Encoding Quadrilateral Meshes

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Abstract

An important problem in geometric compression is to find succinct representations (encoding schemes) for the conectivity of polygonal meshes. In this note, we show that the encoding scheme discussed in [1] for quadrilateral mesh connectivity can be improved from 3.5 bits per vertex to less than 3 bits per vertex. We also show that an easy equivalence between the labelling schemes of King et al [2] and of Kronrod-Gotsman [1], improves this further to 2.67 bits per vertex. The same upper bound has also been reported in [2], making an involved use of the CLRES labelling scheme.

1 Introduction

Following the publication of Deering's paper [3], geometric compression has become a very active field of research [2], [1]. The emphasis of the research has primarily been on finding efficient schemes for encoding the geometry (connectivity) of polygonal meshes. The efficiency of such schemes are quantified by the number of bits needed per vetex of the mesh or the numbers bits required per edge of the mesh. The practical significance of this is that it enables one to store and transmit such meshes (over the Internet) succinctly. In this note we show that the encoding scheme discussed in [1] for meshes made up of quadrilaterals only (quad mesh, for short) can be improved to less than 3 bits per vertex. We also show that an easy equivalence between the labelling schemes of King et al [2] and of Kronrod & Gotsman [1], improves this further to 2.67 bits per vertex. The same upper bound has also been reported in [2], making an involved use of the CLRES scheme.

2 Kronrod-Gotsman scheme

The Kronrod-Gotsman scheme [1] generalizes the CLRES labelling scheme of [2] to non-triangular meshes. Their main observation is that as we traverse a mesh (with or without boundary) in depth-first order, the interaction of each polygon with the rest of the mesh can be enumerated in a finite number of ways. For example, in a quad mesh each quad interacts with the rest of the mesh in one of 13 ways (Fig.1, arrows indicate the current gate) and hence this interaction can be coded in a unique manner. It is easy to enumerate all these cases if we note that each of the remaining three edges of the current quad either belongs to the mesh boundary or doesn't, and so also for the remaining two vertices.

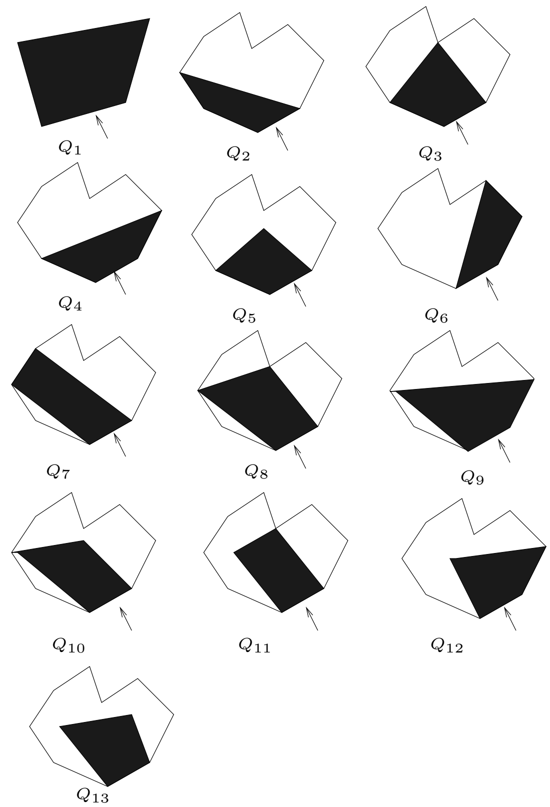

Figure 1: *Interaction of a quad with the mesh*

The compression process traverses the mesh in depth-first order, starting with a quad, at least one of whose edges is part of the mesh boundary. Note that if a mesh is closed we can create a boundary by removing one of the polygons. In the following discussion, the term *gate* will mean an edge of a quad that we are currently visiting, and one that it shares with the current mesh boundary.

For example, if we traverse the quad mesh of Fig.2, starting with the thick edge, and always choose the next gate to be the edge of the current quad that is counterclockwise with respect to the current gate then we get the following output string: $Q_{13}Q_6Q_6Q_5Q_{12}Q_{12}Q_6Q_6Q_{12}Q_{12}Q_9Q_1Q_6Q_1$

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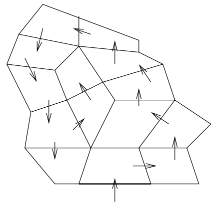

Figure 2: *Traversing a quad mesh*

3 Encoding scheme with less than 3.0 bpv

Kronrod & Gotsman [1] proposed a prefix-free variable length encoding scheme for such a string that needs at most 3.5 bits per quad. We show that this can be improved to less than 3 bits per vertex.

As we process a quad, we introduce new edges and vertices. These new edges and new vertices are free edges and vertices that become part of the mesh boundary when we process and remove the current quad. An edge or vertex of a quad is *free* if it doesn't belong to the mesh boundary. Table 1 summarizes this information for each of the thirteen types of quad that is encountered during mesh traversal.

We also have the following important observation about the encoding process.

Claim 1 *In a manifold mesh, a quad of type Q_5, Q_{10}, Q_{12}, or Q_{13} is never followed by a quad of type Q_1, Q_2, Q_3, Q_4, Q_5, if while traversing the mesh we choose the next gate to be situated counterclockwise with respect to the present one.*

Proof: This is immediate as each quad of type Q_5, Q_{10}, Q_{12} or Q_{13} leaves a free vertex and the choice of the next gate makes it impossible for the next quad to have the edge previous to the gate on the active edge boundary as is required to have a quad of type Q_1, Q_2, Q_3, Q_4, Q_5.

Type	# of new edges	# of new vertices
Q_1	0	0
Q_2	1	0
Q_3	2	0
Q_4	1	0
Q_5	2	1
Q_6	1	0
Q_7	2	0
Q_8	3	0
Q_9	2	0
Q_{10}	3	1
Q_{11}	3	1
Q_{12}	2	1
Q_{13}	3	2

Table 1: *Mesh-Quad Interactions*

Note 1 *In the paper by King et al [2], they point out an exception to the above claim when the quad mesh has a an internal valence-two vertex (that is, a vertex on which exactly two quads are incident) and deals with this situation separately, resulting in an encoding scheme that has an upper bound of more than 3 bits per vertex. For a manifold mesh, the claim is true without any exception.*

In view of the above observation, we borrow an idea from the coding scheme of [2],to set the code for each interaction-type as in the table below to obtain a variable-length prefix-free coding scheme.

We argue below why the above coding scheme provides an upper bound of 3.0 bits per vertex for the Kronrod-Gotsman scheme.

We assume that we have a quad mesh homeomorphic to a sphere. Let E be the number of its edges, V the number of its vertices and Q the number of quads it has. Since each edge is shared by exactly two quads, $E = 2Q$. Combining this with Euler's formula, we get

$$V = Q + 2 \tag{1}$$

For a very large mesh, $Q >> 2$; therefore, ignoring the additive term in (3) above, we can assume that $V = Q$.

Let $|Q_i|$ denote the number of quads of type Q_i and $|Q_{i-j}| = |Q_i| + \ldots + |Q_j|, j > i$. From $V = Q$ above and Table 2, it follows that

$$2|Q_{13}| + |Q_5| + |Q_{10-12}| \quad = \quad Q \tag{2}$$

as the left-hand side counts the number of vertices in the quad mesh.

Again, as $V = Q$ (approximately), the number of quads which have two free vertices must be equal to the number of quads which have no free vertices. Thus from Table 1, it follows that,

$$|Q_{13}| \quad = \quad |Q_{1-4}| + |Q_{6-9}| \tag{3}$$

Each branch in a quad spanning tree ends in a quad of type Q_1 (a leaf node), and each branch begins either at the root gate or at a quad of type Q_3 or Q_{7-11}. With each quad of type Q_3, Q_7, Q_9, Q_{10}, and Q_{11}, one more quad of type Q_1 is associated. With each quad of type Q_8, two more quads of type Q_1's are associated. Therefore, we have the following constraint.

$$|Q_3| + |Q_{7-11}| + |Q_8| = |Q_1| - 1 \tag{4}$$

Using the constraint of equation (3), we can pair a quad of each of the types Q_{1-4} and Q_{6-9} with a quad of the type Q_{13} as shown in Table 3 (Here and subsequently, we have taken the liberty of denoting a quad by its type).

Encoding	Current Quad	Next Quad	Code	Num. of bits
Quad started with Q6-13	Q6	Q1-5	11111	5
		Q6-13	11110	5
	Q7	Q1-5	11101	5
		Q6-13	11100	5
	Q8	Q1-5	11011	5
		Q6-13	11010	5
	Q9	Q1-5	11001	5
		Q6-13	11000	5
	Q10	Q6-13	10111	5
	Q11	Q1-5	10110	5
		Q6-13	10101	5
	Q12	Q6-13	100	3
	Q13	Q6-13	0	1
Quad started with Q1-5	Q1	Q1-5	00	2
		Q6-13	01	2
	Q2	Q1-5	1100	4
		Q6-13	1101	4
	Q3	Q1-5	1010	4
		Q6-13	1011	4
	Q4	Q1-5	1000	4
		Q6-13	1001	4
	Q5	Q6-13	111	3

Table 2: *Coding Scheme*

Code1	bits in Code1	Code2	bits in Code2	Average bits
Q_1	2	Q_{13}	1	1.5
Q_2	4	Q_{13}	1	2.5
Q_3	4	Q_{13}	1	2.5
Q_4	4	Q_{13}	1	2.5
Q_6	5	Q_{13}	1	3.0
Q_7	5	Q_{13}	1	3.0
Q_8	5	Q_{13}	1	3.0
Q_9	5	Q_{13}	1	3.0

Table 3: *Code bits analysis for Q_1 to Q_9*

Thus the grouping of quads of type Q_{1-9} with quads of type Q_{13} yield an average bit count of at most 3. Next, we use the constraint of equation (4) to refine this analysis even further. This constraint implies that $|Q_{7-11}| + |Q_8| < |Q_1| - 1$. Therefore, quads of each of the types from Q_{7-11} and Q_8 can be associated with at most one quad of type Q_1.

Since quads of type Q_3 have been taken care of, we don't have to find pairs for quads of this type. Since quads of type Q_1 have been grouped with quads of type Q_{13} already, and a quad of type Q_1 is associated with a quad of each one of the types $Q_7, Q_9, Q_{10} and Q_{11}$, while two quads of type Q_1 are associated with a quad of type Q_8, we need to associate a a quad group (Q_1, Q_{13}) with each one of the quad groups/quads (Q_7, Q_{13}), (Q_9, Q_{13}), Q_{10}, and Q_{11}. Further, we need to associate two quad groups (Q_1, Q_{13}) with one quad group (Q_8, Q_{13}). The grouping details are shown in

The above grouping ensures that $Q_7, Q_8, Q_9, Q_{10}, Q_{11}$ can be grouped to achieve an upper bound of at most 3 bits per vertex. Interaction-types Q_5 and Q_{12} do not need to be grouped, since these are already assigned 3 bits each. We conclude that quadmesh connectivity can be encoded in less than 3 bits per vertex. Table 5 summarizes the final groupings.

From Table 3, the total cost of the encoding is

$$3Q - (Q_2| + |Q_3| + |Q_4|) - 3(|Q_7| + |Q_9|) - 6|Q_8| - |Q_{10}| - |Q_{11}| - 3|Q_3| - 3.$$

Since $V = Q + 2$ (exactly) for a quad mesh, the total cost is therefore guaranteed to be less than 3 bits per vertex.

Group	Total bits	Group	Total bits	Average bits
(Q_7, Q_{13})	6	(Q_1, Q_{13})	3	2.25
(Q_8, Q_{13})	6	$(Q_1, Q_{13}, Q_1, Q_{13})$	6	2.0
(Q_9, Q_{13})	6	(Q_1, Q_{13})	3	2.25
Q_{10}	5	(Q_1, Q_{13})	3	2.67
Q_{11}	5	(Q_1, Q_{13})	3	2.67

Table 4: *Code bits analysis for Q_7 to Q_{11}*

Grouping	Total Cost	quads in group	Amortized Cost	bits saved	occurrences of this group
(Q_2, Q_{13}) or (Q_3, Q_{13}) or (Q_4, Q_{13})	5	2	2.5	1	$\|Q_2\| + \|Q_3\| + \|Q_4\|$
$(Q_7, Q_{13}, Q_1, Q_{13})$ or $(Q_9, Q_{13}, Q_1, Q_{13})$	9	4	2.25	3	$\|Q_7\| + \|Q_9\|$
$(Q_8, Q_{13}, Q_1, Q_{13}, Q_1, Q_{13})$	12	6	2.0	6	$\|Q_8\|$
(Q_{10}, Q_1, Q_{13})	8	3	2.67	1	$\|Q_{10}\|$
(Q_{11}, Q_1, Q_{13})	8	3	2.67	1	$\|Q_{11}\|$
remaining(Q_1, Q_{13})	3	2	1.5	3	$\|Q_3\|+1$
Q_5	3	1	3	0	$\|Q_5\|$
$(Q_6,\, Q_{13})$	6	2	3	0	$\|Q_6\|$
Q_{12}	3	1	3	0	$\|Q_{12}\|$

Table 5: *Amortization Analysis*

KG	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}	Q_{13}
KRS	LE	LL	LS	LR	LC	LE	SL	SS	SR	SC	CS	CR	CC

Table 6: *Correspondence between the labelling schemes*

4 Improving the upper bound to 2.67 bpv

Table 6 shows the connection between the labelling schemes of Kronrod-Gotsman[KG] and King et al[KRS] The first row contains the quad labels, while the second row contains the equivalent combinations of CLRES labels. This correspondence is obtained by noticing that in the scheme of King et al [2] a quad is implicitly split by a diagonal into two triangles so that the next gate is situated counterclockwise with respect to the current one.

From the above table of equivalence of labels, we can obtain an encoding scheme that uses less than 2.67 bits per vertex, using the encoding schemes of King et al [2].

5 Conclusions

In this note, we show that a scheme proposed by Kronrod & Gotsman [1] can be improved to have an upper bound of less than 3 bits per vertex. Also an easy equivalence between the labelling schemes of [1] and [2] shows that this can be further improved to 2.67 bits per vertex for manifold meshes. We believe that the upper bound can be further improved and this is the main open question.

References

[1] Boris Kronrod and Craig Gotsman. *Efficient coding of non-triangular mesh connectivity*, Proc. 16th European Workshop on Computational Geometry, pp. 24-26, 2000.

[2] Davis King, Jarek Rossignac and Andrzej Szmczak. *Connectivity compression for irregular quadrilateral meshes*, GVU Tech Report GVU-GIT-99-36.

[3] M. Deering. *Geometry Compression*, Proc. ACM SIGGRAPH'95, pp. 13-20, August 1995.