2$

Closest Pair: Divide and Conquer

ClosestPair(P)

if $|P| \leq 1$
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Split P into two sets L and R containing the leftmost $\lceil n/2 \rceil$ and the rightmost $\lfloor n/2 \rfloor$ points, respectively.
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(p_m, q_m) = ClosestPairLR(L, R)
return the pair (p_i, q_i), $i \in \{\ell, r, m\}$, that minimizes $\|p_i - q_i\|_2$

Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

Closest Pair: Divide and Conquer

ClosestPair(P)

```
if |P| ≤ 1
    then return Nothing
Split P into two sets L and R containing the leftmost ⌈n/2⌉ and the
rightmost ⌊n/2⌋ points, respectively.
(p_ℓ, q_ℓ) = ClosestPair(L)
(p_r, q_r) = ClosestPair(R)
(p_m, q_m) = ClosestPairLR(L, R)
return the pair (p_i, q_i), i ∈ {ℓ, r, m}, that minimizes ||p_i − q_i||_2
```

Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n^2)$

Closest Pair: Divide and Conquer

ClosestPair(P)

if $|P| \le 1$
    then return Nothing
Split P into two sets L and R containing the leftmost $\lceil n/2 \rceil$ and the rightmost $\lfloor n/2 \rfloor$ points, respectively.
$(p_\ell, q_\ell) = \text{ClosestPair}(L)$
$(p_r, q_r) = \text{ClosestPair}(R)$
$(p_m, q_m) = \text{ClosestPairLR}(L, R)$
return the pair (p_i, q_i), $i \in \{\ell, r, m\}$, that minimizes $\|p_i - q_i\|_2$

Naïve implementation of ClosestPairLR: $\Theta(n^2)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n^2) \in \Theta(n^2)$

Closest Pair: Divide and Conquer

ClosestPair(P)

```
if |P| ≤ 1
    then return Nothing
Split P into two sets L and R containing the leftmost ⌈n/2⌉ and the
rightmost ⌊n/2⌋ points, respectively.
(p_ℓ, q_ℓ) = ClosestPair(L)
(p_r, q_r) = ClosestPair(R)
(p_m, q_m) = ClosestPairLR(L, R)
return the pair (p_i, q_i), i ∈ {ℓ, r, m}, that minimizes ||p_i − q_i||_2
```

Better implementation of ClosestPairLR: $\Theta(n)$ time

Running time: $T(n) = 2T(n/2) + \Theta(n) \in \Theta(n \lg n)$

Closest Pair: Preprocessing

```
ClosestPair(P)
    Make two copies X and Y of P
    Sort the points in X by their x-coordinates
    Sort the points in Y by their y-coordinates
    return ClosestPairRec(X, Y)
```

Closest Pair: Preprocessing

ClosestPair(P)

```
Make two copies X and Y of P
Sort the points in X by their x-coordinates
Sort the points in Y by their y-coordinates
return ClosestPairRec(X, Y)
```

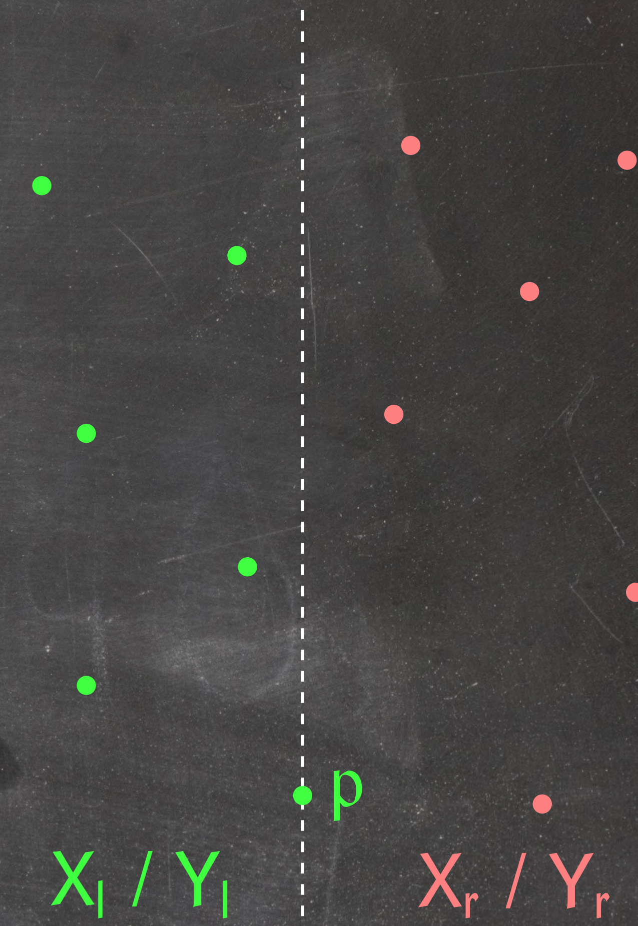
We prove that ClosestPairRec takes $O(n \lg n)$ time.

$\Rightarrow$ The whole algorithm takes $O(n \lg n)$ time.

Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

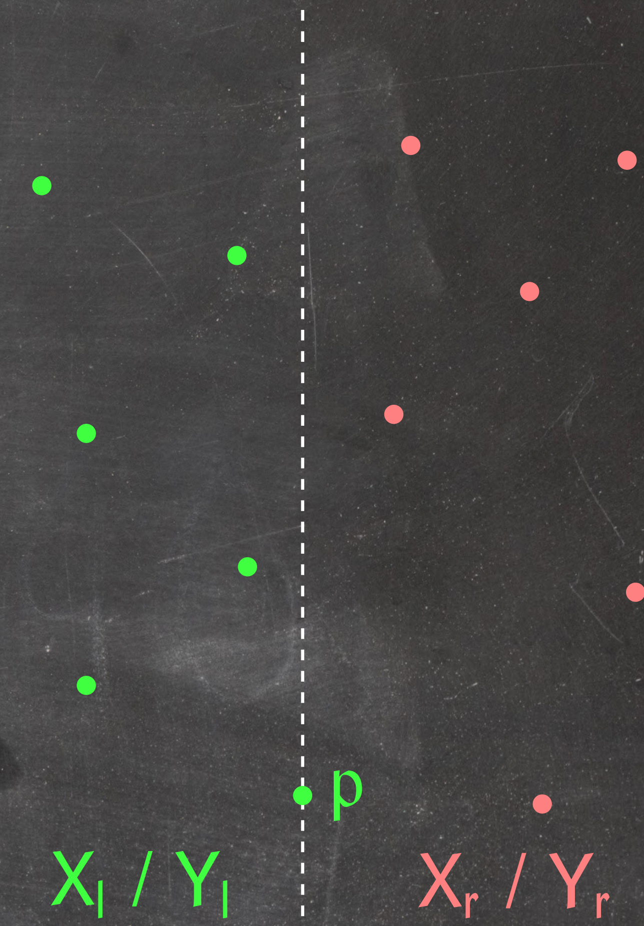
if $|Y| \le 1$
    then return (Nothing, ∞)
p = the middle element in X
X_ℓ = the part of X up to and including p
X_r = the part of X after x
$Y_\ell = \langle q \in Y \mid q.x \le p.x \rangle$
$Y_r = \langle q \in Y \mid q.x > p.x \rangle$
$(pair_\ell, \delta_\ell)$ = ClosestPairRec(X_ℓ, Y_ℓ)
$(pair_r, \delta_r)$ = ClosestPairRec(X_r, Y_r)
$(pair_m, \delta_m)$ = ClosestPairLR$(Y_\ell, Y_r, p.x, \min(\delta_\ell, \delta_r))$
return the pair $(pair_i, \delta_i)$, $i \in \{\ell, r, m\}$, that minimizes δ_i



Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

if $|Y| \le 1$
    then return (Nothing, ∞)
p = the middle element in X
X_ℓ = the part of X up to and including p
X_r = the part of X after x
$Y_\ell = \langle q \in Y \mid q.x \le p.x \rangle$
$Y_r = \langle q \in Y \mid q.x > p.x \rangle$
$(\text{pair}_\ell, \delta_\ell) = \text{ClosestPairRec}(X_\ell, Y_\ell)$
$(\text{pair}_r, \delta_r) = \text{ClosestPairRec}(X_r, Y_r)$
$(\text{pair}_m, \delta_m) = \text{ClosestPairLR}(Y_\ell, Y_r, p.x, \min(\delta_\ell, \delta_r))$
return the pair $(\text{pair}_i, \delta_i)$, $i \in \{\ell, r, m\}$, that minimizes δ_i



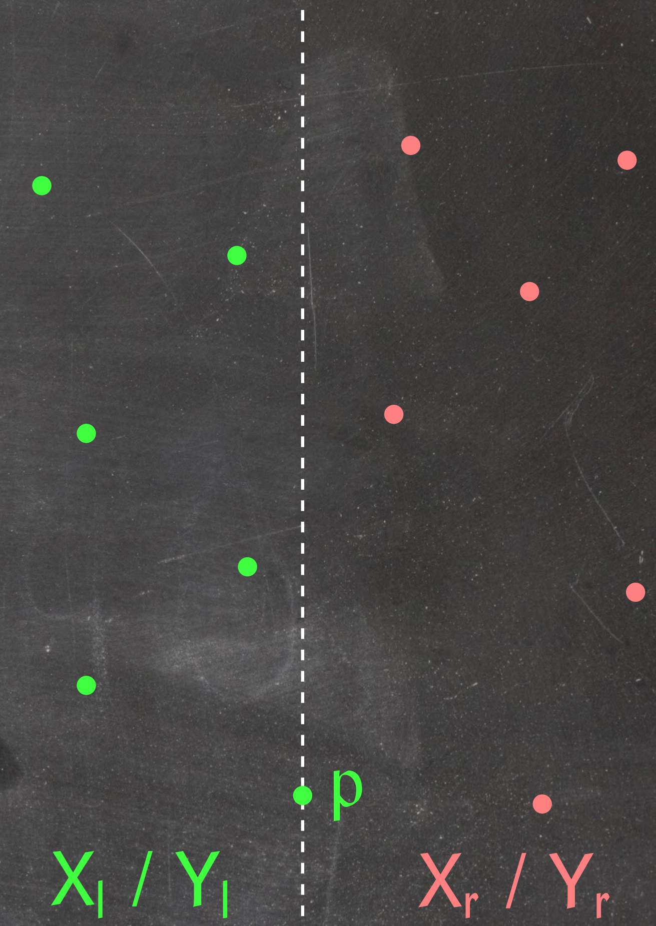
We already have a pair with distance $\delta = \min(\delta_\ell, \delta_r)$.

$\Rightarrow$ only need to look for pairs with distances $< \delta$.

Closest Pair: Divide and Conquer

ClosestPairRec(X, Y)

if $|Y| \le 1$
    then return (Nothing, ∞)
p = the middle element in X
X_ℓ = the part of X up to and including p
X_r = the part of X after x
$Y_\ell = \langle q \in Y \mid q.x \le p.x \rangle$
$Y_r = \langle q \in Y \mid q.x > p.x \rangle$
$(\text{pair}_\ell, \delta_\ell) = \text{ClosestPairRec}(X_\ell, Y_\ell)$
$(\text{pair}_r, \delta_r) = \text{ClosestPairRec}(X_r, Y_r)$
$(\text{pair}_m, \delta_m) = \text{ClosestPairLR}(Y_\ell, Y_r, p.x, \min(\delta_\ell, \delta_r))$
return the pair $(\text{pair}_i, \delta_i)$, $i \in \{\ell, r, m\}$, that minimizes δ_i

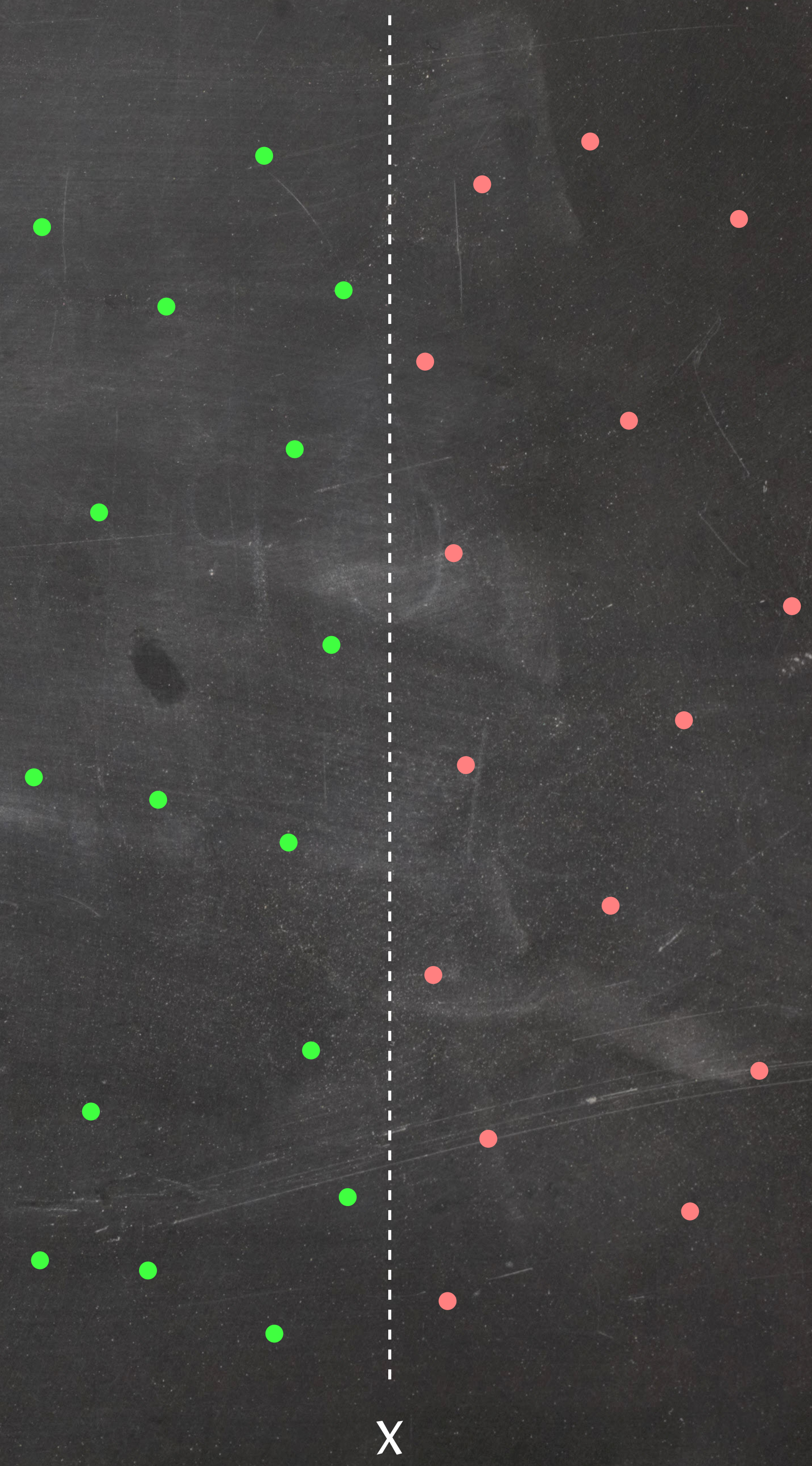


We prove that $\text{ClosestPairLR}(Y_\ell, Y_r, x, \delta)$ takes $O(n)$ time.

$\Rightarrow$ $T(n) = 2T(n/2) + O(n) \in O(n \lg n)$

Closest Pair: One Left, One Right

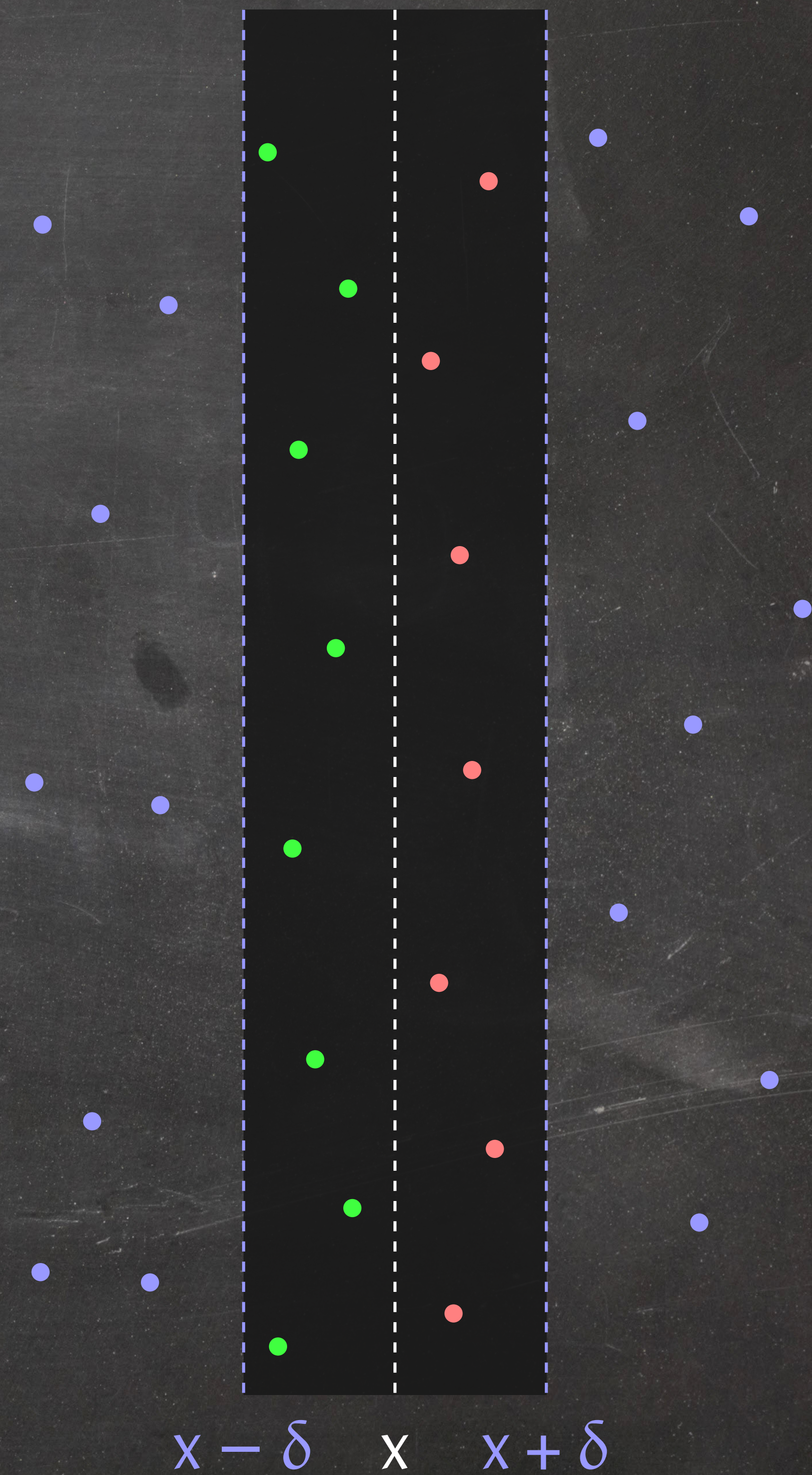
```
ClosestPairLR(Yℓ, Yr, x, δ)

Zℓ = ⟨p ∈ Yℓ | x − p.x ≤ δ⟩
Zr = ⟨p ∈ Yr | p.x − x ≤ δ⟩
pair = Nothing
δ′ = ∞
j = 1
for i = 1 to |Zℓ|
    do while j < |Zr| and Zr[j].y < Zℓ[i].y − δ
           do j = j + 1
       k = j
       while k ≤ |Zr| and Zr[k].y ≤ Zℓ[i].y + δ
           do if ||Zℓ[i] − Zr[k]|| < δ′
                then δ′ = ||Zℓ[i] − Zr[k]||
                     pair = (Zℓ[i], Zr[k])
              k = k + 1
return (pair, δ′)
```



Closest Pair: One Left, One Right

ClosestPairLR(Y_ℓ, Y_r, x, δ)

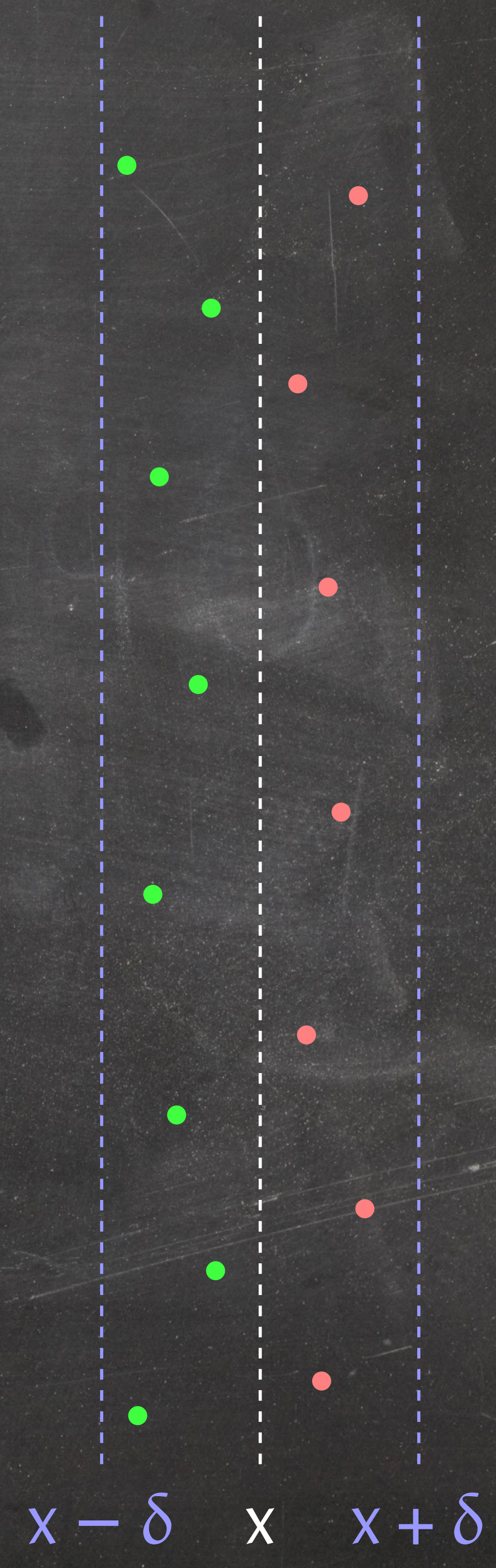
```
Zℓ = ⟨p ∈ Yℓ | x − p.x ≤ δ⟩
Zr = ⟨p ∈ Yr | p.x − x ≤ δ⟩
pair = Nothing
δ′ = ∞
j = 1
for i = 1 to |Zℓ|
    do while j < |Zr| and Zr[j].y < Zℓ[i].y − δ
           do j = j + 1
       k = j
       while k ≤ |Zr| and Zr[k].y ≤ Zℓ[i].y + δ
           do if ||Zℓ[i] − Zr[k]|| < δ′
                  then δ′ = ||Zℓ[i] − Zr[k]||
                       pair = (Zℓ[i], Zr[k])
              k = k + 1
return (pair, δ′)
```



Closest Pair: One Left, One Right

ClosestPairLR(Y_ℓ, Y_r, x, δ)

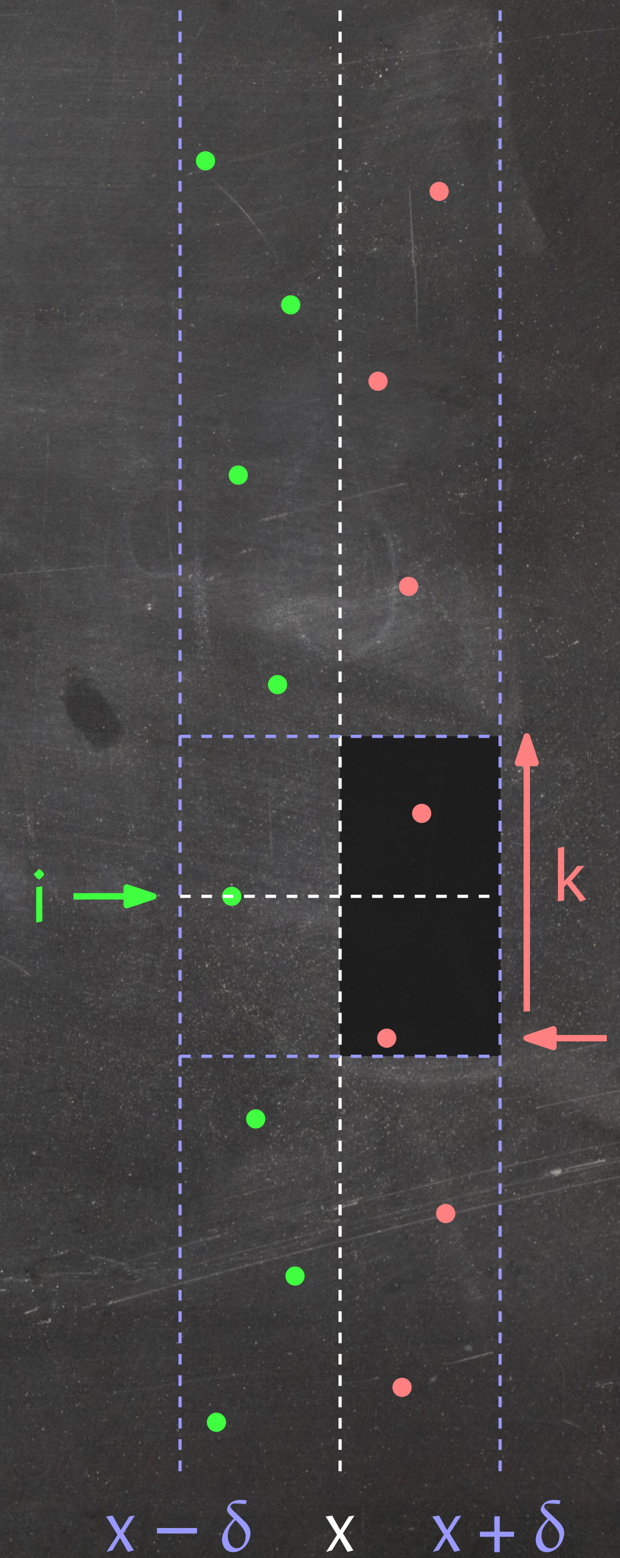
```
Zℓ = ⟨p ∈ Yℓ | x − p.x ≤ δ⟩
Zr = ⟨p ∈ Yr | p.x − x ≤ δ⟩
pair = Nothing
δ′ = ∞
j = 1
for i = 1 to |Zℓ|
    do while j < |Zr| and Zr[j].y < Zℓ[i].y − δ
            do j = j + 1
        k = j
        while k ≤ |Zr| and Zr[k].y ≤ Zℓ[i].y + δ
            do if ‖Zℓ[i] − Zr[k]‖ < δ′
                then δ′ = ‖Zℓ[i] − Zr[k]‖
                    pair = (Zℓ[i], Zr[k])
            k = k + 1
return (pair, δ′)
```



Closest Pair: One Left, One Right

ClosestPairLR(Y_ℓ, Y_r, x, δ)

```
Zℓ = ⟨p ∈ Yℓ | x − p.x ≤ δ⟩
Zr = ⟨p ∈ Yr | p.x − x ≤ δ⟩
pair = Nothing
δ′ = ∞
j = 1
for i = 1 to |Zℓ|
    do while j < |Zr| and Zr[j].y < Zℓ[i].y − δ
            do j = j + 1
        k = j
        while k ≤ |Zr| and Zr[k].y ≤ Zℓ[i].y + δ
            do if ||Zℓ[i] − Zr[k]|| < δ′
                    then δ′ = ||Zℓ[i] − Zr[k]||
                         pair = (Zℓ[i], Zr[k])
                k = k + 1
return (pair, δ′)
```



Closest Pair: One Left, One Right

Lemma: For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

Closest Pair: One Left, One Right

Lemma: For every point $p \in Z_l$, there exist at most 8 points $q \in Z_r$ such that $|q.y - p.y| \leq \delta$.

Corollary: The running time of ClosestPairLR is in $O(n)$.

Closest Pair: One Left, One Right

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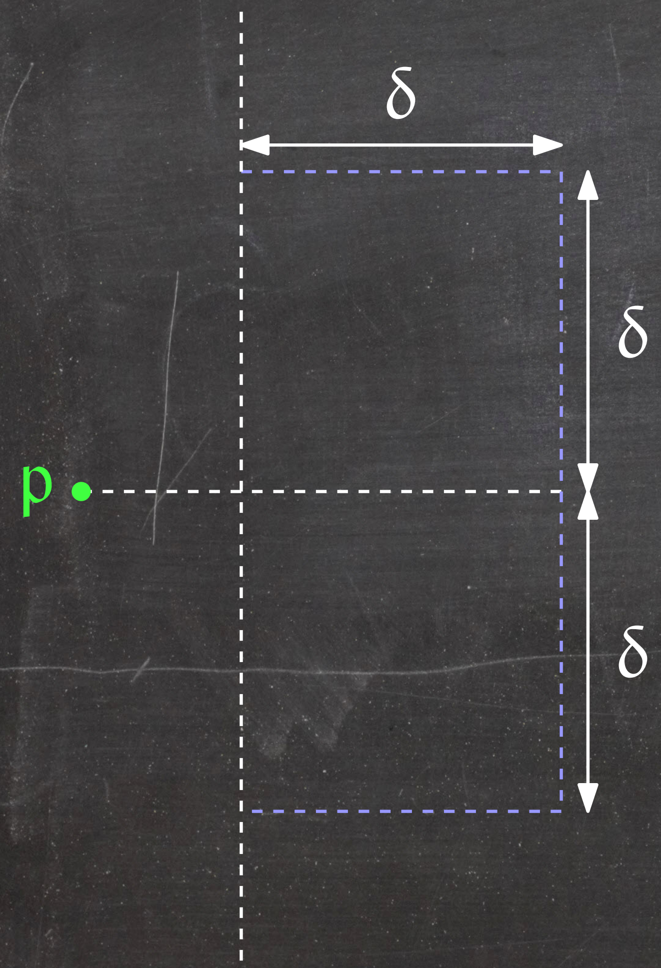

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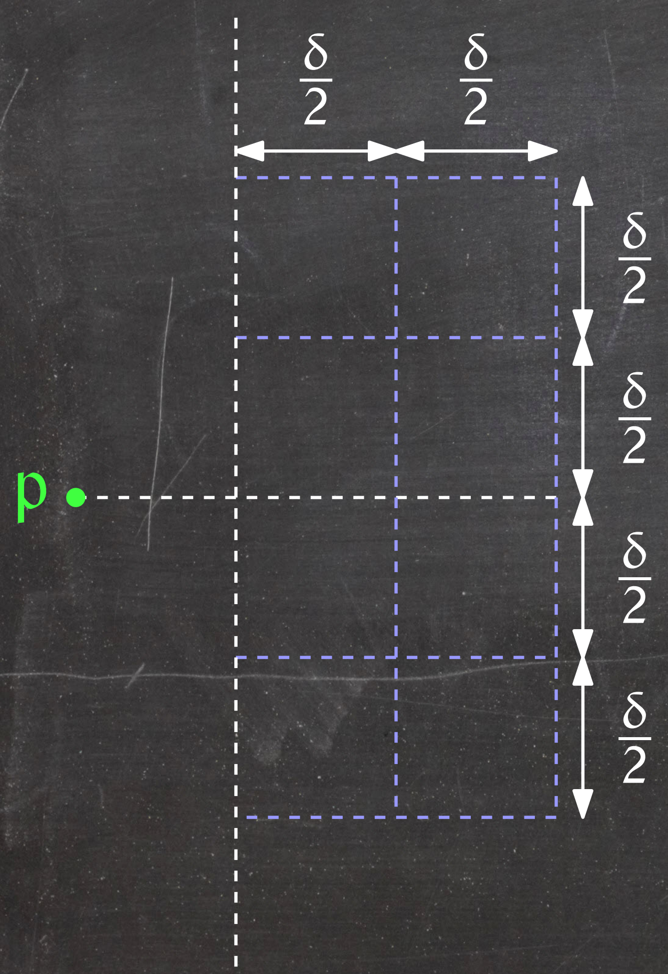

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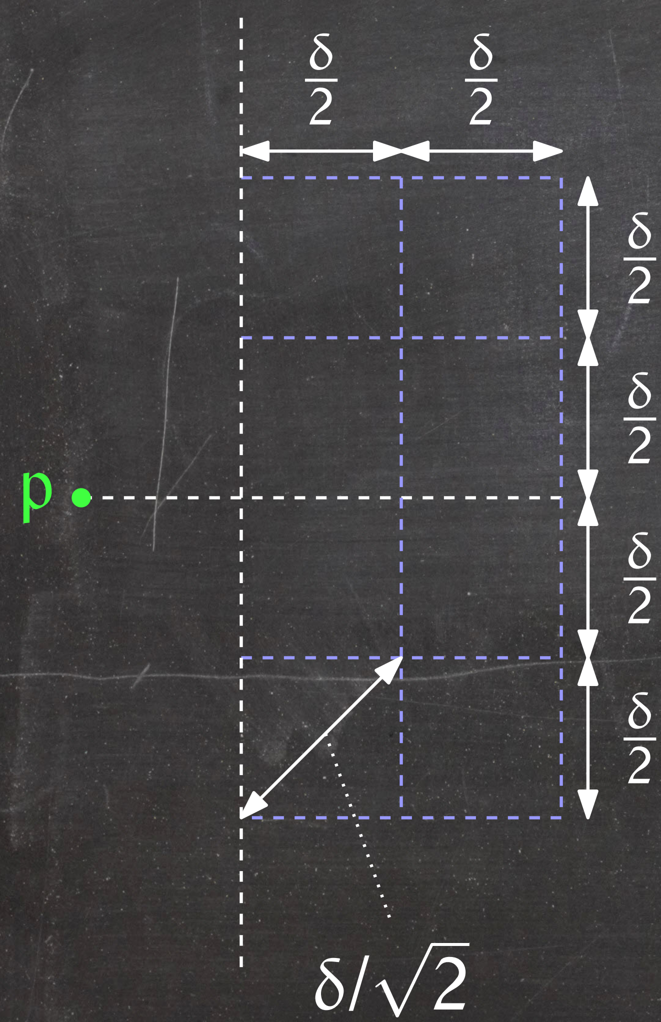

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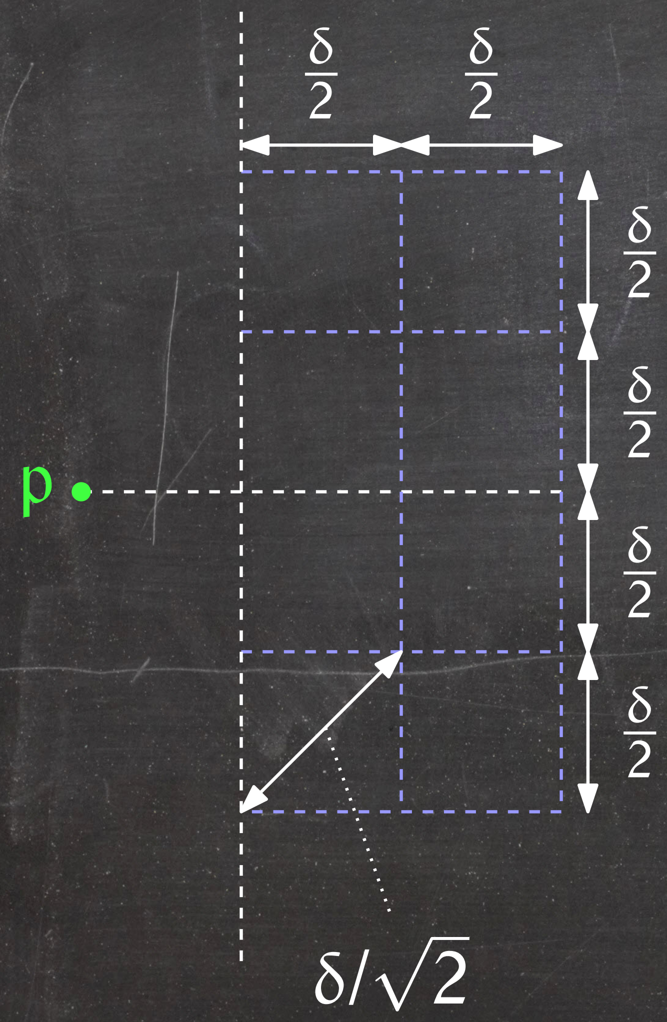

Two points in the same square have distance at most $\delta/\sqrt{2} < \delta$ from each other.

$\Rightarrow$ At most one point in each of the 8 squares.

Summary

The Divide and Conquer paradigm:

- Divide the input into smaller instances of the same problem.
- Solve these instances recursively.
- Combine the obtained solutions to obtain a solution to the original input.

Divide-and-conquer algorithms always recurse on smaller inputs.

$\Rightarrow$ Natural expression of running time using recurrence relations.
$\Rightarrow$ Natural strategy to prove correctness is induction.

Solving recurrence relations:

- Substitution
- Recursion trees
- Master Theorem