A Note on Communication-Efficient Deterministic Parallel Algorithms for Planar Point Location and 2d Voronoï Diagram[*]

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Abstract

In this note we describe deterministic parallel algorithms for planar point location and for building the Voronoï Diagram of n co-planar points. These algorithms are designed for BSP/CGM-like models of computation, where p processors, with $O(\frac{n}{p}) \gg O(1)$ local memory each, communicate through some arbitrary interconnection network. They are communication-efficient since they require, respectively, $O(1)$ and $O(\log p)$ communication steps and $O(\frac{n \log n}{p})$ local computation per step. Both algorithms require $O(\frac{n}{p}) = \Omega(p)$ local memory.

1 Introduction

The planar point location and the Voronoï diagram problems are among the most fundamental problems in computational geometry, widely used in various application areas like image processing, robotics and others [16]. On the other hand, distributed memory multicomputers, i.e. BSP/CGM like machines, have emerged as the preeminent commercially available parallel architectures. Therefore, there has been a growing interest in coarse grained computational models [4, 7, 12, 18] and the design of coarse grained algorithms [5–9, 11, 15], for classic problems in computational geometry. In this mixed sequential/parallel setting, there are three important measures of any coarse grained algorithm, namely, the amount of local computation, the number and type of global communication phases required and the scalability of the algorithm.

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In this note we describe how to use known algorithmic ideas for solving planar point location and 2D voronoi diagrams, originally developed for regular architectures (the mesh [13] and the hypercube [17]), as the basis for communication efficient CGM deterministic algorithms. Given n as the input size, and p as the number of processors of a CGM, then our planar point location algorithm requires local storage $\frac{n}{p} = \Omega(p)$ and is optimal with respect to local computation ($O(\frac{n \log n}{p})$) and communication phases ($O(1)$). This algorithm is then used as a procedure in our Voronoï Diagram algorithm, which also requires local storage $\frac{n}{p} = \Omega(p)$, but uses $\lceil \log p \rceil$ communication phases with $O(\frac{n \log n}{p})$ local computation per phase.

It should be remarked that the principal challenge in deterministically computing the Voronoï diagram in a distributed memory setting is that the problem does not exhibit much locality: the introduction or deletion of a single point may require that every Voronoï cell be recomputed. Note, however, that this lack of locality is also a factor in many graph problems and the most efficient coarse-grained algorithms known for these problems ([2]) also present a $O(\log p)$-rounds communication complexity. Notwithstanding, such algorithms are very efficient in practice: since p is independent of and very small when compared to n, for most practical applications this factor is not even noticed ([10]).

2 The coarse-grained model

In a "bulk synchronous" processing model, an input of size n is distributed evenly across a p-processor parallel computer [18]. In a single *computation round* or *superstep* each processor may send and receive h messages and then perform an internal computation on its internal memory cells using the messages it has just received. To avoid conflicts that might be caused by asynchronies in the network (whose topology is left undefined) the messages sent out in a round t by some processor cannot depend upon any messages that the processor receives in round t.

In this note we use the Coarse-Grained Multicomputer model, or *CGM*(n,p) for short, introduced in [7]. The CGM(n,p) is a BSP model consisting of a set of p processors with $O(\frac{n}{p})$ local memory each. The term "coarse grained" refers to the fact that (as in practice) $O(\frac{n}{p})$ is defined to be "considerably larger" then $O(1)$. The definition of "considerably larger" is $\frac{n}{p} \geq p^{\epsilon}$, where ϵ depends on the proposed algorithms; in this paper $\epsilon = 1$.

Furthermore, it was shown that, given $n^{1-\frac{1}{c}} > p$ ($c \geq 1$), sorting $O(n)$ elements distributed evenly over p processors in the CGM, BSP or LogP models can be achieved in $O(\log n / \log(h+1))$

communication rounds and $O(n \log n/p)$ local computation time, for $h = \Theta(\frac{n}{p})$, i.e. with optimal local computation and $O(1)$ h-relations, when $\frac{n}{p} = \Omega(p)$ [11]. Therefore, using this sort, the communication operations of the $CGM(s, p)$ can be realized in the BSP or LogP models in a constant number of h-relations, where $h = \Theta(\frac{s}{p})$.

The algorithms proposed for the CGM are independent of the communication network. Moreover, it was proved that the main collective communication operations can be implemented by a constant number of calls to global sort ([7]). Hence, by the result recalled above, these operations take a constant number of communication rounds. However, in practice these operations will be implemented through built-in, optimized system-level routines([10]).

3 Planar Point Location

The planar point location problem is stated as follows: Given a straight line planar graph with n vertices (a planar subdivision) and an arbitrary query point q, determine the region of the subdivision containing q.

In this note we address the case where the planar subdivision is convex and where many points are to be located instead of only one.The problem of planar *multi*-point location on a convex subdivision is then stated as follows: Locate $O(n)$ points in a planar convex subdivision defined by $O(n)$ edges. Each edge is labeled with the regions to its left and its right, and regions are defined by coordinates of one interior point (called the center of the region).

3.1 Previous work in planar point location

Many algorithms (sequential or parallel) have been proposed for solving the multi-point version of this problem [1, 14], where $O(n)$ query points are located in a planar convex subdivision with n vertices. The sequential complexity of the problem is $\Theta(n \log n)$ time with $O(n)$ space. In the fine grained parallel setting, algorithms have been described for many architectures including the CREW PRAM [3], the Hypercube [17] and the Mesh [13]. Except for the PRAM, these algorithms are not work-optimal (using n processors, time in $O(\sqrt{n})$ and $O(\log^2 n)$ for the Mesh and the Hypercube, respectively).

To locate a point in the planar subdivision, we design a coarse-grained algorithm based on the chain method originally described in the sequential setting ([14]) and then utilized in the fine-grained parallel setting for MCC ([13]) and hypercubes ([17]).

3.2 The chain method

The idea of the chain method is to perform planar point location via a binary search on a balanced binary tree whose nodes represent a chain of edges of the planar subdivision. The tree is built as follows.

First the regions are sorted by x-coordinate of their centers. There is a chain of edges which share half regions to left and half to right (left and right regions correspond to centers lying to left or right of the chain). The same is applied to left and right half of regions recursively and a monotone complete set of chains is obtained (i.e. the set of chains so that for any two chains c_1 and c_2 the vertices of c_1 that are not on c_2 are on the same side of c_2). These chains are the nodes of the balanced binary tree mentioned above.

The leaves of this tree correspond to regions of the subdivision (see Figure 1). Chains may share common edges. If an edge e belongs to more than one chain then it belongs to all members of a set of consecutive chains. There is a unique member c of this set which, in the binary search tree, is a common ascendant of all the other members of the set (the highest chain, in the hierarchy, to which e belongs). In order to avoid duplication of edges, we assign e to such a member c. By $O(\log n)$ discriminations (deciding on which side of chain c a query point lies) each query point can be located.

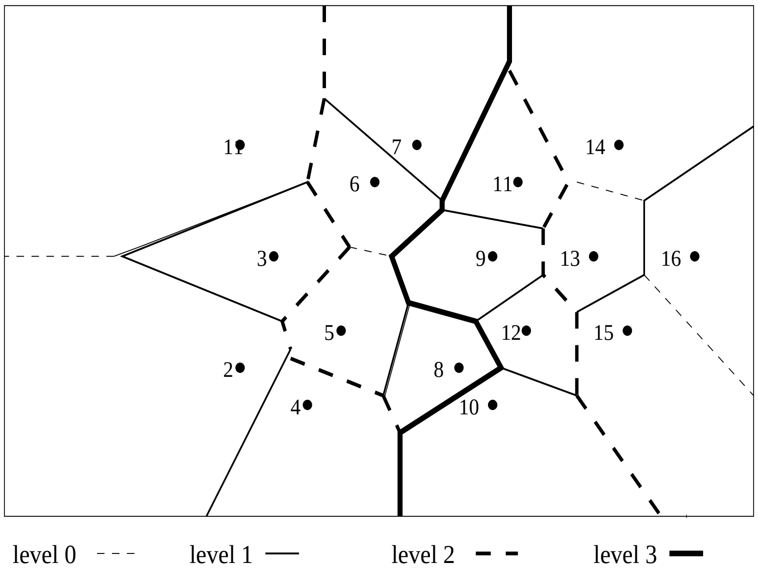


Figure 1: Construction of the chains.

Each chain has a level and an index. The level of a chain is the height were the chain is located in the tree (the root has the highest level). The index is the rank of the chain in the chains of a given level, ranked from left to right. And as described above, each edge is assigned to exactly one chain. The level and the index of an edge are those of the chain to which it belongs to. The levels and indices of the edges can be determined in constant time using the rules described in [13]: for a given edge e, find the "bit exclusive or", say ψ, of the binary indices of centers of e. The level of e, say l_e, is $l_e = \lfloor \log \psi \rfloor$. The index of e is $((2's\ complement(2^{l_e}) - 2^{l_e}) \wedge (index\ of\ center\ of\ e))/2^{l_e+1}$.

3.3 Coarse-Grained Planar Multi-Point Location

We describe in this subsection a planar multi-point location algorithm that requires a constant number of communication rounds. The entire data for a given problem is assumed to be initially distributed across the local memories and remains there until the problem is solved. Given a set Q of n query points, a planar convex subdivision of the plane into n regions (e.g. a Voronoï diagram) and a p processor coarse grained multicomputer we show how to locate the query points into the subdivision.

The basic approach is as follows: First divide the plane into the p regions or vertical slabs $V_1, V_2, \dots V_p$ defined by the $p-1$ highest level chains. Then, for each point $q \in Q$ determine $vs(q) \in \{V_1, V_2, \dots V_p\}$ the vertical slab q is located in. (This is done by forming horizontal slabs from the chains computed in Step 1 and performing a point location within these horizontal slabs after first having load balanced the points and slabs.) Next, load-balance the vertical slabs and the points such that each processor stores $O(1)$ vertical slabs of total size $O(n/p)$ and $O(n/p)$ points that must be located in them. Finally, locally execute planar multi-point location on all processors.

The main challenge lies in computing for each point which vertical slab it is in in a constant number of communication phases and under the constraint given by the memory size. The idea will be to partition the vertical slabs into p horizontal slabs that are bounded by lines rather than polygonal chains. Our Planar Multi-Point Location algorithm is described in detail below.

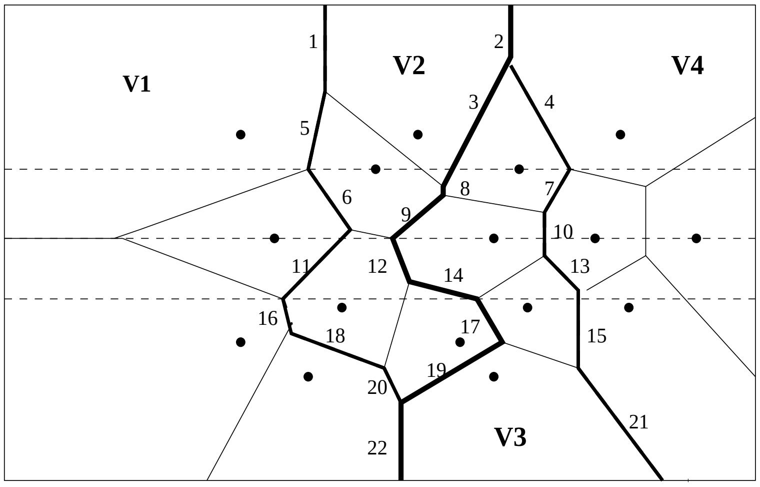
CGM's Planar Multi-Point Location(Q, $Vor(S)$)

Input: A set Q of $O(n)$ query points and a planar subdivision defined by $O(n)$ edges.
Output: The $O(n)$ query points labeled by the center of the region to which they belong.

1. For each edge, determine to which chain it belongs using the chain method described above, which involves sorting the regions' centers by their x-coordinate. Recall that using this method, each edge belongs to only one chain. Note that we are only interested in the $p-1$ higher level chains, these chains partition the plane into p "vertical" slabs $V_1, V_2, \ldots V_p$ (Figure. 2). Let C denotes the set of the edges that define the $p-1$ chains.
2. Sort the edges in C by their largest y-coordinates. Each processor i receives $O(n/p)$ edges denoted H_i and can determine a horizontal line that defines its upper boundary by looking for the largest received y-coordinate (Figure. 2). Perform an all-to-all broadcast of these horizontal lines so that every processor stores a copy of H, the set of these p horizontal lines.
3. Each processor determines for each edge $c \in C$ it stores the elements of H it intersects, denoted $range(c)$. Note that, because the chains are y-monotonic, $range(c)$ is a (contiguous) interval that can be computed by binary search in H, each edge is intersected by at most p horizontal lines and each element of H intersects at most p elements of C. Perform a personalized all-to-all broadcast such that each edge c, for which $range(c) = [i, j]$ is not empty, is broadcast to processors i through j.
4. For each point $q \in Q$ determine $hs(q) \in \{H_1, H_2, \ldots H_p\}$ the horizontal slab q is located in and for each horizontal slab H_i, compute $C(H_i) = \lceil \frac{|\{q \in Q : hs(q) = H_i\}|}{\frac{n}{p}} \rceil$, for $1 \leq i \leq p$. Create $C(H_i)$ copies of H_i and distribute them such that each processor stores at most two horizontal slabs. Redistribute Q such that each point $q \in Q$ is stored on a processor that also stores a copy of $hs(q)$.
5. Each processor locally executes Kirkpatrick's planar multi-point location algorithm ([16]). When a point is located to the right or the left of an edge, the vertical slab to which it belongs, $vs(q)$ is obtained by consulting the rank of the center of the region associated to the edge, in the sorted list.
6. For each vertical slab V_i, compute $C(V_i) = \lceil \frac{|\{q \in Q : vs(q) = V_i\}|}{\frac{n}{p}} \rceil$, for $1 \leq i \leq p$. Create $C(V_i)$ copies of V_i and distribute them such that each processor stores at most two vertical slabs. Redistribute Q such that each point $q \in Q$ is stored on a processor that also stores a copy of $vs(q)$
7. All processors now locally execute Kirkpatrick's planar multi-point location ([16]). The location is done in the vertical slab into which the points are located and each point is now precisely located.

Theorem 1 *Algorithm* **CGM's Planar Multi-Point Location()** *locates $O(n)$ query points in a planar convex subdivision defined by $O(n)$ edges in $O(\frac{n \log n}{p})$ time. It requires $\frac{n}{p} = \Omega(p)$ local memory space and a constant number of communication rounds.*

Proof: The correctness of algorithm **CGM's Planar Multi-Point Location()** follows from the correctness of the chain method, the correctness of Kirkpatrick's sequential planar multi-point location method [16, pages 56-58], and the following observations. (1) Both the vertical and horizontal slabs have a size of $O(n/p)$. (2) The total number of slabs created in Steps 4 and 6 is $O(p)$. (3) The total number of queries moved in steps 4 and 6 is $O(n/p)$. The space requirement is thus $O(\frac{n}{p}+p) = O(\frac{n}{p})$ per processor. In each step, the local computation time is at most $O(\frac{n}{p} \log n)$. The


V1
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22
P=4, horizontal cutting, vertical cutting,
H_1 =edges {1, ...,6}, H_2 =edges {7, ...,12}, H_3 =edges {13,...,17}
H_4 =edges {18,...,22}

Figure 2: Horizontal and Vertical Cuttings.

global communication in each step reduces to a constant number global sorts and communications operations. □

4 Building a 2D-Voronoï Diagram on a CGM

The Voronoï diagram of a finite set S of points in the plane is defined as a partition in which each region is composed of the points which are closer to a point in S in the region than to any other point in S. Important problems such as Delauney triangulation or 3D convex hull are directly equivalent to the Voronoï diagram problem, whose sequential complexity is $\Theta(n \log n)$. [16].

4.1 Previous work in parallel Voronoï diagram

The only time-optimal parallel algorithm (although not work-optimal since it runs in $O(\sqrt{n})$ time with n processors) was proposed in [13] for the Mesh. The same technique (to be explored further in this text) was used in [17] to design a $O(\log^3 n)$ time algorithm for the Hypercube. Finally, the best existing PRAM algorithm requires $O(\log n \log \log n)$ time with $O(n \log^2 n)$ work, or $O(\log^2 n)$ time with $O(n \log n)$ work [3]. With respect to the CGM, no efficient deterministic algorithm exist. The *randomized* algorithm from [5] builds the Voronoï diagram in time $O(\frac{n \log n}{p})$, *with high probability*, and requires $n/p = \Omega(p^2)$.

4.2 Coarse-grained parallel Voronoï diagram

In this section, we first present an algorithm for merging two Voronoï diagrams on a $CGM(n,p)$ which requires only $O(1)$ communication phases and then show how this algorithm can be used to help build the Voronoï diagram of a set of 2d-points through a divide-and-conquer approach. The merge algorithm in turn uses the planar multi-point location algorithm described in the previous section as a basic subprocedure.

Let a set S of n points (the center of each region) in the plane be given and P and Q be two disjoint subsets of S, of size $\frac{n}{2}$ each, such that all points of P are located to the left of all points of Q. Suppose that the Voronoï diagrams of P and Q are known and denoted by Vor(P) and Vor(Q), respectively. Finally, suppose that Vor(P) and Vor(Q) are each represented by a set of edges distributed evenly over $p/2$ processors.

Our merging algorithm implements, on a CGM, the scheme from [13]. For this we extensively use the multi-point location algorithm, presented in the previous section, in order to build the chain between two Voronoï diagrams (see Figure 3). Since the problem is analogous with respect to P or Q, we will describe the details of the merging from only the point of view of P.

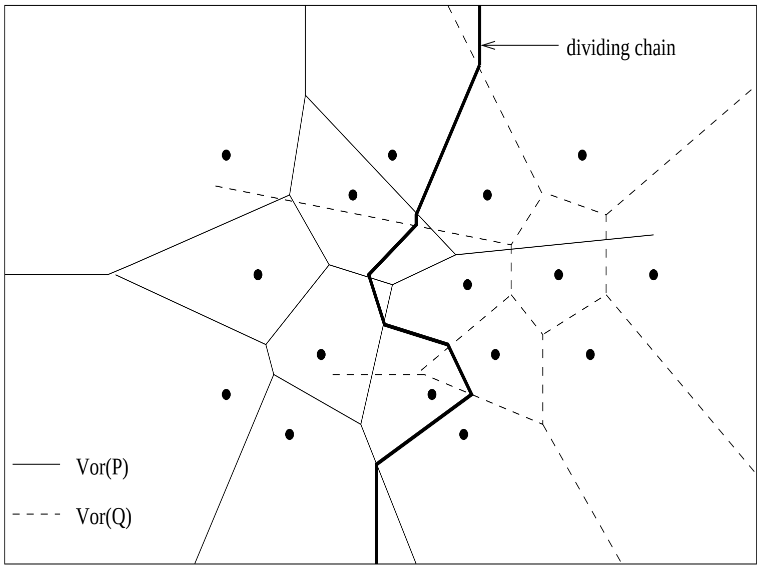

Figure 3: The dividing chain.

The following are the main steps of the algorithm.

Two-Way Merging(Vor(P),Vor(Q))

Input: A distributed representation of Vor(P) and Vor(Q), each over $\frac{p}{2}$ processors.
Output: A distributed representation of Vor($P \cup Q$) over p processors.

1. Partition the edges of Vor(P) into three sets:
   (a) PP, those that have both their endpoints closer to P than to Q,
   (b) PQ, those that have one of their endpoints closer to P than to Q, and the other one closer to Q than to P.
   (c) QQ those that have both their endpoints closer to Q than to P.
2. For each of the sets found above, decide which edges are intersected by the dividing chain (actually the problem is just for QQ).
3. Compute the new endpoints for the edges that are intersected by the dividing chain (intersection point with the dividing chain) and discard the portion of the edge laying in the wrong side.
4. Globally sort all the newly generated endpoints (of the edges of Vor(P) and Vor(Q)) in order to obtain the edges of the dividing chain (for the infinite rays, it suffices to look at the two points, one in P and the other one in Q, that are closer to their finite endpoint to find their slope).
5. Perform Steps 1 through 4, analogously, with respect to Vor(Q).
6. All the current edges form Vor(S). Distribute them over the p processors.

Theorem 2 *Given two sets P and Q of $\frac{n}{p}$ points in the plane, $P \cup Q = S$, such that all points in P are on the left of all points in Q, and a distributed representation of the two Voronoï diagrams Vor(P) and Vor(Q), each distributed over $\frac{p}{2}$ processors, then algorithm* **Two-Way Merging()** *merges Vor(P) and Vor(Q) to form Vor(S) in $O(\frac{n \log n}{p})$ time. It requires $\frac{n}{p} = \Omega(p)$ local memory space and a constant number of communication rounds.*

Proof: In Step 1, partitioning the edges into the sets PP, PQ and QQ can be computed for the finite edges by performing a planar multi-point location of the endpoints of the edges. For the semi-infinite edges, Jeong [13] has established the following lemma:

Lemma 1 *Suppose that all the semi-infinite edges of Vor(Q) are sorted by their slope δ. For the infinite endpoint v_i and the semi-infinite edges e_i of Vor(P), and two consecutive semi-infinite edges e_j and e_{j+1} of Vor(Q), v_i is laying in the unbounded region bordered by e_j and e_{j+1} if and only if $\delta_{e_j} \leq \delta_{e_i} \leq \delta_{e_{j+1}}$.*

Using this lemma, we can find the center of the region, in Vor(Q), containing the endpoint at infinity and thus see to which set it is closer to by just computing the bisector between the closest point in P and the closest one in Q and then see if the semi-infinite edge crosses this bisector. Hence, the time complexity of this step is also dominated by calls to the planar point location algorithm, that is $O(\frac{n \log n}{p})$.

For Step 2, it was shown in [13] that the edges in PP do not cross the dividing chain, the edges in PQ cross it once, and for the edges in QQ we have two cases: if they cross the dividing chain they cross it twice, or else they do not cross it at all (see Figure 4). A simple technique to distinguish these two cases involves again a planar multi-point location: The point location concerns, for each edge of QQ, a unique and precise point X on the concerned edge. Each edge which is determined to be intersected twice is split into two edges of type PQ at the point X. For an edge e in QQ, X is the intersection point between e and the horizontal line passing through one of the centers of the two regions associated to e. The chosen center is the one with the greatest x-coordinate ([13]). Here again, the time complexity of this step is also dominated by calls to the planar point location algorithm, that is $O(\frac{n \log n}{p})$.

Step 3 computes one intersection point per edge since the edges that are intersected twice are now split into two edges of type PQ. The computation of the intersection point can be done in constant time by computing the bisector between the point in P (the one with the greatest x-coordinate) closest to the first endpoint and the point in Q closest to the second endpoint, and then computing the intersection of the edge with this bisector.

Step 4 is composed by a global sort. Once the new endpoints are sorted (using their y-coordinate as principal key), the dividing chain is built. Recall that this chain is y-monotonic, i.e., it is crossed at most once by all horizontal lines. The time complexity of this step is thus $O(\frac{n \log n}{p})$.

Finally, Step 5 is a communication phase in which the newly built dividing chain is distributed over the appropriate processors.

Note that all of the steps consist of at most $O(\frac{n \log n}{p})$ local computation and a constant number of calls to the planar point location algorithm, therefore the time complexity follows. The correctness follows from [13]. $\square$

Using **Two-Way Merging()** we can now easily describe a CGM algorithm for building the Voronoï diagram. Recall that $p = 2^k$ for some integer k.

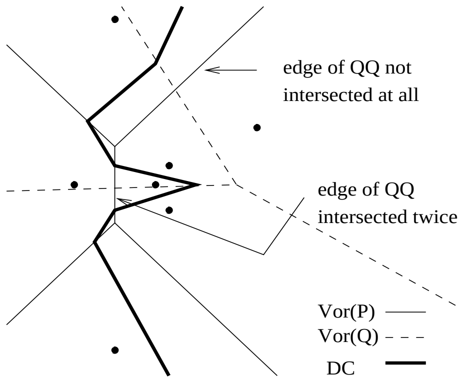
edge of QQ not intersected at all
edge of QQ intersected twice
Vor(P)
Vor(Q)
DC

Figure 4: Example of an edge that is intersected twice.

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Voronoï diagram(S)

Input: Each processor stores a set of $\frac{n}{p}$ points drawn arbitrarily from $S$.
Output: A distributed representation of the Voronoï diagram of $S$.

  1. Globally sort the points in $S$ by x-coordinate. Let $S_i$ denote the set of $\frac{n}{p}$ sorted
     points now stored on processor $i$.
  2. Independently and in parallel, each processor $i$ computes the Voronoï diagram
     of the set $S_i$. Let Vor($S_i^1$) denote the result on processor $i$.
  3. For $j = 1$ to $\log p$ in parallel do
         Vor($S_i^{j+1}$) $\leftarrow$ Two-Way Merging(Vor($S_{2i}^j$), Vor($S_{2i+1}^j$)),
         ($i$ from 0 to $\frac{p}{2^j} - 1$).
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Theorem 3 *Algorithm* **Voronoï diagram()** *computes the Voronoï diagram of a set S of n points in the plane, Vor(S), on a CGM(n,p). It requires $\frac{n}{p} = \Omega(p)$ local memory space, $\lceil \log p \rceil$ communication rounds, and $O(\frac{n \log n}{p})$ local computation time per round.*

Proof: Step 1 is composed of a global sort. The time complexity of this step is then $O(\frac{n \log n}{p})$. In Step 2, local Voronoï diagrams are computed. Since the sequential complexity of the Voronoï diagram problem is $O(n \log n)$, the time complexity of this step is $O(\frac{n \log n}{p})$. Finally, in Step 3 **Two-Way Merging()** is called $\log p$ times. At each call the time complexity is $O(\frac{n \log n}{p})$. Thus, the global time complexity of this step is $O((\frac{n \log n}{p}) \log p)$. $\square$

5 Conclusion

Distributed memory multicomputers, i.e. BSP/CGM like machines, have emerged as the preeminent commercially available parallel architectures. In this note we described how to use known algorithmic ideas for solving planar point location and 2D voronoï diagrams, originally developed for regular architectures (i.e. mesh, hypercube), as the basis for communication efficient CGM algorithms.

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