# Sufficiently Fat Polyhedra are Not 2-Castable*

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## Abstract

In this note we consider the problem of manufacturing a convex polyhedral object via casting. We consider a generalization of the sand casting process where the object is manufactured by gluing together two identical facets of parts cast with a single piece mold. In this model we show that the class of convex polyhedra which can be enclosed between two concentric spheres with the ratio of their radii less than 1.07 cannot be manufactured using only two cast parts.


## 1 Introduction

Casting is a common manufacturing process where some molten substance is poured or injected into a cavity (called a *mold*), and then allowed to solidify. In many applications (see e.g. [2, 3]) it is desirable to remove the cast object from the mold without destroying the mold (or, obviously the recently manufactured object). In general this requires the mold to be partitioned into several parts, which are then translated away from the cast object. In the simplest case (prevalent in sand casting), a mold for polyhedron $P$ is partitioned into two parts using a plane. If a successful partition (i.e. both parts can be removed by translations without collisions), we say that $P$ is 2-castable.

Guided by intuition about smooth objects, one might suspect that all convex polyhedra are 2-castable. This turns out not to be the case. Bose, Bremner and van Kreveld [1] gave an example of a 12 vertex convex polyhedron that is not 2-castable. Unfortunately the proof of non-castability relies on a computer based exhaustive search. Majhi, Gupta and Janardan [5] gave a simpler example with only 6 vertices; here the proof of non-castability is left as an exercise. In neither case can one draw any general conclusions (beyond the tautological) about what sort of convex polyhedra are 2-castable.

In the present note we provide a general class of convex polyhedra that are not 2-castable. In particular, we establish that if the polyhedron has vertices and facets in general position and is a sufficiently close approximation of a sphere, then it is not 2-castable.

*Research funded in part by NSERC Canada

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## 2 Background

We will actually consider a slightly more general definition of 2-castability. We start with the definition of a castable polyhedron. Consider a 3-dimensional half-space $H$. Let $P$ be a polyhedron that lies in $H$, while one of its facets $F$ lies on the boundary of $H$. The set $M = H \setminus P$ is called a *mold* for $P$. We say that $P$ is *castable with respect to a facet* $F$ (or equivalently *with respect to a mold* $M$), if we can pull $P$ out of $M$ by moving it along some vector $d$ without collisions (e.g. interior intersections). If there exists such a facet of $P$, then $P$ is called *castable*. A polyhedron $P$ that can can be divided into $k$ castable parts is called *k-castable*. Note that for $k = 2$ it is often required by manufacturing processes that the two halves are castable with respect to the same mutual facet. This constraint is relaxed here, however we do require that the halves are separated by a plane (i.e. have a mutual facet).

**Definition 1** *A convex polyhedron $P$ is called $(R_i, R_o)$-*fat *if there are two concentric spheres $D_i$ and $D_o$ of radii $R_i$ and $R_o$, such that $D_i \subset P \subset D_o$.*

**Definition 2** *A convex polyhedron is said to be in* general position *if*

- *there are at most 2 facets parallel to a line*
- *if three of its edges lie in a plane then they belong to the same facet*

Let $P$ be a general position $(R_i, R_o)$-fat polyhedron. By scaling $P$ we can assume $R_i = 1$ and $R_o = R > 1$. For the rest of the paper we use *fat* to mean $(1, R)$-fat. Let $O$ denote the center of concentric spheres from Definition 1. The following lemma gives bounds for various elements of a fat polyhedron.

**Lemma 1** *Let $P$ be a fat polyhedron. Every edge of $P$ has length at most $l^* = 2\sqrt{R^2 - 1}$, every facet has area at most $S^* = \pi(R^2 - 1)$ and its volume is bounded $\frac{4}{3}\pi < V(P) < \frac{4}{3}\pi R^3$.*

*Proof.* Let $AB$ be an edge of $P$ and $O'$ be the projection of $O$ to $AB$. Since $|OA| \le R$ and $|OO'| \ge 1$, we have $|AO'| \le \sqrt{R^2 - 1}$, so $|AB| \le |AO'| + |O'B| < 2\sqrt{R^2 - 1}$.

Every facet $F$ of $P$ defines a slice $C$ of the outer ball (enclosed by the outer sphere $D_o$), indeed $F$ is contained within the disk $C$. By the previous consideration, the radius of $C$ is at most $l^*/2$. Therefore, we have the bound $S(F) \leq \pi(R^2 - 1)$.

Since $D_i \subset P \subset D_o$, we have the bounds on the volume of $P$ given by the lemma. □

The following observation is simple but important for the rest of the paper. If we restrict class of polyhedra to $(1, R)$-fat and choose $R$ sufficiently close to 1 then $l^*$ can be made arbitrary small. That is, $l^* \to +0$ as $R \to 1 + 0$. The following lemma gives an upper bound to the volume of a castable polyhedron.

**Lemma 2** *Suppose that $P$ is castable through a facet $F$ of area $S$. Let $H$ be the plane containing $F$ and $h$ be the maximum distance from a point $P$ to $H$. Then $V(P) \leq Sh$.*

*Proof.* Let $v$ be the inner normal unit vector to $F$ and $F(t)$ be the area of $P \cap H + tv$, for $t \geq 0$. Since $P$ is castable through $F$, the area $F(t)$ is at most $S$. Thus $V(P) = \int_0^h F(t) \leq Sh$. □

Let us call $h$ the *thickness* of $P$ with respect to $F$. Note that thickness is bounded by the diameter of $P$, thus it cannot exceed $2R$.

## 3 The proof of non-castability

We use the following method to prove that $P$ is not 2-castable. We will consecutively assume that certain polyhedra are castable under some restrictions and then argue that this implies certain lower bounds on $R$. Assume that all the possible situations are covered and let $R > R^*$ be the loosest bound on $R$. Then $P$ is not 2-castable provided $R < R^*$. In the following, let $S(F)$ denote the area of polygon $F$.

**(I)** First assume that $P$ is *1-castable* through some facet $F$. Using Lemma 2 and Lemma 1, we derive the following inequality.

$$\pi(R^2 - 1)2R \geq S(F)h \geq V(P) > \frac{4}{3}\pi$$

This inequality implies the bound $R > 1.240$.

**(II)** Suppose that $P$ is 2-*castable*. Let $P$ be sliced by a plane. Denote the larger part by $P_1$, the smaller by $P_2$ and their mutual facet by $C$, so that $V(P_1) \geq V(P)/2 \geq V(P_2)$. To simplify the presentation here, without loss of generality assume that $O = (0, 0, 0)$, $C$ is horizontal, $P_1$ lies above $C$ and $P_2$ below. Let the plane containing $C$ be given by the equation $z = z_0$. Each of $P_1$ and $P_2$ has to be 1-castable.

**(IIa)** Assume that $P_1$ is castable through a facet $F \neq C$. As before, we have

$$\pi(R^2 - 1)2R \geq S(F)h \geq V(P_1) \geq \frac{V(P)}{2} > \frac{2}{3}\pi$$

In this case, $R > 1.137$.

**(IIb)** Assume that $P_1$ is castable through $C$, and $P_2$ is castable through a facet $F \neq C$. First consider the case, where $z_0 \geq 0$. Using Lemma 2 for $P_2$ we derive

$$\pi(R^2 - 1)2R > S(F)h > V(D_i)/2 = 2\pi/3 \quad (1)$$

since $P_2$ contain the lower part of the inner sphere. Solving (1) numerically we obtain $R > 1.137$.

Now consider the case of $z_0 < 0$. Since $P_1$ contain the disk $D_i \cap \{z = 0\}$, we require that the diameter of the slice $C$ is at least 1, and hence

$$\sqrt{R^2 - z_0^2} > 1 \quad (2)$$

Using Lemma 2 for $P_2$ gives

$$2\pi(R^2 - 1)\sqrt{R^2 - z_0^2} \geq S(F)h \geq V(P_2)$$
$$> \pi \int_{-1}^{z_0} 1 - t^2 dt = \pi(2/3 + z_0 - z_0^3) \quad (3)$$

since the diameter of $P_2$ is at most $2\sqrt{R^2 - z_0^2}$. The numerical solution of the system of (2) and (3) gives $R > 1.072$.

Summarize the argument so far with the following lemma.

**Lemma 3** *If $P$ is 2-castable then either both $P_1$ and $P_2$ are castable through the common facet $C$ or $R > 1.072$.*

Assume that both $P_1$ and $P_2$ are castable through $C$. For each edge $e$ of $C$ consider its incident facets $F_1$ in $P_1$ and $F_2$ in $P_2$ other than $C$. We *mark* $e$, $F_1$ and $F_2$ if these facets constitute a facet of $P$.

**Lemma 4** *There are at most 2 unmarked edges.*

*Proof.* Each unmarked edge corresponds to an edge of $P$ that lies in $C$. The lemma follows from the general position assumption. □

Consider the set of feasible casting directions $d = (d_x, d_y, d_z)$ for a polytope $P$. Without loss of generality assume that $d_z = -1$ for $P_1$ and $d_z = 1$ for $P_2$.

It is known [4] that each facet $F$ of $P_1$ ($P_2$) implies a linear constraint on $d$, namely $(\mu, d) \leq 0$, where $\mu$ is the outward normal to $F$ with respect to $P_1$ ($P_2$). We restrict ourselves to the facets of $P_1$ ($P_2$) which are incident with marked edges of $C$. Let $\mathrm{LP}_1$ ($\mathrm{LP}_2$) be the

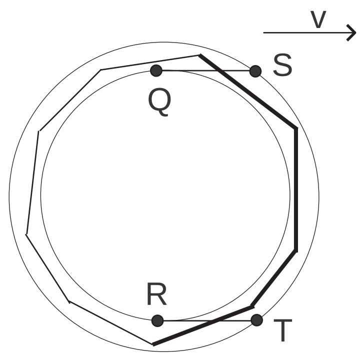


Figure 1: The facet $C$, bold edges are unmarked and form the chain

corresponding 2-dimensional linear programs. Castability of $P_1$ ($P_2$) implies feasibility of $\mathrm{LP}_1$ ($\mathrm{LP}_2$). Note that if a facet $F_1$ of $P_1$ contributes to $\mathrm{LP}_1$ then the incident facet $F_2$ of $P_2$ contributes to $\mathrm{LP}_2$. The corresponding inequalities are:

$$l_1(d) = \mu_x d_x + \mu_y d_y - \mu_z \leq 0 \quad (4)$$
$$l_2(d) = \mu_x d_x + \mu_y d_y + \mu_z \leq 0 \quad (5)$$

Let us define feas($\mathrm{LP}_i$) to be the feasible region of a program $\mathrm{LP}_i$, i.e. the intersection of the constraints $l_i(d) \leq 0$ for each $\mu$. We study these programs in more detail. First note that feasible regions of these programs have to have inner points. For otherwise some there are either:

- Two constraints of the form $(\mu, d) \leq 0$ and $(-\mu, d) \leq 0$ (their bounding lines coincide). Then the corresponding facets are parallel.

- Three constraints, such that their bounding lines on the plane $d_z = \pm 1$ intersect in a point. This means that the corresponding facets are parallel to a line.

Let $d$ and $e$ be inner points in feas($\mathrm{LP}_1$) and feas($\mathrm{LP}_2$) respectively. Then for every $l_1 \in \mathrm{LP}_1$, we have $l_1(d) < 0$. So $l_2(-d) = -l_1(d) > 0$ and $-d$ satisfies no constraints of $\mathrm{LP}_2$. Consider the ray $r = (-d)+\lambda(e-(-d))$ starting at $-d$ towards $e$. The segment between $-d$ and $e$ of this ray has to intersect every bounding line in $\mathrm{LP}_2$, since one of its endpoints satisfies all the constraints while the other satisfies none of them. Therefore the remainder of $r$ (beyond $e$, $\lambda \geq 1$) can not intersect any any of the bounding lines of $\mathrm{LP}_2$. We conclude that feas($\mathrm{LP}_2$) is unbounded with respect to $r$. Similarly we can prove, that feas($\mathrm{LP}_1$) is also unbounded. Suppose that feas($\mathrm{LP}_1$) is unbounded along a ray $r' = p + \lambda v$, $\lambda \geq 0$.

Consider the plane containing the convex polygon $C$ illustrated in Figure 1. Suppose there exists a marked edge $e$ with the outward normal $n$, such that $(n, v) > 0$. Let $\mu$ be the outward normal of the corresponding facet of $P$, whose projection to $\{z = 0\}$ is $n$. Then

$$\begin{aligned} &\mu_x(p_x + \lambda v_x) + \mu_y(p_y + \lambda v_y) + \mu_z \\ &= \lambda(n, v) + \mu_x p_x + \mu_y p_y + \mu_z \leq 0 \end{aligned}$$

for every $\lambda \geq 0$, which is a contradiction. So all marked edges have outward normals $n$ such that $(n, v) \leq 0$. Consider the edges of $C$ with the outward normal $n$, such that $(n, v) > 0$. All such edges are unmarked and form a chain since $C$ is convex. Define the segments $QS$ and $RT$ to be the segments parallel to the vector $v$ and touching the interior circle at the points $Q$ and $R$ and the points $S$ and $T$ lie on the exterior circle (see Figure 1). The first and last edge of $C$, that intersect the interior of $QRTS$ have to belong to this chain. So the chain connects $QS$ and $RT$, hence its length is at least 2. But we know that there are at most two unmarked edges, thus we one of them has to be longer then 1. This means that $l^* = 2\sqrt{R^2 - 1} > 1$, an thus $R > \sqrt{5/4} > 1.118$. Bringing all of the bounds on $R$ together, we conclude with

**Theorem 1** *If $P$ is a $(R_i, R_o)$-fat polyhedron in general position and $R_o/R_i < 1.072$, then $P$ is not 2-castable.*

## References


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